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DYNAMICS OF 2D NAVIER-STOKES EQUATIONS WITH RAYLEIGH'S FRICTION AND DISTRIBUTED DELAY

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Abstract. This article concerns the long time dynamics of a 2D incompressible Navier-Stokes equation with Rayleigh's friction and distributed delay. Under appropriate assumptions on the external force and delay term, we obtain global well-posedness in new phase spaces with delay. Using uniform estimates and compact embedding, we obtain a global attractor.

1. Introduction

The Navier-Stokes equation is a well-known model to describe the essential law of fluid flow. Its asymptotic dynamics can be used to construct mathematical analysis of turbulence for fluid flow, see for example [6, 7, 16, 28, 29, 31, 32, 33] and the references therein. The influence of the delay is originated from engineer and can be expressed by ordinary differential equation with delay terms such as control feedback; see [17] and Hale and Lunel [13]. Time variable delay and memory terms arise in many fields, such as physics, chemistry, biology, economic phenomena, control theory and so on. Moreover, a delay term is a source of instability, which means that the research on asymptotic dynamics for dissipative evolutionary equations with delay is significant in engineer and mathematical analysis.

This article is concerned with asymptotic dynamics for the 2D Navier-Stokes equation with Rayleigh's friction and distributed delay,

$$\begin{gathered}
u_t-\nu\Delta u+(u\cdot\nabla)u+\alpha u+\nabla p=f(x)+\int_{-h}^{0}G(s,u(t+s))ds,\\
(x,t)\in\Omega\times(\tau,+\infty),\\
\operatorname{div}u=0,\quad (x,t)\in\Omega\times(\tau,+\infty),\\
u(t,x)|_{\partial\Omega}=\varphi,\quad \varphi\cdot n=0,\quad (x,t)\in\partial\Omega\times(\tau,+\infty),\\
u(\tau,x)=u_0(x),\quad x\in\Omega,\\
u(t,x)=\phi(t-\tau,x),\quad (x,t)\in\Omega\times(\tau-h,\tau),\quad h>0,
\end{gathered}\tag{1.1}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain with smooth boundary, $\nu>0$ and $\alpha>0$ denote the viscosity and Ekman dissipative parameter respectively. In addition, u_0 and $\phi(\cdot)$ denote the initial data in time τ and interval $[-h,0]$ respectively. The terms $f(x)$ and $\int_{-h}^{0}G(s,u(t+s))ds$ be the autonomous and distributed delay external

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forces respectively. The Ekman damping αu denotes Rayleigh's friction which is widely used in geophysical hydrodynamics such as oceanic models. Moreover, we assume $\varphi \in L^\infty(\partial\Omega)$ for the analysis of unknown velocity $u = (u_1(t,x), u_2(t,x))$ and pressure $p = p(t,x)$.

Let us recall some known results for the dynamics and stability of the Navier-Stokes equation with delays.

(1) For Navier-Stokes models with finite continuous delays as constant or variable functions, such as $F(t, z(t), z(t-\rho(t)))$ for $\rho(\cdot) \in [-h, 0]$, the global well-posedness and existence of pullback attractors have been studied in [8, 9, 11, 12, 14, 18, 21, 22, 23, 27]. If the delay belongs to infinite interval, which is called infinite continuous delay, such as $F(t, z(t), z(t-\rho(t)))$ for $\rho(\cdot) \in (-\infty, 0]$, the pullback dynamics for Navier-Stokes equation has been investigated in [1, 10, 15, 19, 24].

(2) For Navier-Stokes system with finite distributed delay $\int_{-h}^0 \omega(s)b(t,s,z(t+s))ds$ or infinite one $\int_{-\infty}^0 \omega(s)b(t,s,z(t+s))ds$, we can see the pullback dynamics based on global existence of weak and strong solutions in [1, 2, 3, 4, 20], here $\omega(\cdot)$ can be a function or constant.

(3) A comprehensive survey for the fluid flow model with delays, can be found in [5], which presentes also some open problems.

(4) The distributed delay has some similar form as memory, but the methods to deal the dynamics are different, especially the hypotheses on them, see [5, 15] and references therein.

Most of the above publications pay attentions to the pullback attractors for 2D Navier-Stokes equations or 3D modified systems, however there are fewer results on the forward dynamics, which is our objective here. The main results and features of this paper can be stated as following.

(I) Using background function (see [25, 26]), the inhomogeneous boundary system can be reduced to homogeneous problem, which is main feature for our problem. Using Galerkin's approximate procedure and compact argument, we can derive the existence of global weak solution for 2D Navier-Stokes equation with distributed delay in some new phase spaces.

(II) Since the distributed delay in (1.1) is defined in finite interval, for overcoming the uniqueness of global weak solution, we should assume that the kernel of distributed delay has Lipschitz continuous property, which guarantee that the solution generates a semigroup $\{S(t)\}$ for $\tau \le t \in \mathbb{R}$. By some estimates in the delay phase space, the absorbing set can be obtained. Moreover, the existence of global attractor also attained by using compact embedding.

(III) At last, we also want to see the effect of Rayleigh's friction and distributed delay on the dynamics for 2D Navier-Stokes equation. Comparing with the 2D Navier-Stokes equation with general external force, we can see that the Rayleigh's friction effects the domain of absorbing set, hence the structure of attractors between the above two problems is greatly different.

The plan of this article is the following. In Section 2, we derive the existence of continuous dependence global solution for our problem. The asymptotic compactness of semigroup and the global attractors are concluded in Section 3.

2. Global well-posedness

2.1. **Notation.** We set $E := \{u | u \in (C_0^\infty(\Omega))^2, divu = 0\}$, H is the closure of the set E in $(L^2(\Omega))^2$ topology, $|\cdot|_2$ and $(\cdot,\cdot)$ denote the norm and inner product in H

respectively, i.e.,

$$(u,v)=\sum_{j=1}^{2}\int_\Omega u_j(x)v_j(x)dx,\quad \forall u,v\in (L^2(\Omega))^2.$$

V is the closure of the set E in $(H^1(\Omega))^2$ topology, and $\|\cdot\|$ and $((\cdot,\cdot))$ denote the norm and inner product in V respectively, i.e.,

$$((u,v))=\sum_{i,j=1}^{2}\int_\Omega \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i}dx,\quad \forall u,v\in (H_0^1(\Omega))^2.$$

$\|\cdot\|_*$ is the norm in V', and $\langle\cdot\rangle$ be the dual product between V and V' or H.

The bilinear and trilinear operators are defined respectively as

$$B(u,v):=P((u\cdot\nabla)v),\quad b(u,v,w)=(B(u,v),w)=\sum_{i,j=1}^{2}\int_\Omega u_i\frac{\partial v_j}{\partial x_i}\cdot w_j dx$$

which satisfies

$$b(u,v,v)=0,\quad b(u,v,w)=-b(u,w,v), \tag{2.1}$$

$$\|b(u,v,w)\|\le C|u|^{1/2}\|u\|^{1/2}\|v\||w|^{1/2}\|w\|^{1/2},\quad \forall u,v,w\in V. \tag{2.2}$$

Moreover, we define the function with delay as

$$u_t=u(t+s),\ s\in(-h,0),$$

for any $t\in(\tau,T)$ and the Bochner space $L_H^p=L^p(-h,0;H)$ with $1\le p\le+\infty$, especially $L_H^2=L^2(-h,0;H)$.

Also, we define two Banach spaces $C_H=C([-h,0];H)$ and $C_V=C([-h,0];V)$ with norms

$$\|u\|_{C_H}=\sup_{\theta\in[-h,0]}|u(t+\theta)|,\quad \|u\|_{C_V}=\sup_{\theta\in[-h,0]}\|u(t+\theta)\|,$$

respectively, which is our phase spaces in the sequel.

2.2. **Abstract equivalent equation.** Let ψ be the background function which satisfies

$$\begin{gathered}\operatorname{div}\psi=0,\quad x\in\Omega,\\ \psi=\varphi,\quad x\in\partial\Omega,\\ \|\psi\|_{L^\infty}\le c\|\varphi\|_{L^\infty(\partial\Omega)},\\ u(\tau,x)=u_0(x),\ x\in\Omega,\\ |\psi|_2\le c'\|\varphi\|_{L^\infty(\partial\Omega)},\\ \|\psi\|\le c''\|\varphi\|_{L^\infty(\partial\Omega)}.\end{gathered} \tag{2.3}$$

Denoting $v=u-\psi$, then (1.1) is translated into the following problem

$$\begin{gathered}\frac{\partial v}{\partial t}-\nu\Delta v+(v\cdot\nabla)\psi+(\psi\cdot\nabla)v+\alpha v+\nabla p=\bar f+g_\psi(v_t),\\ \operatorname{div} v=0,\\ v=0,\\ v(\tau,x)=v_0(x),\\ v(t,x)=\phi(t-\tau,x)-\psi(x)=\eta(t-\tau,x),\end{gathered} \tag{2.4}$$

here $\bar{f} = f - \alpha\psi + \nu\Delta\psi - (\psi\cdot\nabla)\psi$, $g_\psi(v_t) = \int_{-h}^{0} G(s, v(t+s)+\psi)ds$.

Defining $Ru = B(u,\psi) + B(\psi,u)$, which is also continuous from $V\times V$ to V', hence the problem (2.4) can be written as the abstract functional equivalent form

$$\begin{gathered} v_t + \nu Av + \alpha v + B(v) + R(v) = P\bar{f} + g_\psi(v_t), \\ v(\tau) = v_0, \\ v(t) = \eta(t-\tau). \end{gathered} \tag{2.5}$$

Next, we shall study well-posedness and dynamics of problem (2.5).

2.3. **Assumptions.** For the well-posedness and forward dynamics, we use the following hypothesis.

- (H1) $G : [-h,0]\times\mathbb{R}^2 \to \mathbb{R}^2$ is measurable;
- (H2) $G(s,0) = 0, s\in[-h,0]$;
- (H3) there exists $\gamma\in L^2(-h,0)$ such that $|G(s,u) - G(s,v)|_{\mathbb{R}^2} \le \gamma(s)|u-v|_{\mathbb{R}^2}$ which is also true for $\Omega\subset\mathbb{R}^2$;
- (H4) $\nu\lambda_1 > 2c_1\lambda_1^{1/2}\|\psi\| + 4C_g^2/\alpha$.

From (H1) and (H3) we have

$$\begin{aligned} |g_\psi(\xi) - g_\psi(\eta)|_2^2 &\le \int_\Omega \Big(\int_{-h}^0 |G(s,\xi(s))(x) - G(s,\eta(s))(x)|_{\mathbb{R}^2} ds\Big)^2 dx \\ &\le \int_\Omega \Big(\int_{-h}^0 \gamma(s)|\xi(s)(x) - \eta(s)(x)|_{\mathbb{R}^2} ds\Big)^2 dx \\ &\le \int_\Omega \|\gamma\|_{L^2(-h,0)}^2 \Big(\int_{-h}^0 |\xi(s)(x) - \eta(s)(x)|_{\mathbb{R}^2} ds\Big)^2 dx \\ &\le L_g\|\xi-\eta\|_{C_H}^2 \end{aligned}$$

for any $\xi,\eta\in C_H$, where $L_g = h\|\gamma\|_{L^2(-h,0)}^2$.

For any $u,v\in C([-h,T];H)$, $t>\tau$, there exists $m_0\ge 0$, we also have for any $m\in[0,m_0]$,

$$\int_\tau^t e^{ms}|g_\psi(u_s) - g_\psi(v_s)|_2^2 ds \le C_g^2\int_{\tau-h}^t e^{ms}|u(s)-v(s)|_2^2 ds,$$

where $C_g^2 = \|\gamma\|_{L^2(-h,0)}^2 he^{m_0h}$.

2.4. **Existence of a global weak solution.**

Lemma 2.1 (Generalized Arzelà-Ascoli Theorem [29])**.** *Let $\{f_\gamma(\theta) : \gamma\in\Gamma\}\subset C([\tau-h,\tau];X)$ is equicontinuous. Then for $\forall\theta\in[\tau-h,\tau]$, the sequence $\{f_\gamma(\theta) : \gamma\in\Gamma\}$ is relatively compact in $C([\tau-h,\tau];X)$.*

Lemma 2.2 (Aubin-Lions Lemma [29, 32])**.** *Let $X\subset\subset H\subset Y$ be Banach spaces, and X is reflective. If u_n is a uniformly bounded sequence in $L^p(\tau,T;X)$, and there exists $1<p<+\infty$ such that $\frac{du_n}{dt}$ is uniformly bounded in $L^p(\tau,T;Y)$, then u_n has a strong convergence subsequence in $C([\tau,T];H)$.*

Theorem 2.3. *Assume that $f\in(L^2(\Omega))^2$, $v_0\in H,\eta\in L_H^2$, and* (H1)–(H4) *hold. Then* (2.4) *possesses a unique solution $v(t)$ satisfying*

$$v(t)\in L^\infty(\tau,T;H)\cap L^2(\tau,T;V), \quad \frac{dv}{dt}\in L^2(\tau,T;V').$$

Proof. Step 1: Approximate solution. Using Faedo-Galerkin method to find the approximation solution $v_n(t) = \sum_{j=0}^{n} a_{nj}(t)w_j$ to (2.4), where $a_{nj}(t)$ is to be determined, we deduce that $v_n(t)$ satisfies a ordinary differential equation

$$\frac{dv_n}{dt} + \nu A v_n + \alpha v_n + B(v_n) + R(v_n) = P_n \bar{f} + g_\psi(v_{nt}), \tag{2.6}$$

$$v_n(\tau) = v_{n0}, \tag{2.7}$$

$$v_n(t) = \eta_n(t-\tau), t \in (\tau - h, \tau), \tag{2.8}$$

By the local existence theory for the ordinary differential equations, we can derive a local solution for problem (2.6).

Step 2: The priori estimate and compact argument. Multiplying (2.6) by $e^{mt}v_n$, we have

$$\begin{aligned}
&(\frac{dv_n}{dt}, e^{mt}v_n) + \nu(Av_n, e^{mt}v_n) + (B(v_n), e^{mt}v_n) + R(v_n, e^{mt}v_n) + \alpha(v_n, e^{mt}v_n)\\
&= \langle P_n\bar{f}, e^{mt}v_n\rangle + \langle g_\psi(v_{nt}), e^{mt}v_n\rangle.
\end{aligned} \tag{2.9}$$

Noting that

$$(B(v_n), e^{mt}v_n) = e^{mt}(B(v_n), v_n) = e^{mt}b(v_n, v_n, v_n) = 0, \tag{2.10}$$

$$\begin{aligned}
|R(v_n, e^{mt}v_n)| &= e^{mt}|R(v_n, v_n)|\\
&= e^{mt}|b(v_n, \psi, v_n)| + e^{mt}|b(\psi, v_n, v_n)|\\
&= e^{mt}|b(v_n, \psi, v_n)|\\
&\le c_1 e^{mt}|v_n|_2\|v_n\|\|\psi\|,
\end{aligned} \tag{2.11}$$

$$|\langle P_n\bar{f}, e^{mt}v_n\rangle| = |\langle \bar{f}, P_n e^{mt}v_n\rangle| = |\langle \bar{f}, e^{mt}v_n\rangle| \le e^{mt}\|\bar{f}\|_*\|v_n\|, \tag{2.12}$$

$$|\langle g_\psi(v_{nt}), e^{mt}v_n\rangle| \le e^{mt}|g_\psi(v_{nt})|_2|v_n|_2, \tag{2.13}$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d(e^{mt}|v_n|_2^2)}{dt} + \nu e^{mt}\|v_n\|^2 + e^{mt}\alpha|v_n|_2^2\\
&\le e^{mt}\|\bar{f}\|_*\|v_n\| + e^{mt}|g_\psi(v_{nt})|_2|v_n|_2 + c_1e^{mt}|v_n|_2\|v_n\|\|\psi\|\\
&\le e^{mt}\Big(\frac{\nu\|v_n\|^2}{2} + \frac{\|\bar{f}\|_*^2}{2\nu}\Big) + e^{mt}\Big(\frac{|g_\psi(v_{nt})|_2^2}{\alpha} + \alpha|v_n|_2^2\Big) + c_1e^{mt}|v_n|_2\|v_n\|\|\psi\|,
\end{aligned}$$

and

$$\frac{d(e^{mt}|v_n|_2^2)}{dt} \le \frac{e^{mt}}{\nu}\|\bar{f}\|_*^2 + \frac{2e^{mt}}{\alpha}|g_\psi(v_{nt})|_2^2 - e^{mt}\Big(\nu\lambda_1 - 2c_1\lambda_1^{1/2}\|\psi\|\Big)|v_n|_2^2. \tag{2.14}$$

Choosing an appropriate parameter $\alpha > 0$ such that $\nu\lambda_1 > 2c_1\lambda_1^{1/2}\|\psi\| + 4C_g^2/\alpha$, integrating (2.14) over $[\tau, t]$, we obtain

$$\begin{aligned}
e^{mt}|v_n(t)|_2^2 - e^{m\tau}|v_{n0}|_2^2 &\le \int_\tau^t \frac{e^{ms}}{\nu}\|\bar{f}\|_*^2 ds + \int_\tau^t \frac{2e^{ms}}{\alpha}|g_\psi(v_{ns})|_2^2 ds\\
&\quad - \int_\tau^t e^{ms}\Big(\nu\lambda_1 - 2c_1\lambda_1^{1/2}\|\psi\|\Big)|v_n(s)|_2^2 ds.
\end{aligned} \tag{2.15}$$

Using the hypotheses (H3), we have

$$
\begin{aligned}
&\int_\tau^t e^{ms}|g_\psi(v_{ns})|_2^2 ds \\
&\le C_g^2 \int_{\tau-h}^t e^{ms}|v_n(s)+\psi|_2^2 ds \\
&\le C_g^2 \int_\tau^t e^{ms}|v_n(s)+\psi|_2^2 ds + C_g^2 \int_{\tau-h}^\tau e^{ms}|\eta_n(s)+\psi|_2^2 ds \\
&\le 2C_g^2 \int_\tau^t e^{ms}|v_n(s)|_2^2 ds + 2C_g^2 \int_\tau^t e^{ms}|\psi|_2^2 ds + C_g^2 \int_{\tau-h}^\tau e^{ms}|\phi_n|_2^2 ds \\
&\le 2C_g^2 \int_\tau^t e^{ms}|v_n(s)|_2^2 ds + \frac{2C_g^2}{m}(e^{mt}-e^{m\tau})|\psi|_2^2 + C_g^2 e^{m\tau}\int_{-h}^0 |\phi_n|_2^2 ds.
\end{aligned} \tag{2.16}
$$

Combining (2.15) and (2.16), we have

$$
\begin{aligned}
&e^{mt}|v_n(t)|_2^2 - e^{m\tau}|v_{n0}|_2^2 \\
&\le \frac{e^{mt}}{m\nu}\|\bar f\|_*^2 + \frac{4C_g^2}{\alpha m} e^{mt}|\psi|_2^2 + \frac{4C_g^2}{\alpha}\int_\tau^t e^{ms}|v_n(s)|_2^2 ds \\
&\quad + \frac{2C_g^2 e^{m\tau}}{\alpha}\int_{-h}^0 |\phi_n|_2^2 ds - \frac{4C_g^2 e^{m\tau}}{\alpha m}|\psi|^2 - (\nu\lambda_1 - 2c_1\lambda_1^{1/2}\|\psi\|)\int_\tau^t e^{ms}|v_n(s)|_2^2 ds \\
&= e^{mt}\Big(\frac{\|\bar f\|_*^2}{m\nu} + \frac{4C_g^2}{m\alpha}|\psi|_2^2\Big) + \frac{2C_g^2 e^{m\tau}}{\alpha}\int_{-h}^0 |\phi_n|_2^2 ds \\
&\quad - \frac{4C_g^2 e^{m\tau}}{\alpha m}|\psi|_2^2 - \Big(\nu\lambda_1 - 2c_1\lambda_1^{1/2}\|\psi\| - 4C_g^2/\alpha\Big)\int_\tau^t e^{ms}|v_n(s)|_2^2 ds,
\end{aligned}
$$

which implies

$$
\begin{aligned}
|v_n(t)|_2^2 &\le \frac{\|\bar f\|_*^2}{m\nu} + \frac{4C_g^2}{m\alpha}|\psi|_2^2 \\
&\quad + e^{-mt}\Big(\frac{2C_g^2 e^{m\tau}}{\alpha}\int_{-h}^0 |\phi_n|_2^2 ds - \frac{4C_g^2 e^{m\tau}}{m\alpha}|\psi|_2^2 + e^{m\tau}|v_{n0}|_2^2\Big) \\
&\le \frac{\|\bar f\|_*^2}{m\nu} + \frac{4C_g^2}{m\alpha}|\psi|_2^2 + \frac{2C_g^2}{\alpha}\int_{-h}^0 |\phi_n|_2^2 ds + |v_{n0}|_2^2 \\
&\le \frac{\|\bar f\|_*^2}{m\nu} + \frac{4C_g^2 c'^2}{m\alpha}\|\varphi\|_{L^\infty(\partial\Omega)}^2 + \frac{2C_g^2}{\alpha}\int_{-h}^0 |\phi_n|_2^2 ds + |v_{n0}|_2^2 := K.
\end{aligned} \tag{2.17}
$$

It is sufficient to show $v_n(t) \in L^\infty(\tau,T;H)\cap L^2(\tau,T;V)$ in the following by some estimates. Multiplying (2.6) by v_n, we obtain

$$
\frac{1}{2}\frac{d|v_n|_2^2}{dt} + \nu(Av_n,v_n) + b(v_n,v_n,v_n) + R(v_n,v_n) = \langle P_n\bar f, v_n\rangle + \langle g_\psi(v_{nt}), v_n\rangle,
$$

which yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d|v_n|_2^2}{dt} + \nu\|v_n\|^2 + \alpha|v_n|_2^2 \\
&\le \|\bar f\|_*\|v_n\| + |g_\psi(v_{nt})|_2|v_n|_2 + c_1|v_n|_2\|v_n\|\|\psi\| \\
&\le \frac{\nu\|v_n\|^2}{2} + \frac{\|\bar f\|_*^2}{2\nu} + \frac{|g_\psi(v_{nt})|_2^2}{\alpha} + \alpha|v_n|_2^2 + c_1\lambda_1^{-1/2}\|v_n\|^2\|\psi\|,
\end{aligned}
$$

which implies

$$\frac{d|v_n|_2^2}{dt} \le \frac{\|\bar{f}\|_*}{\nu} + \frac{2}{\alpha}|g_\psi(v_{nt})|_2^2 - (\nu - 2c_1\lambda_1^{-1/2}\|\psi\|)\|v_n\|^2. \tag{2.18}$$

Integrating (2.18) over $[t, t+1]$, we obtain

$$\begin{aligned}
&|v_n(t+1)|_2^2 - |v_n(t)|_2^2 + (\nu - 2c_1\lambda_1^{-1/2}\|\psi\|)\int_t^{t+1} \|v_n\|^2 ds \\
&\le \frac{\|\bar{f}\|_*^2}{\nu} + \frac{2}{\alpha}\int_t^{t+1} |g_\psi(v_{ns})|_2^2 ds.
\end{aligned}$$

From the Hölder inequality and hypotheses (H3), we derive that

$$\begin{aligned}
\int_t^{t+1} |g_\psi(v_{ns})|_2^2 ds &\le \int_t^{t+1}\int_{-h}^0 |G(s, v_n(r+s)+\psi)|_2^2\, dr\, ds \\
&\le \int_t^{t+1}\int_{-h}^0 |\gamma(s)|^2|v_n(r+s)+\psi)|_2^2\, dr\, ds \\
&\le \|\gamma\|_{L^2(-h,0)}^2 \int_t^{t+1}\int_{-h}^0 |v_n(r+s)+\psi|_2^2\, dr\, ds
\end{aligned}$$

and

$$\int_t^{t+1}\int_{-h}^0 |v_n(r+s)+\psi|_2^2\, dr\, ds = \int_{t-h}^{t+1}\int_{-h}^0 |v_n(k)+\psi|_2^2\, dr\, dk.$$

Noting that $v(k)$ dependents only on k, it follows that

$$\begin{aligned}
\int_t^{t+1}\int_{-h}^0 |v_n(r+s)+\psi|_2^2\, dr\, ds &= \int_{t-h}^{t+1}\int_{-h}^0 |v_n(k)+\psi|_2^2\, dr\, dk \\
&= \int_{t-h}^{t+1} |v_n(k)+\psi|_2^2 dk
\end{aligned}$$

and

$$\begin{aligned}
\int_t^{t+1} |g_\psi(v_{ns})|_2^2 ds &\le h\|\gamma\|_{L^2(-h,0)}^2 \int_{t-h}^{t+1} |v_{ns}+\psi|_2^2 ds \\
&= C_g^2 \int_{t-h}^{t+1} |v_{ns}+\psi|_2^2 ds\,.
\end{aligned}$$

Then

$$\int_{t-h}^{t+1} |v_{ns}+\psi|_2^2 ds = \int_{t-h}^{t} |v_{ns}+\psi|_2^2 ds + \int_t^{t+1} |v_{ns}+\psi|_2^2 ds.$$

Since $v(t,x) = \phi(t-\tau, x) - \psi(x)$ for arbitrary $(t,x) \in (\tau - h, \tau)\times\Omega$, it yields

$$\int_{t-h}^{t} |v_{ns}+\psi|_2^2 ds = \int_{-h}^0 |\phi_n(s) - \psi + \psi|_2^2 ds = \int_{-h}^0 |\phi_n|_2^2 ds,$$

hence, we have

$$\begin{aligned}
\int_t^{t+1} |v_{ns}+\psi|_2^2 ds &\le 2\int_t^{t+1} |v_{ns}|_2^2 ds + 2\int_t^{t+1} |\psi|_2^2 ds \\
&= 2\int_t^{t+1} |v_{ns}|_2^2 ds + 2|\psi|_2^2,
\end{aligned}$$

and

$$\int_t^{t+1} |v_{ns}+\psi|_2^2 ds \le 2\lambda_1^{-1}\int_t^{t+1} \|v_{ns}\|^2 ds + 2c'^2\|\varphi\|_{L^\infty(\partial\Omega)}^2.$$

Thus we conclude that

$$\int_t^{t+1} |g_\psi(v_{ns})|_2^2 ds \le C_g^2\Big(2\lambda_1^{-1}\int_t^{t+1}\|v_n\|^2 ds + 2c'^2\|\varphi\|_{L^\infty(\partial\Omega)}^2 + \int_{-h}^0 |\phi_n|_2^2 ds\Big),$$

which implies

$$\begin{aligned}
&(\nu - 2c_1\lambda_1^{-1/2}\|\psi\|)\int_t^{t+1}\|v_n\|^2 ds\\
&\le \frac{\|\bar f\|_*^2}{\nu} + K + \frac{2C_g^2}{\alpha}\Big(2\lambda_1^{-1}\int_t^{t+1}\|v_n\|^2 ds + 2c'^2\|\varphi\|_{L^\infty(\partial\Omega)}^2 + \int_{-h}^0 |\phi_n|_2^2 ds\Big),
\end{aligned}$$

and

$$\begin{aligned}
&\Big(\nu - 2c_1\lambda_1^{-1/2}\|\psi\| - \frac{4C_g^2\lambda_1^{-1}}{\alpha}\Big)\int_t^{t+1}\|v_n\|^2 ds\\
&\le \frac{\|\bar f\|_*^2}{\nu} + K + \frac{4C_g^2c'^2}{\alpha}\|\varphi\|_{L^\infty(\partial\Omega)}^2 + \frac{2C_g^2}{\alpha}\int_{-h}^0 |\phi_n|_2^2 ds,
\end{aligned}$$

i.e.,

$$\int_t^{t+1}\|v_n\|^2 ds \le K', \tag{2.19}$$

where

$$K' = \frac{\frac{\|\bar f\|_*^2}{\nu} + K + \frac{4C_g^2c'^2}{\alpha}\|\varphi\|_{L^\infty(\partial\Omega)}^2 + \frac{2C_g^2}{\alpha}\int_{-h}^0|\phi_n|_2^2 ds}{\nu - 2c_1\lambda_1^{-1/2}\|\psi\| - \frac{4C_g^2\lambda_1^{-1}}{\alpha}}.$$

By the above estimates, we conclude that $v_n(t) \in L^\infty(\tau,T;H)\cap L^2(\tau,T;V)$. From Lemma 2.1, there exists a subsequence (relabeled as $v_n(t)$ without confusion) such that

$$v_n \to^* v text in L^\infty(\tau,T;H), \quad v_n \to v text in L^2(\tau,T;V),$$

i.e., $v\in L^\infty(\tau,T;H)\cap L^2(\tau,T;V)$. Since

$$\frac{dv_n}{dt} = -\nu A v_n - B(v_n) - R(v_n) - \alpha v_n + P_n\bar f + g_\psi(v_{nt}),$$

and $v_n\in L^2(\tau,T;V)$, we have $\nu A v_n, \alpha v_n, g_\psi(v_{nt}) \in L^2(\tau,T;V')$ and

$$\begin{aligned}
\|(P_nB(v_n),v_n)\|_{L^2(0,T;V^*)}^2 &\le \int_0^T \|(B(v_n,v_n)\|_*^2 ds\\
&= \int_0^T \|(v_n\cdot\nabla)v_n\|_*^2 ds\\
&\le c_5\int_0^T |v_n|_2^2\|v_n\|^2 ds\\
&\le c_5\|v_n\|_{L^\infty(0,T;H)}^2\|v_n\|_{L^2(0,T;H)}^2.
\end{aligned}$$

i.e., $P_nB(v_n)\in L^2(\tau,T;V')$.

Passing to the limit as $n\to+\infty$, we conclude that

$$v_n \to v \text{ in } L^2(\tau,T;H), \quad v_n(\tau) = P_n v_{n0} \to v(\tau) = v_0,$$

which implies $\frac{dv_n}{dt} \in L^2(\tau, T; V')$. Using Lemma 2.2, we can derive the existence of a strong convergent subsequence which is the solution for our problem.

Step 3: The uniqueness and continuous dependence on initial data. Assume that v_1 and v_2 are two solutions to the system (2.6)–(2.8), and denote $w = v_1 - v_2$, then w satisfies

$$\frac{dw}{dt} - \nu Aw + B(v_1, v_1) - B(v_2, v_2) + R(w) + \alpha w = g_\psi(v_{1t}) - g_\psi(v_{2t}).$$

Noting that

$$\begin{aligned} B(v_1, v_1) - B(v_2, v_2) &= B(v_1 - v_2, v_1) - B(v_2, v_1 - v_2) \\ &= B(w, v_1) + B(v_2, w), \end{aligned}$$

we have

$$\frac{dw}{dt} - \nu Aw + B(w, v_1) - B(v_2, w) + R(w) + \alpha w = g_\psi(v_{1t}) - g_\psi(v_{2t}). \tag{2.20}$$

Multiplying (2.20) by $e^{mt}w$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d(e^{mt}|w|_2^2)}{dt} + (\nu Aw, e^{mt}w) + (B(w, v_1), e^{mt}w) + (B(v_2, w), e^{mt}w) \\ &+ (Rw, e^{mt}w) + (\alpha w, e^{mt}w) \\ &= \langle g_\psi(v_{1t}) - g_\psi(v_{2t}), e^{mt}w\rangle \end{aligned}$$

Using (2.1)-(2.2) and the Hölder inequality, we derive

$$\begin{aligned} &\frac{1}{2}\frac{d(e^{mt}|w|^2)}{dt} + (\nu Aw, e^{mt}w) \\ &\le |e^{mt}b(w, v_1, w)| + |e^{mt}(Rw, w)| + e^{mt}\langle g_\psi(v_{1t}) - g_\psi(v_{2t}), w\rangle \\ &\le e^{mt}(c_2|w|_2\|w\|\|v_1\| + c_1|w|_2\|w\|\|\psi\| + |g_\psi(v_{1t}) - g_\psi(v_{2t})|_2|w|_2) \\ &\le e^{mt}\Big(\frac{\nu}{2}\|w\|^2 + \frac{c_2^2}{2\nu}|w|_2^2\|v_1\|^2\Big) + e^{mt}\Big(\frac{\nu}{2}\|w\|^2 + \frac{c_1^2}{2\nu}|w|_2^2\|\psi\|^2\Big) \\ &\quad + e^{mt}\Big(\frac{|g_\psi(v_{1t}) - g_\psi(v_{2t})|_2^2}{2} + \frac{|w|_2^2}{2}\Big), \end{aligned}$$

i.e.,

$$\frac{d(e^{mt}|w|_2^2)}{dt} \le e^{mt}\Big[\Big(\frac{c_1^2}{\nu}\|\psi\|^2 + \frac{c_2^2}{\nu}\|v_1\|^2 + 1\Big)|w|_2^2 + |g_\psi(v_{1t}) - g_\psi(v_{2t})|_2^2\Big]. \tag{2.21}$$

Integrating (2.21) over $[\tau, t]$, we obtain

$$\begin{aligned} &e^{mt}|w(t)|_2^2 - e^{m\tau}|w(0)|_2^2 \\ &\le \int_\tau^t e^{ms}\Big(\frac{c_1^2}{\nu}\|\psi\|^2 + \frac{c_2^2}{\nu}\|v_1\|^2 + 1\Big)|w|^2 ds + \int_\tau^t e^{ms}|g_\psi(v_{1t}) - g_\psi(v_{2t})|_2^2 ds \\ &\le \int_\tau^t e^{ms}\Big(\frac{c_1^2}{\nu}\|\psi\|^2 + \frac{c_2^2}{\nu}\|v_1\|^2 + 1\Big)|w|_2^2 ds + C_g^2\int_{\tau-h}^t e^{ms}|v_1(s) - v_2(s)|_2^2 ds \\ &\le \int_\tau^t \Big(\frac{c_1^2}{\nu}\|\psi\|^2 + \frac{c_2^2}{\nu}\|v_1\|^2 + 1\Big)e^{ms}|w|_2^2 ds \\ &\quad + C_g^2\Big(\int_{\tau-h}^\tau e^{ms}|v_1(s) - v_2(s)|_2^2 ds + \int_\tau^t e^{ms}|v_1(s) - v_2(s)|_2^2 ds\Big) \end{aligned}$$

$$
\begin{aligned}
&\le e^{mt}\int_\tau^t\Big(\frac{c_1^2}{\nu}\|\psi\|^2+\frac{c_2^2}{\nu}\|v_1\|^2+1\Big)|w|_2^2ds\\
&\quad+C_g^2\Big(e^{m\tau}\int_{\tau-h}^{\tau}|v_1(s)-v_2(s)|_2^2ds+e^{mt}\int_\tau^t|v_1(s)-v_2(s)|_2^2ds\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
|w(t)|^2-|w(0)|_2^2&\le\int_\tau^t\Big(\frac{c_1^2}{\nu}\|\psi\|^2+\frac{c_2^2}{\nu}\|v_1\|^2+1\Big)|w|_2^2ds\\
&\quad+C_g^2\Big(\int_{-h}^0|v_1(r)-v_2(r)|_2^2dr+\int_\tau^t|v_1(r)-v_2(r)|_2^2dr\Big).
\end{aligned}
$$

It follows that

$$
|w(t)|_2^2\le|w(0)|_2^2+C_g^2\|\eta_1-\eta_2\|_{L_H^2}^2+\int_0^t\Big(\frac{c_1^2}{\nu}\|\psi\|^2+\frac{c_2^2}{\nu}\|v_1\|^2+C_g^2+1\Big)|w|^2ds,
$$

by the Gronwall inequality, we conclude that

$$
|w(t)|_2^2\le\Big(|w(0)|^2+C_g^2\|\eta_1-\eta_2\|_{L_H^2}^2\Big)e^{\int_0^t(\frac{c_1^2}{\nu}\|\psi\|^2+\frac{c_2^2}{\nu}\|v_1\|^2+C_g^2+1)|w|_2^2ds},
$$

which leads to the uniqueness and continuous dependence on initial data for our global weak solution. □

3. Long-time asymptotic dynamics

3.1. **Existence of absorbing set.** In this subsection, from Theorem 2.3, we see that the global weak solution generates a continuous semigroup $S(t)(v_0,\eta)=v_t(\cdot;(v_0,\eta))$ for any $(v_0,\eta)\in H\times L_H^2$ which satisfies

$$
\|(v_0,\eta)\|_{H\times L_H^2}^2=|v_0|_2^2+\int_{-h}^0|\eta(s)|_2^2ds.
$$

Theorem 3.1. *Assume that $f\in(L^2(\Omega))^2$, $(v_0,\eta)\in H\times L_H^2$, and* (H1)–(H4) *hold. Then the semigroup $S(t)$ possesses an absorbing ball in C_H for the system* (2.4).

Proof. Let $D\subset H\times L_H^2$ be any bounded set with radius d for $(v_0,\eta)\in D$ which satisfies

$$
|v_0|_2^2+\|\eta\|_{L_H^2}^2\le d^2. \tag{3.1}
$$

Multiplying (2.5) by $e^{mt}v$, we obtain

$$
\begin{aligned}
&(\frac{dv}{dt},e^{mt}v)+\nu(Av,e^{mt}v)+(B(v),e^{mt}v)+R(v,e^{mt}v)+\alpha(v,e^{mt}v)\\
&=\langle P_n\bar f,e^{mt}v\rangle+\langle g_\psi(v_t),e^{mt}v\rangle.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
(B(v),e^{mt}v)&=e^{mt}(B(v),v)=e^{mt}b(v,v,v)=0,\\
|R(v,e^{mt}v)|&=e^{mt}|R(v,v)|=e^{mt}|b(v,\psi,v)|+e^{mt}|b(\psi,v,v)|\\
&=e^{mt}|b(v,\psi,v)|\\
&\le c_1e^{mt}|v|_2\|v\|\|\psi\|,\\
|\langle P_n\bar f,e^{mt}v\rangle|&=|\langle\bar f,P_ne^{mt}v\rangle|=|\langle\bar f,e^{mt}v\rangle|\le e^{mt}\|\bar f\|_*\|v\|,
\end{aligned}
$$

and

$$
|\langle g_\psi(v_t),e^{mt}v\rangle|=e^{mt}|g_\psi(v_t)|_2|v|_2, \tag{3.2}
$$

we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d(e^{mt}|v|_2^2)}{dt}+\nu e^{mt}\|v\|^2+e^{mt}\alpha|v|_2^2\\
&\le e^{mt}\|\bar f\|_*\|v\|+e^{mt}|g_\psi(v_t)|_2|v|_2+c_1e^{mt}|v|_2\|v\|\|\psi\|\\
&\le e^{mt}\Big(\frac{\nu\|v\|^2}{2}+\frac{\|\bar f\|_*^2}{2\nu}\Big)+e^{mt}\Big(\frac{|g_\psi(v_t)|_2^2}{\alpha}+\alpha|v|_2^2\Big)+c_1e^{mt}|v|_2\|v\|\|\psi\|,
\end{aligned}
$$

and

$$
\frac{d(e^{mt}|v|_2^2)}{dt}\le\frac{e^{mt}}{\nu}\|\bar f\|_*^2+\frac{2e^{mt}}{\alpha}|g_\psi(v_t)|_2^2-e^{mt}\Big(\nu\lambda_1-2c_1\lambda_1^{1/2}\|\psi\|\Big)|v|_2^2 \tag{3.3}
$$

from the Poincaré inequality, where $\alpha>0$ is an appropriate constant satisfying $\nu\lambda_1>2c_1\lambda_1^{1/2}\|\psi\|+4C_g^2/\alpha$.

Integrating (3.3) over $[\tau,t]$, we obtain

$$
\begin{aligned}
&e^{mt}|v(t)|_2^2-e^{m\tau}|v(0)|_2^2\\
&\le\int_\tau^t\frac{e^{ms}}{\nu}\|\bar f\|_*^2ds+\int_\tau^t\frac{2e^{ms}}{\alpha}|g_\psi(v_s)|_2^2ds-\int_\tau^t e^{ms}\big(\nu\lambda_1-2c_1\lambda_1^{1/2}\|\psi\|\big)|v|_2^2ds\\
&\le\frac{1}{m\nu}(e^{mt}-e^{m\tau})\|\bar f\|_*^2+\frac{2}{\alpha}\int_\tau^t e^{ms}|g_\psi(v_s)|_2^2ds\\
&\quad-\int_\tau^t e^{ms}(\nu\lambda_1-2c_1\lambda_1^{1/2}\|\psi\|)|v|_2^2ds.
\end{aligned}
\tag{3.4}
$$

Using hypothesis (H3), we have

$$
\begin{aligned}
\int_\tau^t e^{ms}|g_\psi(v_s)|_2^2ds\le 2C_g^2\int_\tau^t e^{ms}|v(s)|_2^2ds+\frac{2C_g^2}{m}(e^{mt}-e^{m\tau})|\psi|_2^2\\
+C_g^2e^{m\tau}\int_{-h}^0|\eta(s)+\psi|_2^2ds.
\end{aligned}
\tag{3.5}
$$

From (3.4)–(3.5), we conclude that

$$
\begin{aligned}
&e^{mt}|v(t)|_2^2-e^{m\tau}|v(0)|_2^2\\
&\le\frac{e^{mt}}{m\nu}\|\bar f\|_*^2+\frac{4C_g^2}{\alpha m}e^{mt}|\psi|_2^2+\frac{4C_g^2}{\alpha}\int_\tau^t e^{ms}|v(s)|_2^2ds+\frac{2C_g^2e^{m\tau}}{\alpha}\int_{-h}^0|\eta(s)+\psi|_2^2ds\\
&\quad-(\nu\lambda_1-2c_1\lambda_1^{1/2}\|\psi\|)\int_\tau^t e^{ms}|v(s)|_2^2ds\\
&=e^{mt}\Big(\frac{\|\bar f\|_*^2}{m\nu}+\frac{4C_g^2}{m\alpha}|\psi|_2^2\Big)+\frac{2C_g^2e^{m\tau}}{\alpha}\int_{-h}^0|\eta(s)+\psi|_2^2ds\\
&\quad-\Big(\nu\lambda_1-2c_1\lambda_1^{1/2}\|\psi\|-4C_g^2/\alpha\Big)\int_\tau^t e^{ms}|v(s)|_2^2ds.
\end{aligned}
$$

Choosing an appropriate constant such that $\nu\lambda_1-2c_1\lambda_1^{1/2}\|\psi\|-4C_g^2/\alpha\ge0$, we obtain

$$
|v(t)|_2^2\le\frac{\|\bar f\|_*^2}{m\nu}+\frac{4C_g^2}{m\alpha}|\psi|_2^2+e^{-mt}\Big(\frac{2C_g^2e^{m\tau}}{\alpha}\int_{-h}^0|\eta(s)+\psi|_2^2ds+e^{m\tau}|v_0|_2^2\Big),
$$

where

$$\int_{-h}^{0}|\eta(s)+\psi|_2^2ds \le 2\int_{-h}^{0}|\eta(s)|_2^2ds + 2\int_{-h}^{0}|\psi|_2^2ds = 2h|\eta(s)|_2^2+2h|\psi|_2^2.$$

From (3.1), we have

$$|v_0|_2^2+\frac{4C_g^2he^{m\tau}}{\alpha}|\eta(s)|_2^2 \le \Big(1+\frac{4C_g^2he^{m\tau}}{\alpha}\Big)d^2,$$

i.e.,

$$|v(t)|^2 \le \frac{\|\bar f\|_*^2}{m\nu}+\frac{4C_g^2c'^2}{m\alpha}\|\varphi\|^2_{L^\infty(\partial\Omega)}+e^{-mt}\Big[\frac{4C_g^2he^{m\tau}}{\alpha}|\psi|^2+(1+\frac{4C_g^2he^{m\tau}}{\alpha})d^2\Big].$$

Hence, for $t>h$ and $\theta\in[-h,0]$, we have

$$\begin{aligned}
&|v(t+\theta)|_2^2-\Big(\frac{\|\bar f\|_*^2}{m\nu}+\frac{4C_g^2c'^2}{m\alpha}\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big)\\
&\le e^{-m(t+\theta)}\Big[\frac{4C_g^2he^{m\tau}}{\alpha}|\psi|_2^2+(1+\frac{4C_g^2he^{m\tau}}{\alpha})d^2\Big]\\
&\le e^{-mt}e^{mh}\Big[\frac{4C_g^2c'^2he^{m\tau}}{\alpha}\|\varphi\|^2_{L^\infty(\partial\Omega)}+(1+\frac{4C_g^2he^{m\tau}}{\alpha})d^2\Big]
\end{aligned}$$

and

$$\begin{aligned}
&\|v_t\|^2_{C_H}-\Big(\frac{\|\bar f\|_*^2}{m\nu}+\frac{4C_g^2c'^2}{m\alpha}\|\varphi\|^2_{L^\infty(\partial\Omega)}\Big)\\
&\le e^{-mt}e^{mh}\Big[\frac{4C_g^2c'^2he^{m\tau}}{\alpha}\|\varphi\|^2_{L^\infty(\partial\Omega)}+(1+\frac{4C_g^2he^{m\tau}}{\alpha})d^2\Big].
\end{aligned}$$

If we take

$$e^{-mt}e^{mh}\le\frac{\alpha\|\bar f\|_*^2+4\nu C_g^2c'^2\|\varphi\|^2_{L^\infty(\partial\Omega)}}{4m\nu hC_g^2c'^2e^{m\tau}\|\varphi\|^2_{L^\infty(\partial\Omega)}+m\nu(\alpha+4hC_g^2e^{m\tau})d^2},$$

i.e.,

$$t\ge T_H=\frac{1}{m}\ln\frac{m\nu\Big[4hC_g^2c'^2e^{m\tau}\|\varphi\|^2_{L^\infty(\partial\Omega)}+(\alpha+4hC_g^2e^{m\tau})d^2\Big]}{\alpha\|\bar f\|_*^2+4\nu C_g^2c'^2\|\varphi\|^2_{L^\infty(\partial\Omega)}},$$

and denoting

$$\rho_H^2=2(\frac{1}{m\nu}\|\bar f\|_*^2+\frac{2C_g^2}{\alpha}|\psi|_2^2),$$

then it is sufficient to show that

$$\|v\|^2_{C_H}\le\rho_H^2 \tag{3.6}$$

for any $(v_0,\eta)\in D\subset H\times L^2_H$, where $B_H(0,\rho_H)$ denotes an absorbing ball with center 0 and radius ρ_H in C_H, the proof is complete. □

Theorem 3.2. *Assume $\bar f\in(L^2(\Omega))^2$, $(v_0,\eta)\in H\times L^2_H$, and (H1)–(H4) hold. Then the semigroup $S(t)$ possesses an absorbing ball in C_V to system (2.4).*

Proof. Multiplying (2.5) by v, we obtain

$$\frac{1}{2}\frac{d|v|_2^2}{dt}+\nu(Av,v)+b(v,v,v)+R(v,v)=\langle P_n\bar f,v\rangle+\langle g_\psi(v_{nt}),v\rangle,$$

hence,

$$\begin{aligned}&\frac{1}{2}\frac{d|v|_2^2}{dt}+\nu\|v\|^2+\alpha|v|_2^2\\&\le\|\bar{f}\|_*\|v\|+|g_\psi(v_{nt})||v|_2+c_1|v|_2\|v\|\|\psi\|\\&\le\frac{\nu\|v\|^2}{2}+\frac{\|\bar{f}\|_*^2}{2\nu}+\frac{|g_\psi(v_{nt})|_2^2}{\alpha}+\alpha|v|_2^2+c_1\lambda_1^{-1/2}\|v\|^2\|\psi\|,\end{aligned}$$

which implies

$$\frac{d|v|_2^2}{dt}\le\frac{\|\bar{f}\|_*}{\nu}+\frac{2}{\alpha}|g_\psi(v_t)|_2^2-(\nu-2c_1\lambda_1^{-1/2}\|\psi\|)\|v\|^2. \tag{3.7}$$

Integrating over $[t,t+1]$, we obtain

$$\begin{aligned}&|v(t+1)|_2^2-|v(t)|_2^2+(\nu-2c_1\lambda_1^{-1/2}\|\psi\|)\int_t^{t+1}\|v\|^2ds\\&\le\frac{\|\bar{f}\|_*^2}{\nu}+\frac{2}{\alpha}\int_t^{t+1}|g_\psi(v_s)|_2^2ds.\end{aligned} \tag{3.8}$$

Noting that

$$\begin{aligned}&\int_t^{t+1}|g_\psi(v_s)|_2^2ds\\&\le C_g^2\int_{t-h}^{t+1}|v+\psi|_2^2ds\\&\le 2C_g^2d^2+2hC_g^2c'^2\|\varphi\|_{L^\infty(\partial\Omega)}^2+2C_g^2c'^2\|\varphi\|_{L^\infty(\partial\Omega)}^2+2C_g^2\lambda_1^{-1}\int_t^{t+1}\|v\|^2ds.\end{aligned} \tag{3.9}$$

we can derive

$$\begin{aligned}&|v(t+1)|_2^2-|v(t)|_2^2+\Big(\nu-2c_1\lambda_1^{-1/2}\|\psi\|\Big)\int_t^{t+1}\|v\|^2ds\\&\le\frac{\|\bar{f}\|_*}{\nu}+\frac{2}{\alpha}\Big(2C_g^2d^2+2hC_g^2c'^2\|\varphi\|_{L^\infty(\partial\Omega)}^2+2C_g^2c'^2\|\varphi\|_{L^\infty(\partial\Omega)}^2\\&\quad+2C_g^2\lambda_1^{-1}\int_t^{t+1}\|v\|^2ds\Big)\end{aligned}$$

from (3.8)–(3.9), i.e.,

$$\begin{aligned}&\Big(\nu-2c_1\lambda_1^{-1/2}\|\psi\|-\frac{4C_g^2\lambda_1^{-1}}{\alpha}\Big)\int_t^{t+1}\|v\|^2ds\\&\le\frac{\|\bar{f}\|_*}{\nu}+\frac{4C_g^2d^2}{\alpha}+\frac{4C_g^2c'^2}{\alpha}(1+h)\|\varphi\|_{L^\infty(\partial\Omega)}^2+K,\end{aligned}$$

where K is defined as in (2.17).

From the above estimates, we deduce that

$$\int_t^{t+1}\|v\|^2\le I_v, \tag{3.10}$$

where

$$I_v=\frac{\frac{\|\bar{f}\|_*}{\nu}+\frac{4C_g^2d^2}{\alpha}+\frac{4C_g^2c'^2}{\alpha}(1+h)\|\varphi\|_{L^\infty(\partial\Omega)}^2+K}{\nu-2c_1\lambda_1^{-1/2}\|\psi\|-\frac{4C_g^2\lambda_1^{-1}}{\alpha}}. \tag{3.11}$$

Multiplying (2.5) by Av, we have

$$\frac{1}{2}\frac{d\|v\|^2}{dt}+\nu(Av,Av)+B(v,Av)+R(v,Av)+(\alpha v,Av)=\langle\bar f,Av\rangle+\langle g_\psi(v_t),Av\rangle,$$

i.e.,

$$\begin{aligned}&\frac{1}{2}\frac{d\|v\|^2}{dt}+\nu|Av|_2^2+\alpha\|v\|^2\\&\le|\langle\bar f,Av\rangle|+|\langle g_\psi(v_t),Av\rangle|+|b(v,v,Av)|+|R(v,Av)|.\end{aligned}\tag{3.12}$$

Noting that

$$\begin{aligned}|\langle\bar f,Av\rangle|+|\langle g_\psi(v_t),Av\rangle|&\le|\bar f|_2|Av|_2+|g_\psi(v_t)|_2|Av|_2\\&\le\frac{\nu}{6}|Av|_2^2+\frac{3}{2\nu}|\bar f|_2^2+\frac{\nu}{6}|Av|_2^2+\frac{3}{2\nu}|g_\psi(v_t)|_2^2\\&=\frac{\nu}{3}|Av|_2^2+\frac{3}{2\nu}|\bar f|_2^2+\frac{3}{2\nu}|g_\psi(v_t)|_2^2,\end{aligned}\tag{3.13}$$

$$\begin{aligned}|b(v,v,Av)|&\le c_1|v|_{L^\infty}\|v\||Av|_2\\&\le C|v|_2^{1/2}|Av|_2^{1/2}\|v\||Av|_2\\&\le\frac{\nu}{3}|Av|_2^2+\frac{C}{\nu}|v|_2^2\|v\|^4,\end{aligned}\tag{3.14}$$

$$\begin{aligned}&|R(v,v,Av)|\\&\le|b(v,\psi,Av)|+|b(\psi,v,Av)|\\&\le c_3|v|_2^{1/2}|Av|_2^{3/2}\|\varphi\|_{L^\infty(\partial\Omega)}+c_4\|\varphi\|_{L^\infty(\partial\Omega)}\|v\||Av|_2\\&\le\frac{\nu}{3}|Av|_2^2+(\frac{6}{\nu})^3c_3^4|v|_2^2\|\varphi\|_{L^\infty(\partial\Omega)}^4+\frac{3}{2\nu}c_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2\|v\|^2,\end{aligned}\tag{3.15}$$

and using (3.13)-(3.15) in (3.12), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d\|v\|^2}{dt}+\nu|Av|_2^2+\alpha\|v\|^2\\&\le|\bar f|_2|Av|_2+|g_\psi(v_t)|_2|Av|_2\\&\le\frac{\nu}{3}|Av|_2^2+\frac{3}{2\nu}(|\bar f|_2^2+|g_\psi(v_t)|_2^2)+\frac{\nu}{3}|Av|_2^2+(\frac{3}{\nu})^3c_1^4|v|_2^2\|v\|^4\\&\quad+\frac{\nu}{3}|Av|_2^2+(\frac{6}{\nu})^3c_3^4|v|_2^2\|\varphi\|_{L^\infty(\partial\Omega)}^4+\frac{3}{2\nu}c_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2\|v\|^2\\&=\nu|Av|_2^2+\frac{3}{2\nu}(|\bar f|_2^2+|g_\psi(v_t)|_2^2)+(\frac{3}{\nu})^3c_1^4|v|_2^2\|v\|^4\\&\quad+(\frac{6}{\nu})^3c_3^4|v|_2^2\|\varphi\|_{L^\infty(\partial\Omega)}^4+\frac{3}{2\nu}c_4^2\|\varphi\|_{L^\infty(\partial\Omega)}^2\|v\|^2.\end{aligned}$$

Hence, the above estimate yields

$$\begin{aligned}\frac{1}{2}\frac{d\|v\|^2}{dt}+\alpha\|v\|^2&\le\frac{3}{2\nu}(|\bar f|_2^2+|g_\psi(v_t)|_2^2)+(\frac{6}{\nu})^3c_3^4|v|_2^2\|\varphi\|_{L^\infty(\partial\Omega)}^4\\&\quad+(\frac{3}{\nu})^3c_1^4|v|_2^2\|v\|^4+\frac{3c_4^2}{2\nu}\|\varphi\|_{L^\infty(\partial\Omega)}^2\|v\|^2,\end{aligned}$$

i.e.,

$$\frac{d\|v\|^2}{dt}\le\frac{3}{\nu}(|\bar f|_2^2+|g_\psi(v_t)|_2^2)+2(\frac{6}{\nu})^3c_3^4|v|_2^2\|\varphi\|_{L^\infty(\partial\Omega)}^4$$

$$+ 2(\frac{3}{\nu})^3 c_1^4 |v|_2^2 \|v\|^4 + \frac{3c_4^2}{\nu} \|\varphi\|^2_{L^\infty(\partial\Omega)} \|v\|^2 - 2\alpha \|v\|^2,$$

for $t \geq T_H$.

Denoting

$$a_1 = 2(\frac{3}{\nu})^3 c_1^4 \rho_H^2 I_v + \frac{3c_4^2}{\nu} \|\varphi\|^2_{L^\infty(\partial\Omega)} \|v\|^2,$$

$$a_2 = \frac{3}{\nu}\Big(|\bar{f}|_2^2 + 2C_g^2(\rho_H^2 + c'^2(1+h)\|\varphi\|^2_{L^\infty(\partial\Omega)} + d^2)\Big) + 2(\frac{6}{\nu})^3 c_3^4 \rho_H^2 \|\varphi\|^4_{L^\infty(\partial\Omega)},$$

$$a_3 = 2\alpha I_v,$$

where I_v is defined as in (3.10). By Gronwall's inequality, we conclude that for $t \geq T_H + h = T_V$,

$$\|v(t)\|^2 \leq (a_2 + a_3)e^{a_1},$$

i.e.,

$$\sup_{\theta\in[-h,0]} \|v(t+\theta)\|^2 \leq (a_2 + a_3)e^{a_1} = \rho_V^2,$$

and

$$\|v(t)\|^2_{C_V} \leq \rho_V^2, \tag{3.16}$$

where $B_V(0, \rho_V)$ denotes an absorbing ball with center 0 and radius ρ_V in C_V, then the proof has been completed. □

3.2. **Existence of global attractors.**

Theorem 3.3. *Assume $\bar{f} \in (L^2(\Omega))^2$, $(v_0, \eta) \in H \times L^2_H$, and* (H1)–(H4) *hold. Then* (2.4) *possesses a global attractor $\mathcal{A}$.*

Proof. From Theorem 2.3 we know the semigroup is continuous, and the existence of absorbing balls $B(0, \rho_H)$ and $B_V(0, \rho_V)$ in C_H and C_V respectively, in Theorems 3.1 and 3.2, is established. If we can show that $B(0, \rho_V)$ is compact in C_H, then we can declare $S(t)$ possesses global attractors in C_H, which is deduced to prove the next two conclusions by the generalized Arzelà-Ascoli theorem:

- (a) $\overline{\cup_{\phi\in B_V} S(t)(\phi)(\theta)}$ is relatively compact in H for all $\theta \in [-h, 0]$. This conclusion holds since $V \subset\subset H$ is compact.
- (b) $S(t)B_V(0, \rho_V)$ is equicontinuous.

Our next objective is to show that (b) holds. Since

$$|S(t)(\phi)(\theta_1) - S(t)(\phi)(\theta_2)|_2 = |v(t+\theta_1; \phi) - v(t+\theta_2; \phi)|_2,$$

where $t \in \mathbb{R}$, $\theta_1, \theta_2 \in [-h, 0]$, $s \geq T_V$, $\phi \in B_V(0, \rho_V)$.

Let $\theta_2 > \theta_1$, from Theorem 3.2 and the Poincaré inequality, we derive

$$\begin{aligned} \frac{1}{2}\frac{d\|v\|^2}{dt} + \alpha\lambda_1^{-1}|Av|_2^2 &\leq \frac{3}{2\nu}(|\bar{f}|_2^2 + |g_\psi(v_t)|_2^2) + (\frac{6}{\nu})^3 c_3^4 |v|_2^2 \|\varphi\|^4_{L^\infty(\partial\Omega)} \\ &\quad + (\frac{3}{\nu})^3 c_1^4 |v|_2^2 \|v\|^4 + \frac{3c_4^2}{2\nu} \|\varphi\|^2_{L^\infty(\partial\Omega)} \|v\|^2. \end{aligned} \tag{3.17}$$

Integrating (3.17) over $[t+\theta_1, t+\theta_2]$, we have

$$\begin{aligned} \int_{t+\theta_1}^{t+\theta_2} |Av(r)|_2^2 dr &\leq \frac{1}{\alpha}\int_{t+\theta_1}^{t+\theta_2} \Big\{ \frac{3\lambda_1}{2\nu}(|\bar{f}|_2^2 + |g_\psi(v_t)|_2^2) + (\frac{6}{\nu})^3 \lambda_1 c_3^4 |v|_2^2 \|\varphi\|^4_{L^\infty(\partial\Omega)} \\ &\quad + (\frac{3}{\nu})^3 \lambda_1 c_1^4 |v|_2^2 \|v\|^4 + \frac{3\lambda_1 c_4^2}{2\nu} \|\varphi\|^2_{L^\infty(\partial\Omega)} \|v\|^2 dr \Big\} \end{aligned}$$

$$
\begin{aligned}
&- \frac{\lambda_1}{2\alpha}\int_{t+\theta_1}^{t+\theta_2} \frac{d\|v\|^2}{dr}dr\\
&\le \beta_1|\theta_1-\theta_2|_2+\beta_2,
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_1 = \frac{1}{\alpha}\Big[&\frac{3\lambda_1}{2\nu}\Big(|\bar{f}|_2^2 + 2C_g^2(\rho_H^2 + c'^2(1+h)\|\varphi\|^2_{L^\infty(\partial\Omega)} + d^2)\Big)\\
&+ (\frac{6}{\nu})^3\lambda_1 c_3^4|v|_2^2\|\varphi\|^4_{L^\infty(\partial\Omega)} + (\frac{3}{\nu})^3\lambda_1 c_1^4|v|_2^2\|v\|^4 + \frac{3\lambda_1 c_4^2}{2\nu}\|\varphi\|^2_{L^\infty(\partial\Omega)}\|v\|^2\Big],\\
\beta_2 &= \frac{\lambda_1}{2\alpha}I_v^2.
\end{aligned}
$$

Noting that $|v(t+\theta_1)-v(t+\theta_2)| = |\int_{t+\theta_1}^{t+\theta_2} v'(r)dr|$, we obtain

$$
\begin{aligned}
&|v(t+\theta_1)-v(t+\theta_2)|_2\\
&\le \int_{t+\theta_1}^{t+\theta_2} |v'(r)|_2 dr\\
&\le \int_{t+\theta_1}^{t+\theta_2} (\nu|Av(r)|_2 + \alpha|v(r)|_2 + |B(v(r))|_2 + |\bar{f}|_2 + |g_\psi(v_r)|_2 + |R(v(r))|_2)dr.
\end{aligned}
$$

Since

$$
|Bv|_2 \le c_1|Av|_2\|v\|, |Rv|_2 \le c_1|Av|_2^{\vartheta}\|v\|^{1-\vartheta} \le |Av|_2 + c_1^{\frac{1}{1-\vartheta}}\|v\|,
$$

holds for $\vartheta\in[0,1)$, we obtain

$$
|v(t+\theta_1)-v(t+\theta_2)|_2 \le \hbar|\theta_1-\theta_2|^{1/2},
$$

where

$$
\hbar = \big(c_1^{\frac{1}{1-\vartheta}}I_v + L_g\rho_H + |\bar{f}|_2\big)|\theta_1-\theta_2|^{1/2} + \big(\alpha + c_1 I_v + 1\big)(\beta_1|\theta_1-\theta_2| + \beta_2),
$$

which means $\{S(t)\}$ is equi-continuous, and it is also asymptotically compact in C_H. From the theory of global attractor in [6, 16, 33], we conclude that the semigroup $\{S(t)\}$ possesses a global attractor $\mathcal{A}$ in C_H. □

Conclusion. From the result above which prove the existence of a global attractor for problem (1.1), we can see the effect of distributed delays on forward dynamics. A natural question is what about the robustness if the delay disappears? which is similar to the subsonic limit in [30].

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