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HOMOCLINIC SOLUTIONS OF DISCRETE NONLINEAR SCHRÖDINGER EQUATIONS WITH PARTIALLY SUBLINEAR NONLINEARITIES

GENGHONG LIN, JIANSHE YU, ZHAN ZHOU

Abstract. We consider a class of discrete nonlinear Schrödinger (DNLS) equations in m dimensional lattices with partially sublinear nonlinearity f. Combining variational methods and a priori estimate, we give a general sufficient condition on f for type (A), that is, a sequence of nontrivial homoclinic solutions accumulating to zero. By using a compact embedding technique, we overcome the loss of compactness due to the problem being set on the unbounded domain $\mathbb{Z}^m$. Another obstacle caused by the local definition of f is solved by using the cutoff methods to recover the global property of f. To the best of our knowledge, this is the first time to obtain infinitely many homoclinic solutions for the DNLS equations with partially sublinear nonlinearity. Moreover, we prove that if f is not sublinear, the zero solution is isolated from other homoclinic solutions. Our results show that the sublinearity and oddness of f yield type (A). Without the oddness assumption, we still can prove that this problem has at least a nontrivial homoclinic solution if f is local sublinear, which improves some existing results.

1. Introduction

The discrete nonlinear Schrödinger (DNLS) equation serves widely the field of nonlinear science, ranging from condensed matter physics to biology [12, 18, 24]. Breathers (one kind of solutions with periodic time behavior) exist in the DNLS equations and have been observed in experiments [7, 13, 14]. In the past decade, the existence of breathers of the DNLS equations has been a very hot topic [1, 2, 3, 16, 26, 43]. Methods such as the principle of anticontinuity [2, 26], centre manifold reduction [16] and variational methods [1, 3, 31] were used. It is worth mentioning that variational methods are powerful for obtaining the existence of solutions of difference equations [1, 3, 19, 28, 29, 41]. Based on variational approaches, the existence and multiplicity results of breathers for the DNLS equations have been proved under different assumptions on the nonlinearities; see [17, 25, 27, 30, 35, 36, 40] for the superlinear nonlinearity, [5, 15, 19, 31, 41, 42] for the saturable (asymptotically linear) nonlinearity and [20, 21, 22] for mixed nonlinearity. However, only a few results were obtained on the existence of breathers for the DNLS equations with sublinear nonlinearity [6, 9, 34]. Since it appears in inflation cosmology and

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supersymmetric field theories, quantum mechanics and nuclear physics [4, 8, 10], the sublinear nonlinearity is of much interest in physics. How does the sublinear nonlinearity affect the existence of breathers for the DNLS equations remains to be fully understood.

Assume that m is a positive integer. We consider the DNLS equation in m dimensional lattices with attractive self-interaction

$$i\dot{\psi}_n = -\Delta\psi_n + v_n\psi_n - f_n(\psi_n), \quad n = (n_1, n_2, \dots, n_m) \in \mathbb{Z}^m, \tag{1.1}$$

where

$$\begin{aligned}\Delta\psi_n = {}& \psi_{(n_1+1,n_2,\dots,n_m)} + \psi_{(n_1,n_2+1,\dots,n_m)} + \dots \\ & + \psi_{(n_1,n_2,\dots,n_m+1)} - 2m\psi_{(n_1,n_2,\dots,n_m)} \\ & + \psi_{(n_1-1,n_2,\dots,n_m)} + \psi_{(n_1,n_2-1,\dots,n_m)} + \dots + \psi_{(n_1,n_2,\dots,n_m-1)}\end{aligned}$$

is the discrete Laplacian in m spatial dimension, and $\{v_n\}$ is a real-valued sequence satisfying the assumption

(A1) the discrete potential $V = \{v_n\}_{n\in\mathbb{Z}^m}$ is bounded from below.

Under this condition, the discrete potential $V = \{v_n\}$ is allowed to change sign or to be unbounded from above. We assume further that the nonlinearity $f_n(u)$ is gauge invariant, i.e.,

$$f_n(e^{i\theta}u) = e^{i\theta}f_n(u), \quad \theta \in \mathbb{R}. \tag{1.2}$$

Since breathers are spatially localized time-periodic solutions and decay to zero at infinity. Thus ψ_n has the form

$$\psi_n = u_n e^{-i\omega t},$$

and

$$\lim_{|n|\to\infty} \psi_n = 0,$$

where $\{u_n\}$ is a real-valued sequence, $\omega \in \mathbb{R}$ is the temporal frequency, and $|n| = |n_1| + |n_2| + \dots + |n_m|$ is the length of multi-index n. Then (1.1) becomes

$$Lu_n - \omega u_n = f_n(u_n), \quad n \in \mathbb{Z}^m, \tag{1.3}$$

and

$$\lim_{|n|\to\infty} u_n = 0, \tag{1.4}$$

where $L = -\Delta + V$ is a self-adjoint operator [33] defined by

$$Lu_n = -\Delta u_n + v_n u_n, \quad n \in \mathbb{Z}^m.$$

If $f_n(0) \equiv 0$, then $u_n \equiv 0$ is a solution of (1.3), which is called the trivial solution. As usual, we say that a solution $u = \{u_n\}$ of (1.3) is homoclinic (to 0) if (1.4) holds. To find breathers of (1.1), we just need to seek the homoclinic solutions of (1.3).

In this paper, we use the variant of Clark's theorem posted by Liu-Wang [23] to the problem above and prove the existence of a sequence of nontrivial homoclinic solutions converging to the zero solution if the nonlinear term $f_n(u)$ is sublinear and odd with respect to u, i.e., $f_n(-u) = f_n(u)$. We call $f_n(u)$ sublinear if it satisfies

$$\lim_{u\to 0} \frac{F_n(u)}{u^2} = +\infty \quad \text{where } F_n(u) = \int_0^u f_n(s)ds, \tag{1.5}$$

meaning roughly that the nonlinear term, like $f_n(u) = |u|^{p-2}u$ with $0 < p < 2$, has a growth order less than one in a neighborhood of $u = 0$. Condition (1.5) is weaker than the condition

$$\lim_{u\to 0} \frac{f_n(u)}{u} = +\infty. \tag{1.6}$$

It is easy to verify that (1.6) implies (1.5) by using the L'Hospital rule. Moreover, we prove that if $f_n(u)$ is not sublinear, the zero solution is isolated from other homoclinic solutions. The oddness assumption on $f_n(u)$ is important, since it is necessary for applying the variant Clark's theorem.

We first introduce some notation. Let

$$l^p \equiv \big\{u = \{u_n\}_{n\in\mathbb{Z}^m} : u_n \in \mathbb{R},\, n \in \mathbb{Z}^m,\, \|u\|_p = \Big(\sum_{n\in\mathbb{Z}^m} |u_n|^p\Big)^{1/p} < \infty\big\}. \tag{1.7}$$

Then the following embedding between l^p spaces holds,

$$l^q \subset l^p, \quad \|u\|_p \le \|u\|_q, \quad 1 \le q \le p \le \infty. \tag{1.8}$$

It follows from (A1) that the self-adjoint operator L is bounded from below in l^2. Thus, the spectrum $\sigma(L)$ of L is also bounded from below. We denote

$$\alpha = \inf \sigma(L).$$

In this article, we focus on the homoclinic solutions of (1.3) for the case where $\omega < \alpha$.

Equation (1.3) with the zero solution can been classified into the following two types:

(A) the zero solution is an accumulation point of the set of all homoclinic solutions,

(I) the zero solution is an isolated point of the set of all homoclinic solutions.

In the above statement, we adopt the l^∞-topology. Then types (A) and (I) are rewritten as

(A) there exists a sequence of nontrivial homoclinic solutions for (1.3) whose l^∞-norm converges to zero,

(I) there exists a constant $C > 0$ such that $\|u\|_\infty \ge C > 0$ for all nontrivial homoclinic solutions u of (1.3).

Unlike type (A), many existing results concentrated on the existence of a sequence of solutions going to infinity [6, 17, 22, 36, 40]. However, we mainly focus on types (A) and (I). The most typical example of type (I) is a discrete nonlinear Emden-Fowler equation,

$$-\Delta u_n + u_n = \gamma_n |u_n|^{p-2}u_n,\ n \in \mathbb{Z}^m, \quad \lim_{|n|\to\infty} u_n = 0, \tag{1.9}$$

with a superlinear exponent $2 < p < \infty$ and $0 < \gamma_n < \infty$ for $n \in \mathbb{Z}^m$. In fact, if u is a nontrivial homoclinic solution of (1.9) in l^2, then we take the l^2 inner product $(\cdot,\cdot)$ of (1.9) with u to obtain

$$\|u\|_2^2 \le ((-\Delta + 1)u, u) = \sum_{n\in\mathbb{Z}^m} \gamma_n |u_n|^p \le \gamma^* \|u\|_\infty^{p-2} \|u\|_2^2,$$

where $\gamma^* = \sup\{\gamma_n\}$. Dividing both sides by $\|u\|_2^2$, we have $1 \le \gamma^* \|u\|_\infty^{p-2}$, which is a priori lower estimate of all nontrivial homoclinic solutions. Hence the zero solution of (1.9) with a superlinear exponent p is isolated from other homoclinic solutions in l^2.

A typical example of type (A) is (1.9) with $1 < p < 2$ and a positive-valued sequence $\gamma = \{\gamma_n\} \in l^{2/(2-p)}$. To explain this assertion, we have the following result, which will be proved as a corollary of our main theorem (Theorem 2.1 in Section 2).

Theorem 1.1. *Assume that $f_n(u)$ is an odd continuous function with respect to u on $[-\epsilon, \epsilon]$ with some $\epsilon > 0$ for $n \in \mathbb{Z}^m$. Assume further that there exist a constant $1 < \nu < 2$ and a positive-valued sequence $a = \{a_n\} \in l^{2/(2-\nu)}$ such that*

$$|f_n(u)| \le a_n |u|^{\nu-1}, \quad u \in [-\epsilon, \epsilon], \quad n \in \mathbb{Z}^m. \tag{1.10}$$

If $\omega < \alpha$ and (1.5) holds uniformly for $n \in \mathbb{Z}^m$, then there exists a sequence of nontrivial homoclinic solutions for (1.3) whose l^∞-norm converges to zero.

As a direct corollary of Theorem 1.1, we have the next result of (1.9) on the accumulation of the zero solution.

Corollary 1.2. *Equation (1.9) has a sequence of nontrivial homoclinic solutions whose l^∞-norm converges to zero if $1 < p < 2$ and the positive-valued sequence $\gamma = \{\gamma_n\}$ belongs to $l^{2/(2-p)}$.*

It is easy to verify that (1.9) with $1 < p < 2$ and the positive-valued sequence $\gamma = \{\gamma_n\} \in l^{2/(2-p)}$ implies (1.5) and (1.10). Hence Corollary 1.2 follows from Theorem 1.1.

The purpose of this paper is to weaken the assumptions in Theorem 1.1 and to find a general sufficient condition on $f_n(u)$ for type (A). Our results show that the sublinearity and oddness of $f_n(u)$ of (1.3) yield a sequence of nontrivial homoclinic solutions converging to zero. To the best of our knowledge, this is the first time to obtain infinitely many homoclinic solutions for (1.3) with partially (local) sublinear nonlinearity. The oddness assumption on $f_n(u)$ is only used for applying the variant Clark's theorem, which is a very powerful tool for obtaining the multiplicity results. Even if there is no oddness assumption on $f_n(u)$, we still can prove the existence of homoclinic solutions of (1.3). Moreover, we will give a criterion to make sure that (1.3) is of type (I).

The main idea in this paper is as follows. Note that the DNLS equation (1.3) will face to the loss of compactness due to the problem is set on the unbounded domain $\mathbb{Z}^m$. To overcome this difficulty, a compact embedding technique has been used according to a rather mild condition (A3) (given in Section 2). Another obstacle of (1.3) is caused by the local definition of sublinear terms, which may let the functional J (defined by (3.2) in Section 3) do not make sense. We use a cutoff method to recover the global property of sublinear terms, allowing us to apply the critical point theory.

The remaining of this paper is organized as follows: Section 2 is the statement of our main results and its explanation. In Section 3, we first establish the variational framework associated with (1.3), transferring the problem of the homoclinic solutions of (1.3) into that of critical points of the corresponding functional J, and then present the proof of our main results. Some discussion will be made in Section 4.

To obtain the main results, for the reader's convenience, we include this section by citing some basic notations and some known results from the critical point theory.

Definition 1.3 ([32]). Let E be a real Banach Space and $J \in C^1(E,\mathbb{R})$. A sequence $\{u_n\} \subset E$ is called a Palais-Smale sequence (P.S. sequence for short) for J if $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$. We say J satisfies the Palais-Smale condition (P.S. condition for short) if any P.S. sequence for J possesses a convergent subsequence.

Lemma 1.4 ([32]). *Let E be a real Banach space and $J \in C^1(E,\mathbb{R})$ satisfy the P.S. condition. If J is bounded from below, then $c = \inf_E J$ is a critical value of J.*

Lemma 1.5 ([23]). *Let E be a real Banach space and $J \in C^1(E,\mathbb{R})$. Assume J satisfies the P.S. condition, is even and bounded from below, and $J(0) = 0$. If for any $k \in \mathbb{N}$, there exists a k-dimensional subspace E_k of E and $\rho_k > 0$ such that $\sup_{E_k \cap S_{\rho_k}} J < 0$, where $S_\rho = \{u \in X| \, \|u\| = \rho\}$, then at least one of the following conclusions holds.*

(i) *There exists a sequence of critical points $\{u_k\}$ satisfying $J(u_k) < 0$ for all k and $\|u_k\| \to 0$ as $k \to \infty$.*
(ii) *There exists $r > 0$ such that for any $0 < a < r$ there exists a critical point u such that $\|u\| = a$ and $J(u) = 0$.*

2. Main results

We first present some assumptions in order to establish our results in this paper.

(A2) $f_n(u)$ is an odd continuous function with respect to u on $[-\epsilon,\epsilon]$ with some $\epsilon > 0$ for $n \in \mathbb{Z}^m$.
(A3) There exist two constants $1 \le \nu_1 < \nu_2 < 2$ and two positive-valued sequences $a_i = \{a_{i,n}\} \in l^{2/(2-\nu_i)}$ of $i = 1, 2$, such that
$$|f_n(u)| \le a_{1,n}|u|^{\nu_1-1} + a_{2,n}|u|^{\nu_2-1} \tag{2.1}$$
for $u \in [-\epsilon,\epsilon]$ and $n \in \mathbb{Z}^m$.
(A4) There exists an infinite sequence $I = \{n^{(i)}\} \subseteq \mathbb{Z}^m$ such that (1.5) holds uniformly for $n \in I$.

Under the above assumptions, we have the following result.

Theorem 2.1. *Assume that* (A1) *holds, and $f_n(u)$ satisfies* (A2)–(A4). *If $\omega < \alpha$, then* (1.3) *is of type (A), that is, there exists a sequence of nontrivial homoclinic solutions for* (1.3) *whose l^∞-norm converges to zero.*

Remark 2.2. Under condition (A1), the discrete potential $V = \{v_n\}$ is allowed to change sign or to be unbounded from above in this paper. The existence of a sequence of homoclinic solutions diverging to infinity for (1.3) with unbounded potential ($v_n \to +\infty$ as $|n| \to +\infty$) has been studied in many papers [6, 17, 22, 36, 40]. Unlike these existing results, we can obtain infinitely many homoclinic solutions of (1.3) converging to zero, even for the unbounded potential case.

Remark 2.3. We emphasize that in Theorem 2.1, the conditions on the nonlinear term $f_n(u)$ are supposed near $u = 0$ only and there are no conditions for large $|u|$. This is very important. Indeed, this assumption allows us to study equations having singularity or rapidly-increasing terms as $|u| \to \infty$. For example, let us consider the equation
$$-\Delta u_n + u_n = \frac{|u_n|^{q-1}u_n}{|\sin u_n|(1+|n|)^2} + \frac{|u_n|^{r-2}u_n \exp(u_n^2)}{(1+|n|)^2}, \quad n \in \mathbb{Z}^m, \quad \lim_{|n|\to\infty} u_n = 0,$$

with $1 < q < 2 < r$. The first term in the right hand side has singularities at $k\pi$ with $k \in \mathbb{Z} \setminus \{0\}$, but continuous at $u = 0$. The second term on the right hand side grows exponentially as $|u| \to \infty$. Theorem 2.1 asserts that the equation above is of type (A). Therefore it is important that Theorem 2.1 does not require any condition on $f_n(u)$ for large $|u|$.

Remark 2.4. Many existing results focused on the existence and multiplicity of homoclinic solutions for (1.3) with asymptotically or super linear term $f_n(u)$ at both the origin and infinity [5, 15, 17, 19, 25, 27, 30, 31, 35, 36, 40, 41, 42]. The following condition

(A5) $H_n(u) = f_n(u)u - 2F_n(u) < 0$ if $u \neq 0$, where $F_n(u) = \int_0^u f_n(s)ds$,

plays a very important role in the existence of nontrivial homoclinic solutions for the asymptotically or super linear cases. In fact, (A5) is used to satisfy that the associated functional J (defined by (3.2) in Section 3) has $u = 0$ as the only critical point with the critical value 0. However, there is no assumption on $H_n(u)$ in our results. In our setting, J is allowed to has the nontrivial critical point with the critical value 0, by using the improved Clark's theorem [23].

The oddness assumption on $f_n(u)$ is only used for obtaining multiple results by using the variant Clark's theorem. Even if there is no such an oddness assumption, we still have the following result on the existence of homoclinic solutions of (1.3).

Theorem 2.5. *Assume that* (A1) *holds. The nonlinearity* $f_n(u)$ *satisfies the following condition.*

(A2') $f_n(u)$ *is a continuous function with respect to* u *for* $u \in \mathbb{R}$ *and* $n \in \mathbb{Z}^m$.

(A3') *There exist two constants* $1 \leq \nu_1 < \nu_2 < 2$ *and two positive-valued sequences* $a_i = \{a_{i,n}\} \in l^{2/(2-\nu_i)}$ *of* $i = 1, 2$ *such that* (2.1) *holds for* $u \in \mathbb{R}$ *and* $n \in \mathbb{Z}^m$.

(A4') *There exists a* $n^{(0)} \in \mathbb{Z}^m$ *such that*

$$\limsup_{u \to 0} \frac{F_{n^{(0)}}(u)}{u^2} = \infty. \tag{2.2}$$

If $\omega < \alpha$, *then* (1.3) *has at least a nontrivial homoclinic solution.*

Remark 2.6. There were only a few results of nontrivial homoclinic solutions of the DNLS equations in one spatial dimension ($m = 1$) with sublinear nonlinearity f_n [6, 9, 34]. It was assumed in [6, 9, 34] that $f_n(u) = O(|u|^s)$ as $u \to 0$ with some $0 < s < 1$, which implies (2.2) in our paper. However, there exists some f_n which satisfies (2.2) but $f_n(u) = O(|u|^s)$ does not hold as $u \to 0$ for any $0 < s < 1$. For example, taking $f_n(u) = u \ln(|u|)$ gives us that $f_n(u) = o(|u|^s)$ as $u \to 0$ for any $0 < s < 1$ and so (2.2) holds. Thus, Theorem 2.5 is a new result even for one spatial dimensional case ($m = 1$).

Now we make some preparations for the proof of our main results. Under the assumption $\omega < \alpha$, we have

$$\delta \equiv \alpha - \omega > 0,$$
$$\delta \|u\|_2^2 \leq ((L - \omega)u, u), \quad u \in l^2,$$

where $(\cdot, \cdot)$ is the inner product of l^2. We define the space

$$E \equiv \{u \in l^2 : ((L - \omega)u, u) < \infty\} \tag{2.3}$$

which is a Hilbert space equipped with the norm

$$\|u\| = \sqrt{((L-\omega)u,u)}. \tag{2.4}$$

It is obviously that

$$\|u\|_\infty \le \|u\|_2 \le \frac{1}{\sqrt{\delta}}\|u\|, \quad u \in E. \tag{2.5}$$

We end this section by giving a sufficient condition for type (I) in the following statement.

Proposition 2.7. *Assume $\omega < \alpha$. If*

$$\limsup_{u\to 0}\Big(\sup_{n\in\mathbb{Z}^m}\frac{f_n(u)}{u}\Big) < \alpha-\omega, \tag{2.6}$$

then (1.3) *is of type (I) in E.*

Proof. According to (2.6), we can pick $\beta > 0$ such that

$$\limsup_{u\to 0}\Big(\sup_{n\in\mathbb{Z}^m}\frac{f_n(u)}{u}\Big) < \beta < \alpha-\omega. \tag{2.7}$$

Then we can find a small enough $\varepsilon > 0$ such that

$$f_n(u)u \le \beta u^2, \quad |u| \le \varepsilon, \quad n \in \mathbb{Z}^m. \tag{2.8}$$

Let $u = \{u_n\}$ be any solution of (1.3) in E. Multiplying (1.3) by u_n and summing it over $\mathbb{Z}^m$, we have

$$\|u\|^2 = \sum_{n\in\mathbb{Z}^m} f_n(u_n)u_n \le \beta\|u\|_2^2,$$

provided that $\|u\|_\infty \le \varepsilon$. It follows from (2.5) that $u \equiv 0$. Thus, if $\|u\|_\infty \le \varepsilon$, then u vanishes in E and so type (I) occurs. $\square$

3. Proofs of main results

3.1. **Proof of Theorem 2.1.** Under condition (A3), for each $n \in \mathbb{Z}^m$, the nonlinear term $f_n(u)$ is defined on $[-\epsilon,\epsilon]$. To recover the global property of $f_n(u)$, we define a function $g \in C_0^\infty(\mathbb{R},\mathbb{R})$ such that $0 \le g(u) \le 1$, $g(-u) = g(u)$ for $u \in \mathbb{R}$, $g(u) = 1$ for $|u| \le \epsilon/2$, and $g(u) = 0$ for $|u| \ge \epsilon$. For example,

$$g(u) = \begin{cases} 1, & 0 < |u| \le \frac{\epsilon}{2}, \\ \exp\Big(1 + \frac{1}{\left(\frac{2|u|}{\epsilon}-1\right)^2 - 1}\Big), & \frac{\epsilon}{2} < |u| < \epsilon, \\ 0, & |u| \ge \epsilon, \end{cases}$$

meets our requirement. Instead of (1.3), we consider the equation

$$Lu_n - \omega u_n = f_n(u_n)g(u_n), \quad n \in \mathbb{Z}^m, \tag{3.1}$$

with the boundary condition (1.4). We see from (2.5) that, to prove Theorem 2.1, it is sufficient to show that (3.1) has a sequence $\{u^{(k)}\}$ in E with all $u^{(k)} \ne 0$ such that E-norm of $u^{(k)}$ converges to zero. Then $u^{(k)}$ belongs to l^∞ and the l^∞-norm of $u^{(k)}$ converges to zero. Let the l^∞-norm of $u^{(k)}$ be less than $\epsilon/2$ for k large enough. Then $g(u_n^{(k)}) \equiv 1$ and (3.1) is reduced to (1.3). Thus Theorem 2.1 follows.

Even if $f_n(u)$ is defined on the whole space $\mathbb{R}$ in (A2) from the beginning, we also need the truncation $g(u)$. Otherwise, in case $f_n(u)$ grows up to ∞ very rapidly as $u \to \infty$, the functional $J(u)$ given in (3.2) later on, is not well defined.

In (3.1), we rewrite $f_n(u)g(u)$ as $f_n(u)$. Consider the function J defined on E by

$$J(u)=\frac{1}{2}\left((L-\omega)u,u\right)-\sum_{n\in\mathbb{Z}^m}F_n(u_n). \tag{3.2}$$

which $F_n(u)$ is given in (1.5).

To apply Lemma 1.5 on the existence of critical points of (3.2), we first show that the functional J is a well-defined C^1 functional on E. Moreover, for the derivative of J, we have

$$(J'(u),v)=((L-\omega)u,v)-\sum_{n\in\mathbb{Z}^m}f_n(u_n)v_n,\quad u,v\in E. \tag{3.3}$$

According to (A3), for any $u\in E$, we have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}^m}F_n(u_n)&\le\sum_{n\in\mathbb{Z}^m}\frac{1}{\nu_1}a_{1,n}|u_n|^{\nu_1}+\sum_{n\in\mathbb{Z}^m}\frac{1}{\nu_2}a_{2,n}|u_n|^{\nu_2}\\
&\le\frac{1}{\nu_1}\|a_1\|_{2/(2-\nu_1)}\|u\|_2^{\nu_1}+\frac{1}{\nu_2}\|a_2\|_{2/(2-\nu_2)}\|u\|_2^{\nu_2}\\
&\le\frac{1}{\nu_1\delta^{\nu_1/2}}\|a_1\|_{2/(2-\nu_1)}\|u\|^{\nu_1}+\frac{1}{\nu_2\delta^{\nu_2/2}}\|a_2\|_{2/(2-\nu_2)}\|u\|^{\nu_2}.
\end{aligned}$$

Hence J is well defined on E. Next, we prove that (3.3) holds. For any $u,v\in E$ and any sequence $\{\theta_n\}$ with $\theta_n\in(0,1)$ for $n\in\mathbb{Z}^m$, we have

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^m}\max_{h\in(0,1)}|f_n(u_n+h\theta_nv_n)v_n|\\
&\le\sum_{n\in\mathbb{Z}^m}a_{1,n}\left(|u_n|^{\nu_1-1}+|v_n|^{\nu_1-1}\right)|v_n|+\sum_{n\in\mathbb{Z}^m}a_{2,n}\left(|u_n|^{\nu_2-1}+|v_n|^{\nu_2-1}\right)|v_n|\\
&\le\|a_1\|_{2/(2-\nu_1)}(\|u\|_2^{\nu_1-1}+\|v\|_2^{\nu_1-1})\|v\|_2\\
&\quad+\|a_2\|_{2/(2-\nu_2)}(\|u\|_2^{\nu_2-1}+\|v\|_2^{\nu_2-1})\|v\|_2\\
&\le\frac{1}{\delta^{\nu_1/2}}\|a_1\|_{2/(2-\nu_1)}(\|u\|^{\nu_1-1}+\|v\|^{\nu_1-1})\|v\|\\
&\quad+\frac{1}{\delta^{\nu_2/2}}\|a_2\|_{2/(2-\nu_2)}(\|u\|^{\nu_2-1}+\|v\|^{\nu_2-1})\|v\|<+\infty.
\end{aligned}$$

Combining the above inequality and (3.2), we have

$$\begin{aligned}
(J'(u),v)&=\lim_{h\to0^+}\frac{J(u+hv)-J(u)}{h}\\
&=\lim_{h\to0^+}\frac{1}{2h}[((L-\omega)(u+hv),u+hv)-((L-\omega)u,u)]\\
&\quad-\lim_{h\to0^+}\frac{1}{h}\sum_{n\in\mathbb{Z}^m}[F_n(u_n+hv_n)-F_n(u_n)]\\
&=\lim_{h\to0^+}\Big[((L-\omega)u,v)+\frac{h\|v\|^2}{2}-\sum_{n\in\mathbb{Z}^m}f_n(u_n+h\theta_nv_n)v_n\Big]\\
&=((L-\omega)u,v)-\sum_{n\in\mathbb{Z}^m}f_n(u_n)v_n.
\end{aligned}$$

This shows that (3.3) holds. Thus, (1.3) is the corresponding Euler-Lagrange equation for J. To find nontrivial homoclinic solutions of (1.3), we need only to look for nonzero critical points of J in E.

Now let us prove that J' is continuous. Let $u^{(k)} \to u \in E$ as $k \to +\infty$. Then $\{\|u^{(k)}\|\}$ is bounded, which follows that $\{\|u^{(k)}\|_2\}$ is also bounded. Note that $a_i \in l^{2/(2-\nu_i)}$ for $i = 1, 2$. For any $\varepsilon > 0$, there exists an integer A_ε such that

$$\Big(\sum_{|n|>A_\varepsilon} a_{i,n}^{2/(2-\nu_i)} \Big)^{(2-\nu_i)/2} < \varepsilon, \quad i = 1, 2. \tag{3.4}$$

For any $v \in E$, we have

$$\begin{aligned}
&|(J'(u^{(k)}) - (J'(u), v)| \\
&\le \Big| \big((L-\omega)(u^{(k)} - u), v\big) \Big| + \sum_{n\in\mathbb{Z}^m} \big| f_n(u_n^{(k)}) - f_n(u_n) \big| \, |v_n| \\
&\le o_k(1) + \|v\|_\infty \sum_{|n|>A_\varepsilon} \Big| f_n(u_n^{(k)}) + f_n(u_n) \Big| \\
&\le o_k(1) + \|v\|_\infty \sum_{|n|>A_\varepsilon} \Big(a_{1,n} \Big(|u_n^{(k)}|^{\nu_1 - 1} + |u_n|^{\nu_1 - 1} \Big) \Big) \\
&\quad + \|v\|_\infty \sum_{|n|>A_\varepsilon} \Big(a_{2,n} \Big(|u_n^{(k)}|^{\nu_2 - 1} + |u_n|^{\nu_2 - 1} \Big) \Big) \\
&\le o_k(1) + \|v\|_\infty \Big(\|u^{(k)}\|_2^{\nu_1 - 1} + \|u\|_2^{\nu_1 - 1} + \|u^{(k)}\|_2^{\nu_2 - 1} + \|u\|_2^{\nu_2 - 1} \Big) \varepsilon,
\end{aligned}$$

where $o_k(1)$ means $o_k(1) \to 0$ as $k \to +\infty$. This implies that $J \in C^1(E, \mathbb{R})$.

Next we prove that J is bounded from below. According to (A3), we have

$$\begin{aligned}
J(u) &= \frac{1}{2}((L-\omega)u, u) - \sum_{n\in\mathbb{Z}^m} F_n(u_n) \\
&\ge \frac{1}{2}\|u\|^2 - \frac{1}{\nu_1} \sum_{n\in\mathbb{Z}^m} a_{1,n}|u_n|^{\nu_1} - \frac{1}{\nu_2} \sum_{n\in\mathbb{Z}^m} a_{2,n}|u_n|^{\nu_2} \\
&\ge \frac{1}{2}\|u\|^2 - \frac{1}{\nu_1 \delta^{\nu_1/2}} \|a_1\|_{2/(2-\nu_1)} \|u\|^{\nu_1} - \frac{1}{\nu_2 \delta^{\nu_2/2}} \|a_2\|_{2/(2-\nu_2)} \|u\|^{\nu_2}.
\end{aligned}$$

It follows from $1 \le \nu_1 < \nu_2 < 2$ that $J(u) \to +\infty$ as $\|u\| \to +\infty$. Thus J is bounded from below.

Now we claim that J satisfies the P.S. condition. Assume that $\{u^{(k)}\} \subset E$ is a P.S. sequence. The coerciveness of J gives us that both $\{\|u^{(k)}\|\}$ and $\{\|u^{(k)}\|_2\}$ are bounded. Hence, passing to a subsequence if necessary, we have $u^{(k)} \rightharpoonup u$ in E as $k \to +\infty$ and

$$(J'(u^{(k)}) - J'(u), u^{(k)} - u) \to 0 \quad \text{as } k \to +\infty. \tag{3.5}$$

It is easy to see that for each $n \in \mathbb{Z}^m$, $u_n^{(k)}$ converges to u_n pointwise as $k \to +\infty$. For any $\varepsilon > 0$, there exist two positive integers A_ε and k_ε such that (3.4) holds and

$$\sum_{|n|\le A_\varepsilon} \|f_n(u_n^{(k)}) - f_n(u_n)| \, |u_n^{(k)} - u_n| < \varepsilon, \quad k \ge k_\varepsilon.$$

In addition, we have

$$
\begin{aligned}
&\sum_{|n|>A_\varepsilon} |f_n(u_n^{(k)}) - f_n(u_n)|\,|u_n^{(k)} - u_n| \\
&\le \sum_{|n|>A_\varepsilon} \big[a_{1,n}(|u_n^{(k)}|^{\nu_1-1} + |u_n|^{\nu_1-1})\big](|u_n^{(k)}| + |u_n|) \\
&\quad + \sum_{|n|>A_\varepsilon} \big[a_{2,n}(|u_n^{(k)}|^{\nu_2-1} + |u_n|^{\nu_2-1})\big](|u_n^{(k)}| + |u_n|) \\
&\le 2\sum_{|n|>A_\varepsilon} \big[a_{1,n}(|u_n^{(k)}|^{\nu_1} + |u_n|^{\nu_1}) + a_{2,n}(|u_n^{(k)}|^{\nu_2} + |u_n|^{\nu_2})\big] \\
&\le 2\Big(\|u^{(k)}\|_2^{\nu_1} + \|u\|_2^{\nu_1} + \|u^{(k)}\|_2^{\nu_2} + \|u\|_2^{\nu_2}\Big)\varepsilon,
\end{aligned}
$$

which implies that

$$
\sum_{n\in\mathbb{Z}^m} \big(f_n(u_n^{(k)}) - f_n(u_n)\big)\big(u_n^{(k)} - u_n\big) \to 0 \quad \text{as } k \to +\infty. \tag{3.6}
$$

Note that

$$
(J'(u^{(k)}) - J'(u), u^{(k)} - u) = \|u^{(k)} - u\| - \sum_{n\in\mathbb{Z}^m} \big(f_n(u_n^{(k)}) - f_n(u_n)\big)\big(u_n^{(k)} - u_n\big).
$$

It follows from (3.5) and (3.6) that

$$
u^{(k)} \to u \quad \text{in } E, \quad \text{as } k \to +\infty. \tag{3.7}
$$

Hence, J satisfies the P.S. condition.

We are in a position to show that J satisfies the last part of Lemma 1.5. Assume that $n^{(i)} \in I$ where I is given in (A4). Define $e^{(i)} = \{e_j^{(i)}\}$ by

$$
e_j^{(i)} = \begin{cases} 1, & j = n^{(i)}, \\ 0, & j \neq n^{(i)}. \end{cases}
$$

Let

$$
E_k = \operatorname{span}\{e^{(i)} : i = 1, 2, \dots, k\}, \quad k \in \{1, 2, 3, \dots\},
$$

and L_k be the operator L acting in E_k. Denote $I_k = \{n^{(i)} : n^{(i)} \in I,\ i = 1, 2, \dots, k\}$. According to (A4), for a given $M_k = \|L_k - \omega\|^2$, there exists a ε_k such that

$$
M_k|u|^2 \le F_n(u), \quad |u| \le \varepsilon_k,\ n \in I_k.
$$

Denote $\rho_k = \min\{\varepsilon_k,\, \varepsilon_k/\sqrt{\delta}\}$. For $u \in E_k$ with $\|u\| = \rho_k$, we have $\|u\|_\infty \le \varepsilon_k$ and

$$
\begin{aligned}
J(u) &= \frac{1}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}^m} F_n(u_n) \\
&= \frac{1}{2}\|u\|^2 - \sum_{n\in I_k} F_n(u_n) \\
&\le \frac{1}{2}\|u\|^2 - M_k \sum_{n\in I_k} |u_n|^2 \\
&\le \frac{1}{2}\|u\|^2 - \frac{M_k}{\|L_k - \omega\|^2}\|u\|^2 \\
&= -\frac{1}{2}\rho_k^2 < 0.
\end{aligned}
$$

We have verified the conditions in Lemma 1.5. It follows that (3.1) has a sequence $\{u^{(k)}\}$ in E with all $u^{(k)} \neq 0$ such that $\|u^{(k)}\| \to 0$ as $k \to 0$. According to (2.5), $\|u^{(k)}\|_\infty \to 0$ as $k \to 0$. Let k large enough such that $\|u^{(k)}\|_\infty < \epsilon/2$. Then $g(u_n^{(k)}) \equiv 1$ and (3.1) becomes (1.3). The desired result follows.

3.2. **Proof of Theorem 2.5.** Under the assumptions of Theorem 2.5, it is easy to show that (1.3) is the corresponding Euler-Lagrange equation for J defined by (3.2). To find nontrivial homoclinic solutions of (1.3), we just need to look for nonzero critical points of J in E. Similar to the proof of Theorem 2.1, we can prove that $J \in C^1(E, \mathbb{R})$ is bounded from below and satisfies the P.S. condition. By Lemma 1.4, J possesses a critical value $c = \inf_E J(u)$. Hence, there exists $u_* \in E$ such that $J(u_*) = c$ with $J'(u_*) = 0$. We claim that $u_* \neq 0$. According to (f_2'), there exist $0 < \varepsilon_0 < 1$ and $M_0 > 0$ with $2m + v_{n^{(0)}} - \omega - 2M_0 < 0$ such that

$$F_{n^{(0)}}(\varepsilon_0) \geq M_0 \varepsilon_0^2.$$

Taking $u = u^{(0)} = \{u_n^{(0)}\}$ with

$$u_n^{(0)} = \begin{cases} \varepsilon_0, & n = n^{(0)}, \\ 0, & n \neq n^{(0)}, \end{cases}$$

we have

$$\begin{aligned} J(u^{(0)}) &= \frac{1}{2}((L-\omega)u^{(0)}, u^{(0)}) - \sum_{n \in \mathbb{Z}^m} F_n(u_n^{(0)}) \\ &\leq \frac{1}{2}(2m + v_{n^{(0)}} - \omega)\varepsilon_0^2 - M_0 \varepsilon_0^2 \\ &= \frac{1}{2}(2m + v_{n^{(0)}} - \omega - 2M_0)\varepsilon_0^2 < 0. \end{aligned}$$

It follows that $J(u_*) \leq J(u^{(0)}) < 0$, which implies that u_* is a nontrivial solution of (1.3) in E. Thus, Theorem 2.5 follows.

4. Discussion

The discrete nonlinear Schrödinger (DNLS) equation is one of the most important nonlinear lattice models in the field of nonlinear science [11, 12, 18, 24]. Breathers that have been observed in experiments can exist in the DNLS equations [7, 13, 14]. Indeed, breathers admit one special kind of homoclinic solutions. In the past decade, the existence of homoclinic solutions of the DNLS equations has drawn a great deal of interest [1, 2, 3, 16, 26, 43]. See [17, 25, 27, 30, 35, 36, 40] for the superlinear nonlinearity, [5, 15, 19, 31, 41, 42] for the saturable (asymptotically linear) nonlinearity and [20, 21, 22] for mixed nonlinearity. However, only some results considered the existence of homoclinic solutions for the DNLS equations with sublinear nonlinearity [6, 9, 34]. Considering its importance in physics [4, 8, 10], there needs a further study on the existence of homoclinic solutions for the DNLS equations with sublinear nonlinearity.

In this paper, we consider the DNLS equation (1.3) in m dimensional lattices with attractive self-interaction and give a partially sublinear condition on $f_n(u)$ for type (A), i.e., a sequence of nontrivial homoclinic solutions accumulating to zero. Our results assert that the sublinearity and oddness of $f_n(u)$ admit type (A) for (1.3). The oddness of $f_n(u)$ is only used for applying the variant Clark's theorem. Without this oddness assumption, we still can prove that (1.3) has at least a nontrivial homoclinic solution if $f_n(u)$ is sublinear.

Now, we discuss what we may do for the homoclinic solutions of sublinear DNLS equation (1.3) in the future. We have focused on the homoclinic solutions of (1.3) for the case where $\omega < \alpha$. Under this condition, (1.3) is a positively definite problem, allowing us to find the nonzero critical points with mountain pass geometry. If $\omega \geq \alpha$, this problem is strongly indefinite. It will be of interest to obtain the nontrivial homoclinic solutions with type (A) for the strongly indefinite problem (1.3). Another interesting direction may be the connection between superlinear and sublinear nonlinearities in (1.3). We have shown that the sublinear DNLS equation (1.3) has at least a nontrivial homoclinic solution without any oddness assumption. How do the combined effects of superlinear and sublinear nonlinearities affect the existence of homoclinic solutions for (1.3) needs to be fully understood. Last but not least, we may consider an extension of this topic to a more general equation. For example, it is worth considering the homoclinic solutions of sublinear difference equations with either p-Laplacian [21] or Jacobi operator [33], since these equations have attracted a lot of attentions [5, 15, 17, 19, 20, 21, 22, 25, 27, 30, 31, 35, 36, 37, 38, 39, 40, 41, 42].

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Genghong Lin
Center for Applied Mathematics, Guangzhou University, Guangzhou 510006, China.
School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China
Email address: lin_genghong@163.com

Jianshe Yu
Center for Applied Mathematics, Guangzhou University, Guangzhou 510006, China
Email address: jsyu@gzhu.edu.cn

Zhan Zhou (corresponding author)
Center for Applied Mathematics, Guangzhou University, Guangzhou 510006, China.
School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China
Email address: zzhou0321@hotmail.com