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ENTROPY SOLUTIONS TO NONCOERCIVE NONLINEAR ELLIPTIC EQUATIONS WITH MEASURE DATA

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Abstract. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. In this article, we investigate the existence of entropy solutions to the nonlinear elliptic problem

$$\begin{gathered}-\operatorname{div}\Big(\frac{|\nabla u|^{(p-2)}\nabla u + c(x)u^\gamma}{(1+|u|)^{\theta(p-1)}}\Big) + \frac{b(x)|\nabla u|^\lambda}{(1+|u|)^{\theta(p-1)}} = \mu, \quad x \in \Omega,\\ u(x) = 0, \quad x \in \partial\Omega,\end{gathered}$$

where μ is a diffuse measure with bounded variation on Ω, $0 \le \theta < 1$ is a positive constants, $1 < p < N$, $0 < \gamma \le p-1$, $0 < \lambda \le p-1$, $c(x)$ and $b(x)$ belong to appropriate Lorentz spaces.

1. Introduction and statement of main results

1.1. Model problem. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. We are interested in existence of entropy solutions of quasilinear elliptic problems with principal part having degenerate coercivity. The model case is

$$\begin{gathered}-\operatorname{div}\Big(\frac{|\nabla u|^{(p-2)}\nabla u + c(x)u^\gamma}{(1+|u|)^{\theta(p-1)}}\Big) + \frac{b(x)|\nabla u|^\lambda}{(1+|u|)^{\theta(p-1)}} = \mu, \quad x \in \Omega,\\ u(x) = 0, \quad x \in \partial\Omega,\end{gathered} \tag{1.1}$$

where p is a real number such that $2 - 1/N < p < N$, $c(x) \in L^{\frac{N}{p-1},r}(\Omega)$ with $\frac{N}{p-1} \le r \le +\infty$, $b(x) \in L^{N,1}(\Omega)$, μ is a diffuse measures. $0 \le \theta < 1$ is a positive constants, it is worth pointing out that the ranges of θ lead to different qualitative properties of solutions.

Existence results for noncoercive elliptic problem (1.1) with $\theta = 0$ are well-known when the Radon measure datum μ with bounded variation on Ω. Nonlinear problem of the type (1.1) have been studied by Del Vecchio and Posteraro[20, 21], in all these papers the existence of solution was proved when $\|c(x)\|_{L^{r_1}(\Omega)}$ or $\|b(x)\|_{L^{r_2}(\Omega)}$ small enough for some r_i, $i = 1, 2$ and $p > 2 - 1/N$. The results of [20, 21] was developed by Betta et al. [6] for $1 < p < N$ and $b(x) \equiv 0$, where renormalized solution be introduced. Droniou [23] studied regularity properties of solutions to (1.1)with $\theta \equiv 0$ and $b(x) \equiv 0$ with the help of some new local estimates on sets far from the

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support of the singular part of the right-hand side data μ. Both influence of terms $-\operatorname{div}(c(x)u^\gamma)$ and $b(x)|\nabla u|^\lambda$ were considered in [25, 26]. Some other related results of noncoercive elliptic equations, we refer to [33, 22, 24, 15, 27] and the references therein.

Note that the case $\theta > 0$ is noticeable different from the case $\theta = 0$, since in this case, problem (1.1) is strongly noncoercive even $c(x) \equiv 0$ and $b(x) \equiv 0$. There are a lot of papers devoted to the study of the existence, regularity and uniqueness of solutions to the problem

$$\begin{gathered} -\operatorname{div}\Big(\frac{a(x,\nabla u)}{(1+|u|)^{\theta(p-1)}}\Big) + cu = \mu, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega. \end{gathered} \tag{1.2}$$

Some surprising results, proved by Boccardo and Brezis [12], shown that problem (1.2) with $p = 2$ have solutions provided $\theta > 1$ and $\mu \in L^m(\Omega)$ for some appropriate m, which extended in [1] for $p > 1$. Note that nonexistence result for problem (1.2) with $c \equiv 0$ was proved in [1], even for $\mu \in L^\infty(\Omega)$. Another nonexistence results of problem (1.2) with μ is a bounded Radon measure concentrated on a set of zero harmonic capacity were established in [13]. The surprising results of [12] relies on the fact that the lower order term u provides a regularizing effect on problem (1.2). We refer to [3, 8, 2, 32, 34] and the references therein for regularizing effect of lower order term.

Recently, Porzio and Smarrazzo[36] proved the existence of suitably defined weak Radon measure-valued solutions to problem (1.2) with finite Radon measure data μ. In the same paper, the authors also investigated the uniqueness of very weak solutions to problem (1.2) with $p = 2$ if μ is a diffuse measure with respect to the 2-capacity. Our interest in this article is to investigate the existence of entropy solutions to problem (1.1) and the combined effects of lower order term $-\operatorname{div}(c(x)u^\gamma)$ and $b(x)|\nabla u|^\lambda$.

It is worth to point out that problem (1.2) is a generalization of the model problem

$$\begin{gathered} -\Delta(\phi(u)) + cu = \mu, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.3}$$

where

$$\phi(s) = \int_0^s \frac{1}{(1+|t|)^{\theta(p-1)}} dt, \tag{1.4}$$

correspondence with $0 \le \theta < 1$. Define $v = \phi(u)$, and rewrite problem (1.3) formally as

$$\begin{gathered} -\Delta v + cg(v) = \mu, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.5}$$

where

$$\eta(p-1))s]^{\frac{1}{1-\theta(p-1)}} - 1.$$

Recently, great attention has been devoted to nonlinear problem of the type (1.5) involving absorption and measures as boundary data[7, 19, 28, 29, 17]. A fundamental contribution to this problem is due to Brezis[16], see also [18, 5].

1.2. **The general problem and main results.** In this article, we do not consider only the model problem (1.1), but we prove the existence of positive weak entropy solutions to the general nonlinear elliptic problem

$$\begin{gathered} -\operatorname{div}\Big(\frac{a(x,u,\nabla u)+K(x,u)}{(1+|u|)^{\theta(p-1)}}\Big)+H(x,u,\nabla u)+G(x,u)=\mu, \quad x\in\Omega,\\ u=0, \quad x\in\partial\Omega, \end{gathered} \tag{1.6}$$

where Ω is bounded domain in $\mathbb{R}^N, N\geq 3$, with sufficiently smooth boundary $\partial\Omega$, $1<p<N$ and $0<\theta<1-\frac{N-p}{N(p-1)}$.

The function $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function (that is, $a(\cdot,s,\xi)$ measurable on Ω for every (s,ξ) in $\mathbb{R}\times\mathbb{R}^N$, and $a(x,\cdot,\cdot)$ continuous on $\mathbb{R}\times\mathbb{R}^N$ for almost every x in Ω) satisfying the following assumptions:

$$\begin{gathered} a(x,s,\xi)\xi\geq\alpha_1|\xi|^p, \quad \alpha_1>0,\\ |a(x,s,\xi)|\leq\alpha_2[|\xi|^{p-1}+s^{p-1}+a_0(x)], \quad a_0(x)\in L^{p'}(\Omega),\ \alpha_2>0,\\ \langle a(x,s,\xi)-a(x,s,\eta),\xi-\eta\rangle>0, \quad \xi\neq\eta, \end{gathered} \tag{1.7}$$

for almost every $x\in\Omega$ and for every $s\in\mathbb{R},\xi\in\mathbb{R}^N,\eta\in\mathbb{R}^N$, p' is the Hölder conjugate exponent of p, i.e. $1/p+1/p'=1$.

The function $K:\Omega\times\mathbb{R}\to\mathbb{R}^N$ is a Carathéodory function satisfying

$$\begin{gathered} |K(x,s)|\leq c_0(x)|s|^\gamma+c_1(x),\ 0<\gamma\leq(p-1),\\ c_0(x)\in L^{\frac{N}{p-1},r}(\Omega), \quad \frac{N}{p-1}\leq r\leq+\infty, \quad c_1(x)\in L^{p'}(\Omega), \end{gathered} \tag{1.8}$$

for almost every $x\in\Omega$ and for every $s\in\mathbb{R}$.

The function $H:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a Carathéodory function satisfying

$$\begin{gathered} |H(x,s,\xi)|\leq\frac{b_0(x)|\xi|^\lambda}{(1+|s|)^{\theta(p-1)}}+\frac{b_1(x)}{(1+|s|)^{\theta(p-1)}}, \quad 0\leq\lambda\leq p-1,\\ b_0(x)\in L^{N,1}(\Omega), \quad b_1(x)\in L^1(\Omega), \end{gathered} \tag{1.9}$$

for almost every $x\in\Omega$ and for every $s\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$.

The function $G:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying

$$\begin{gathered} G(x,s)s\geq 0, \quad |G(x,s)|\leq d_0(x)|s|^t+d_1(x),\\ d_0(x)\in L^{z',1}(\Omega), \quad d_1(x)\in L^1(\Omega), \end{gathered} \tag{1.10}$$

for almost every $x\in\Omega$ and for every $s\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$, where

$$0\leq t<\frac{N(p-1)(1-\theta)}{N-p}, \quad z=\frac{N(p-1)(1-\theta)}{N-p}\frac{1}{t}, \quad \frac{1}{z}+\frac{1}{z'}=1. \tag{1.11}$$

Finally, μ is diffuse measures on Ω with bounded total variation, denoted by $\mathfrak{M}_0(\Omega)$, which is decomposed as

$$\mu=f-\operatorname{div}(g), \tag{1.12}$$

with $f\in L^1(\Omega)$, $g\in\Big(L^{p'}(\Omega)\Big)^N$, more details see Sections 2 below.

It is worth pointing out that problem (1.6) has two main features:

Firstly, the left-hand of (1.6) is defined on $W_0^{1,p}(\Omega)$, but it is not coercive on the same space. On the one hand, $\frac{1}{(1+|u|)^{\theta(p-1)}}$ tends to zero as u tends to infinity, which produces a saturation effect, that is, the diffusion goes to zero when u goes to infinity. On the other hand, this differential operator has two lower order terms, which also produce a lack of coercivity even for $\theta = 0$. Therefore standard Leray-Lions surjectivity theorem for solutions to nonlinear elliptic equations cannot be applied. To overcome this problem, we will reason by approximation, introduced in [30, 4], cutting by means of truncatures the nonlinearity $a(x, s, \xi)$ in order to get a pseudomonotone and coercive differential operator on $W_0^{1,p}(\Omega)$. Then establish some a priori estimates on $|\nabla u|^{p-1}$, which be obtained by the estimate of $|\nabla T_k(u)|^{p-1}$ and a result in [6]. Thus, a technical result of almost everywhere convergence for the gradients leads to pass the limit of the approximate solutions.

Secondly, the right-hand side of problem (1.6) is μ, which is a measure, not an element of the dual space of $W^{-1,p'}(\Omega)$. Therefore, the solution cannot be expected to belong to the energy space $W_0^{1,p}(\Omega)$, it is necessary to change the functional setting in order to prove existence results. To overcome this problem, we define a concept of entropy solution and show that problem (1.6) with measure data is well-posed in this generalized class. It is interesting to note that when $p > N$, the Sobolev embedding theorem and a duality argument imply that the space of measures with bounded variation on Ω is a subspace of $W^{-1,p'}(\Omega)$, which reconduces the problem (1.6) to a classical one. Here and elsewhere in the paper, We will consider the case that $2 - 1/N < p < N$.

Definition 1.1. Under hypotheses (1.7)-(1.12), for $\mu \in \mathcal{M}_0$, a function u is an entropy solution for problem (1.6) if the following conditions hold:

(1) $H(x, u, \nabla u) \in L^1(\Omega)$, $G(x, u) \in L^1(\Omega)$,
(2) $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$,
(3) For every $\omega \in W_0^{1,p}(\Omega) \bigcap L^\infty(\Omega)$ and $k > 0$, it holds

$$\begin{aligned} &\int_\Omega \Big(\frac{a(x, u, \nabla u) + K(x, u)}{(1 + |u|)^{\theta(p-1)}}\Big) \cdot \nabla T_k(u - \omega) \\ &+ \int_\Omega H(x, u, \nabla u) T_k(u - \omega) + \int_\Omega G(x, u) T_k(u - \omega) \\ &\leq \langle \mu, T_k(u - \omega)\rangle. \end{aligned} \tag{1.13}$$

Remark 1.2. Every terms in (1.13) is well defined. In fact, for the right hand side of (1.13), since $\mu = f - \operatorname{div}(g)$, where f belongs to $L^1(\Omega)$ and $T_k(u - \omega)$ is in $L^\infty(\Omega)$, and g belongs to $(L_0^{p'}(\Omega))^N$ while $T_k(u - \omega)$ is in $W_0^{1,p}(\Omega)$.

The left-hand side is well defined since the integral is only on the set $|u - \omega| \leq k$, and on this set $|u| \leq k + \|\omega\|_{L^\infty(\Omega)} := M$, it is equal to

$$\begin{aligned} &\int_\Omega \Big(\frac{a(x, u, \nabla u) + K(x, u)}{(1 + |u|)^{\theta(p-1)}}\Big) \cdot \nabla T_k(u - \omega) \\ &= \int_{\{|u-\omega| \leq k\}} \Big(\frac{a(x, u, \nabla u) + K(x, u)}{(1 + |u|)^{\theta(p-1)}}\Big)\ cdot \nabla(u - \omega) \\ &= \int_{\{|u-\omega| \leq k\}} \Big(\frac{a(x, T_M(u), \nabla T_M(u)) + K(x, T_M(u))}{(1 + |T_M(u)|)^{\theta(p-1)}}\Big) \cdot \nabla(T_M(u) - \omega), \end{aligned}$$

which is finite by the growth assumptions (1.7) on $a(x, u, \nabla u)$.

Remark 1.3. The definition of entropy solution is not suitable to deal with general measure, since

$$\int_\Omega T_k(u-\omega)d\mu$$

may not be well defined when μ is a Radon measure.

The concept of entropy solution for $\mu \in \mathcal{M}_0$ can be extended to measure-valued solution for $\mu \in \mathcal{M}_b$. We quote [36] for measure-valued solution to elliptic equations and [38, 35, 37, 39] for measure-valued solution to parabolic equations. The effort to understand the proper concept of measure-valued solution is still work in progress, and further work is needed if we want to apply it to problem (1.6). We leave the subject at this point.

The main results of this paper is the following theorem.

Theorem 1.4. *Under assumptions* (1.7)–(1.12). *there exists at least one entropy solution u to* (1.1) *if one of the following conditions holds:*

(1) $\gamma = \lambda = p-1$, $c_0(x) \in L^{\frac{N}{p-1},r}(\Omega)$, $\frac{N}{p-1} \le r \le +\infty$, $\|b_0(x)\|_{L^{N,1}(\Omega)}$ *is small enough,*
(2) $\gamma = p-1$, $\lambda < p-1$, $c_0(x) \in L^{\frac{N}{p-1},r}(\Omega)$, $\frac{N}{p-1} \le r \le +\infty$,
(3) $\gamma < p-1$, $\lambda \le p-1$, $c_0(x) \in L^{\frac{N}{p-1},r}(\Omega)$.

The plan of this article is as follows. In Section 2, we briefly recall some notation and known results about measures and Lorentz spaces, then we give some technical results to be used in this paper. Section 3 is devoted to the study an approximate problem. Section 4 contains the proof of Theorem 1.4

2. Preliminaries

2.1. **Notation.** In this section, we first recall some notation and definitions. In the following, C will be a constant that may change from an inequality to another, to indicate a dependence of C on the real parameters $N, p, \alpha, \gamma, \lambda$, we shall write $C = C(N, p, \alpha, \gamma, \lambda)$.

For $k > 0$, denote by $T_k : \mathbb{R} \to \mathbb{R}$ the usual truncation at level k, that is,

$$T_k(s) = \max\{-k, \min\{k, s\}\}. \tag{2.1}$$

The remainder of the truncation $T_k(s)$ is defined as $G_k(s) = s - T_k(s)$.

Note that we deal with functions u that may not belong to Sobolev spaces. Thus, we need to give a suitable definition of gradient. Consider a measurable function $u : \Omega \to \mathbb{R}$ which is finite almost everywhere and satisfies $T_k(u) \in W_0^{1,p}(\Omega)$ for every $k > 0$. According to [4, Lemma 2.1], there exists an unique measurable function $v : \Omega \to \overline{\mathbb{R}}^N$ such that, for each $k > 0$,

$$\nabla T_k(u) = v\chi_{|u|\le k} \quad \text{almost everywhere in } \Omega.$$

where $\chi_{|u|\le k}$ is the characteristic function of $\{|u| \le k\}$. We define the gradient ∇u of u as this function v, and denote $\nabla u = v$.

Remark 2.1. The gradient defined in this way is not, in general, the gradient used in the definition of Sobolev spaces. However, v is the distributional gradient of u provided v belongs to $(L^1_{\rm loc}(\Omega))^N$, which also implies that u belongs to $W^{1,1}_{\rm loc}(\Omega)$.

Remark 2.2. As point out in [4], the set of function u such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k>0$ is not a linear space. That is, if u and v are such that both $T_k(u)$ and $T_k(v)$ belong to $W_0^{1,p}(\Omega)$ for every $k>0$, while $\nabla(u+v)$ may not be well defined.

Denote by $|\Omega|$ the N-dimensional Lebesgue measure of a measurable set Ω. Let $f(x), g(x)$ are functions defined in $\mathbb{R}^N$ and a, b are constants, we set

$$\{f(x)>a\} := \{x\in\mathbb{R}^N : f(x)>a\}, \quad \{g(x)\le b\} := \{x\in\mathbb{R}^N : g(x)\le b\},$$
$$\{f(x)>a, g(x)\le b\} := \{f(x)>a\}\cap\{g(x)\le b\}.$$

2.2. **Measures.** Let Ω be a bounded open subset of $\mathbb{R}^N$, $N\ge 2$. We denote by $\mathfrak{M}_b(\Omega)$ the set of all Radon measures with bounded variation on Ω. Denote by $\mathfrak{M}_0(\Omega)$ the set of all measures in $\mathfrak{M}_b(\Omega)$ which are absolutely continuous with respect to the capacity $\operatorname{cap}_{1,p}(\cdot,\Omega)$, i.e., which satisfy $\mu(K)=0$ for every Borel set $K\subset\Omega$ such that $\operatorname{cap}_{1,p}(K,\Omega)=0$. Denote $\mathfrak{M}_s(\Omega)$ as the set of all measures μ in $\mathfrak{M}_b(\Omega)$ which are singular with respect to the $\operatorname{cap}_{1,p}(\cdot,\Omega)$, i.e., the measures for which there exists a Borel set $E\subset\Omega$, with $\operatorname{cap}_{1,p}(K,\Omega)=0$, such that $\mu\lfloor\mu_E$, where $\mu\lfloor\mu_E$ is defined by $\mu\lfloor\mu_E(B)=\mu(E\cap B)$ for any Borel set $B\subseteq\Omega$. Note that if a measure μ in $\mathfrak{M}_b(\Omega)$ is such that $\mu=\mu\lfloor\mu_E$ for a certain Borel set E, the measure μ is said to be concentrated on E, where $\operatorname{cap}_{1,p}$ is defined by

$$\operatorname{cap}_{1,p}(K,\Omega)=\inf\Big\{\int_\Omega|\nabla\phi|^p dx : \phi\in C_0^\infty(\Omega), \phi\ge\chi_E\Big\},$$

for any compact set $E\subset\Omega$.

Proposition 2.3. *For every measure μ in $\mathfrak{M}_b(\Omega)$ there exists a unique pair of measures (μ_0,μ_s), with μ_0 in $\mathfrak{M}_0(\Omega)$ and μ_s in $\mathfrak{M}_s(\Omega)$, such that*

$$\mu=\mu_0+\mu_s.$$

The following decomposition results were given in [14, Theorem 2.1].

Proposition 2.4. *Let μ be a measure in $\mathfrak{M}_b(\Omega)$. Then μ belongs to $\mathfrak{M}_0(\Omega)$ if and only if it belongs to $L^1(\Omega)+W_0^{-1,p'}(\Omega)$. Thus if μ belongs to $\mathfrak{M}_0(\Omega)$, there exists f in $L^1(\Omega)$ and g in $(L^{p'}(\Omega))^N$ such that*

$$\mu=f-\operatorname{div}(g),$$

in the sense of distributions. Moreover every function $v\in W_0^{1,p}(\Omega)$ is measurable with respect to μ and belongs to $L^\infty(\Omega,\mu)$ if v further belongs to $L^\infty(\Omega)$, and one has

$$\int_\Omega v d\mu=\int_\Omega fv+\int_\Omega g\nabla v, \quad \forall v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega).$$

Note that decomposition (2.2) is not unique since $L^1(\Omega)\cap W^{-1,p'}(\Omega)\ne\{0\}$.

Definition 2.5. We say that a sequence of measures $\{\mu_k\}$ in $\mathfrak{M}_b(\Omega)$ converges in the narrow topology to $\mu\in\mathfrak{M}_b(\Omega)$ if

$$\lim_{k\to\infty}\int_\Omega\varphi d\mu_k=\int_\Omega\varphi d\mu,$$

for every bounded and continuous function φ on Ω.

The following technical proposition will be used to prove the stability result.

Proposition 2.6. *Let Ω be a bounded open subset of $\mathbb{R}^N$. Assume that ρ_ε is a sequence of $L^1(\Omega)$ functions converging to ρ weakly in $L^1(\Omega)$ and assume that σ_ε is a sequence of $L^\infty(\Omega)$ functions which is bounded is $L^\infty(\Omega)$ and converges to σ almost everywhere in Ω. Then*

$$\lim_{\varepsilon\to 0}\int_\Omega \rho_\varepsilon\sigma_\varepsilon = \int_\Omega \rho\sigma.$$

2.3. **Lorentz spaces.** For $0 < p < \infty$ and $0 < q \leq \infty$, the Lorentz space $L^{p,q}(\Omega)$ is the set of measurable functions f on Ω such that

$$\|f\|_{L^{p,q}(\Omega)} := \Big[p\int_0^\infty (\alpha^p|\{x\in\Omega : |f(x)| > \alpha\}|)^{q/p}\frac{d\alpha}{\alpha}\Big]^{1/q} < \infty,$$

for $q \neq \infty$. For $q = \infty$ the space $L^{p,\infty}(\Omega)$ is set to be the usual weak L^p or Marcinkiewicz space with quasinorm

$$\|f\|_{L^{p,\infty}(\Omega)} := \sup_{\alpha>0}\alpha|\{x\in\Omega : |f(x)| > \alpha\}|^{1/p}.$$

It is easy to see that when $p = q$, the Lorentz space $L^{p,p}(\Omega)$ is nothing but the Lebesgue space $L^p(\Omega)$.

We recall here only few properties of the Lorentz spaces which will be used later. As regards the other values of the second index q, the Lorentz spaces are intermediate spaces between the Lebesgue spaces in the sense that the following inclusions hold

$$L^{p,1}(\Omega) \subset L^{p,p}(\Omega) = L^p(\Omega) \subset L^{p,\infty}(\Omega) \subset L^{p_1,1}(\Omega),$$

when Ω is bounded, $1 < p_1 < p < \infty$.

A generalized version of the Hölder inequality and the Sobolev embedding hold true in the Lorentz spaces $L^{p,q}(\Omega)$. More precisely,

$$\|fg\|_{L^{p,q}(\Omega)} \leq \|f\|_{L^{p_1,q_1}(\Omega)}\|g\|_{L^{p_2,q_2}(\Omega)},$$

holds for each $f \in L^{p_1,q_1}(\Omega)$ and $g \in L^{p_2,q_2}(\Omega)$, where

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Especially,

$$\int_\Omega |fg| \leq \|f\|_{L^{p,\infty}(\Omega)}\|g\|_{L^{p',1}(\Omega)}, \tag{2.2}$$

holds for $f \in L^{p,\infty}(\Omega)$ and $g \in L^{p',1}(\Omega)$.

Sobolev's embedding

$$\|f\|_{L^{p^*,p}(\Omega)} \leq S_{N,p}\|f\|_{W^{1,p}(\Omega)}, \tag{2.3}$$

holds for

$$p^* = \frac{Np}{N-p}, \quad S_{N,p} = \frac{(\Gamma(1+\frac{N}{2}))^{\frac{1}{N}}}{\sqrt{\pi}}\frac{p}{N-p}. \tag{2.4}$$

2.4. **Technical results.** The following technical lemma plays an important role in the proof of our main theorem, which is a generalization of corresponding results in [6, 23, 4, 31].

Proposition 2.7. *Assume that Ω is an open subset of $\mathbb{R}^N$ with finite measure and that $1 < p < N$. Let ϑ be a positive, not necessarily finite, measure on Ω and u be a measurable function satisfying $T_k(u_n) \in W_0^{1,p}(\vartheta, \Omega)$, for every positive k and $0 < \sigma < p$, such that*

$$\int_\Omega |\nabla T_k(u)|^p d\vartheta < Mk^\sigma + L, \quad \forall k > 0, \tag{2.5}$$

where M and L are given constants. Then $|u|^{p-\sigma}$ belongs to $L_\vartheta^{\frac{p^*}{p},\infty}(\Omega)$, $|\nabla u|^{p-\sigma}$ belongs to $L_\vartheta^{\frac{N}{N-\sigma},\infty}(\Omega)$ and

$$\| |u|^{p-\sigma} \|_{L_\vartheta^{\frac{p^*}{p},\infty}(\Omega)} \le C(N,p)\big[M + \vartheta(\Omega)^{\frac{\sigma}{p^*}} L^{1-\frac{\sigma}{p}}\big], \tag{2.6}$$

$$\| |\nabla u|^{p-\sigma} \|_{L_\vartheta^{\frac{N}{N-\sigma},\infty}(\Omega)} \le C(N,p)\big[M + \vartheta(\Omega)^{\sigma(\frac{1}{p}-\frac{1}{N})} L^{\frac{p-\sigma}{p}}\big], \tag{2.7}$$

where $C(N,p)$ is a constant depending only on N and p. Here and elsewhere in this paper, $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent.

Proof. The proof of this proposition is essentially the same as the proof of [6, Lemma A.1], in which $\sigma = 1$, but, for the sake of completeness, we sketch it here. Thanks to (2.5), we can find C_1, depending on Ω, p, N, such that

$$\begin{aligned} k^{p^*}\vartheta(|\{x \in \Omega : |u| > k\}|) &\le \int_\Omega |T_k(u)|^{p^*} d\vartheta \\ &\le C_1 \|\nabla T_k(u)\|^{p^*}_{L^p_\vartheta(\Omega)} \\ &\le C_1 (Mk^\sigma + L)^{p^*/p}, \end{aligned} \tag{2.8}$$

which gives

$$k^{\frac{p^*}{p-\sigma}} \vartheta(\{x \in \Omega : |u|^{p-\sigma} > k\}) \le C_2 \Big(Mk^{\frac{\sigma}{p-\sigma}} + L\Big)^{p^*/p},$$

equivalently

$$k\vartheta(\{x \in \Omega : |u|^{p-\sigma} > k\})^{p/p^*} \le C_3 \left(M + Lk^{-\frac{\sigma}{p}}\right).$$

Thus

$$\begin{aligned} &\| |u|^{p-\sigma} \|_{L_\vartheta^{\frac{p^*}{p},\infty}(\Omega)} \\ &= \sup_{k>0} k\vartheta(\{x \in \Omega : |u|^{p-\sigma} > k\})^{p/p^*} \\ &= \sup_{0<k\le k_0} k\vartheta(\{x \in \Omega : |u|^{p-\sigma} > k\})^{p/p^*} + \sup_{k>k_0} k\vartheta(\{x \in \Omega : |u|^{p-\sigma} > k\})^{p/p^*} \\ &\le k_0 \vartheta(\Omega)^{p/p^*} + C_3\big(M + Lk^{-\frac{\sigma}{p-\sigma}}\big). \end{aligned} \tag{2.9}$$

Choose $k_0\vartheta(\Omega)^{p/p^*} = Lk_0^{-\frac{\sigma}{p-\sigma}}$, that is

$$k_0 = \Big(\frac{L}{\vartheta(\Omega)^{p/p^*}}\Big)^{\frac{p-\sigma}{p}}. \tag{2.10}$$

Substituting (2.10) into (2.9), leads to (2.6).

Obviously, for every $k > 0$,

$$\begin{aligned} &\vartheta\left(\{x\in\Omega : |\nabla u|^{p-\sigma} > h\}\right) \\ &\le \vartheta\left(\{x\in\Omega : |\nabla u|^{p-\sigma} > h, |u| < k\}\right) + \vartheta\left(\{x\in\Omega : |\nabla u|^{p-\sigma} > h, |u| \ge k\}\right). \end{aligned} \tag{2.11}$$

For the first term of the right-hand side of (2.11), it can be easily seen that

$$h^p\vartheta\left(\{x\in\Omega : |\nabla u| > h, |u| < k\}\right) \le \int_{|u|<k} |\nabla u|^p d\vartheta = \int_\Omega |\nabla T_k u|^p d\vartheta \le Mk^\sigma + L,$$

which implies

$$\vartheta\left(\{x\in\Omega : |\nabla u|^{p-\sigma} > h, |u| < k\}\right) \le \frac{Mk^\sigma + L}{h^{\frac{p}{p-\sigma}}}. \tag{2.12}$$

Thus, taking into account (2.8), (2.11) and (2.12), we obtain

$$\vartheta\left(\{x\in\Omega : |\nabla u|^{p-\sigma} > h\}\right) \le \frac{Mk^\sigma + L}{h^{\frac{p}{p-\sigma}}} + C_1\frac{(Mk^\sigma + L)^{p^*/p}}{k^{p^*}}. \tag{2.13}$$

Decompose k as $k = k_1 + k_2$, where k_1 and k_2 are positive constants, will be made explicit later. Therefore, (2.13) implies that

$$\begin{aligned} &\vartheta\left(\{x\in\Omega : |\nabla u|^{p-\sigma} > h\}\right) \\ &\le \frac{Mk^\sigma + L}{h^{\frac{p}{p-\sigma}}} + C_1\frac{(Mk^\sigma + L)^{p^*/p}}{k^{p^*}} \\ &= \frac{M(k_1+k_2)^\sigma + L}{h^{\frac{p}{p-\sigma}}} + C_1\frac{\left(M(k_1+k_2)^\sigma + L\right)^{p^*/p}}{(k_1+k_2)^{p^*}} \\ &\le \frac{2^\sigma Mk_1^\sigma}{h^{\frac{p}{p-\sigma}}} + \frac{2^\sigma Mk_2^\sigma}{h^{\frac{p}{p-\sigma}}} + \frac{L}{h^{\frac{p}{p-\sigma}}} + C_1 2^{p^*/p}M^{p^*/p}(k_1+k_2)^{\frac{\sigma p^*}{p}-p^*} \\ &\quad + C_1 2^{p^*/p}L^{p^*/p}(k_1+k_2)^{-p^*} \\ &\le \frac{2^\sigma Mk_1^\sigma}{h^{\frac{p}{p-\sigma}}} + \frac{2^\sigma Mk_2^\sigma}{h^{\frac{p}{p-\sigma}}} + \frac{L}{h^{\frac{p}{p-\sigma}}} + C_1 2^{p^*/p}M^{p^*/p}k_1^{\frac{\sigma p^*}{p}-p^*} + C_1 2^{p^*/p}L^{p^*/p}k_2^{-p^*} \\ &\le C_4\Big(\frac{Mk_1^\sigma}{h^{\frac{p}{p-\sigma}}} + \frac{Mk_2^\sigma}{h^{\frac{p}{p-\sigma}}} + \frac{L}{h^{\frac{p}{p-\sigma}}} + M^{p^*/p}k_1^{\frac{\sigma p^*}{p}-p^*} + L^{p^*/p}k_2^{-p^*}\Big), \end{aligned}$$

here we use the fact that $\sigma < p$. Choose

$$k_1 = M^{\frac{1}{N-\sigma}}h^{\frac{N-p}{(p-\sigma)(N-\sigma)}}, \quad k_2 = \Big(\frac{L^{p^*/p}h^{\frac{p}{p-\sigma}}}{M}\Big)^{\frac{1}{p^*+\sigma}}.$$

Therefore,

$$\vartheta\left(\{x\in\Omega : |\nabla u|^{p-\sigma} > h\}\right) \le C\Big(\frac{M^{\frac{N}{N-\sigma}}}{h^{\frac{N}{N-\sigma}}} + \Big(\frac{ML^{\frac{\sigma}{p}}}{h^{\frac{p}{p-\sigma}}}\Big)^{\frac{p^*}{p^*+\sigma}} + \frac{L}{h^{\frac{p}{p-\sigma}}}\Big),$$

equivalently,

$$\begin{aligned} &h\left(\text{meas }\{x\in\Omega : |\nabla u|^{p-\sigma} > h\}\right)^{\frac{N-\sigma}{N}} \\ &\le C_5\Big(M + M^{\frac{p^*(N-\sigma)}{N(p^*+\sigma)}}\Big(\frac{L^{\frac{N-\sigma}{N}}}{h^{\frac{\sigma(N-p)}{N(p-\sigma)}}}\Big)^{\frac{\sigma p^*}{p(p^*+\sigma)}} + \frac{L^{\frac{N-\sigma}{N}}}{h^{\frac{\sigma(N-p)}{N(p-\sigma)}}}\Big). \end{aligned} \tag{2.14}$$

Young's inequality with
$$\Big(\frac{N(p^*+\sigma)}{p^*(N-\sigma)},\frac{p(p^*+\sigma)}{\sigma p^*}\Big),$$
(2.14) implies that
$$\begin{aligned}
&h\vartheta\left(\Big\{x\in\Omega:|\nabla u|^{(p-\sigma)}>h\Big\}\right)^{\frac{N-\sigma}{N}}\\
&\le C_5\Big(M+\frac{p^*(N-\sigma)}{N(p^*+\sigma)}M+\frac{\sigma p^*}{p(p^*+\sigma)}\frac{L^{\frac{N-\sigma}{N}}}{h^{\frac{\sigma(N-p)}{N(p-\sigma)}}}+\frac{L^{\frac{N-\sigma}{N}}}{h^{\frac{\sigma(N-p)}{N(p-\sigma)}}}\Big)\\
&\le C_6\Big(M+\frac{L^{\frac{N-\sigma}{N}}}{h^{\frac{\sigma(N-p)}{N(p-\sigma)}}}\Big).
\end{aligned}$$
Therefore,
$$\begin{aligned}
&\sup_{h>0}h\vartheta\left(\big\{x\in\Omega:|\nabla u|^{p-\sigma}>h\big\}\right)^{\frac{N-\sigma}{N}}\\
&\le\sup_{0<h\le h_0}h\vartheta\left(\big\{x\in\Omega:|\nabla u|^{p-\sigma}>h\big\}\right)^{\frac{N-\sigma}{N}}\\
&\quad+\sup_{h>h_0}h\vartheta\left(\big\{x\in\Omega:|\nabla u|^{p-\sigma}>h\big\}\right)^{\frac{N-\sigma}{N}}\\
&\le h_0|\Omega|^{\frac{N-\sigma}{N}}+C\Big(M+\frac{L^{\frac{N-\sigma}{N}}}{h^{\frac{\sigma(N-p)}{N(p-\sigma)}}}\Big)\\
&\le C_7\Big(M+h_0|\Omega|^{\frac{N-\sigma}{N}}+\frac{L^{\frac{N-\sigma}{N}}}{h^{\frac{\sigma(N-p)}{N(p-\sigma)}}}\Big).
\end{aligned}\tag{2.15}$$
Now choose
$$h_0=\big(\frac{L}{|\Omega|}\big)^{\frac{p-\sigma}{p}}.\tag{2.16}$$
Thus, taking into account (2.15) and (2.16), we obtain
$$\begin{aligned}
&\sup_{h>0}h\vartheta\left(\big\{x\in\Omega:|\nabla u|^{p-\sigma}>h\big\}\right)^{\frac{N-\sigma}{N}}\\
&\le C_8\Big(M+\vartheta(\Omega)^{\sigma\left(\frac{1}{p}-\frac{1}{N}\right)}L^{\frac{p-\sigma}{p}}\Big).
\end{aligned}$$
This completes the proof of (2.9). □

To the proof of almost everywhere convergence of the gradients of the approximate solutions u_n, the following technical result will allow us to pass to the limit in the approximate equations.

Proposition 2.8. *Let u_n be a sequence of solutions of the problem*
$$\begin{gathered}
-\operatorname{div}\Big(\frac{a(x,u_n,\nabla u_n)}{(1+|u_n|)^{\theta(p-1)}}\Big)=f_n-\operatorname{div}(g),\quad x\in\Omega,\\
u_n=0,\quad x\in\partial\Omega,
\end{gathered}$$
with f_n strongly convergent to some f in $L^1(\Omega)$, $g\in\big(L^{p'}(\Omega)\big)^N$,

(1) *u_n is such that $T_k(u_n)$ belongs to $W_0^{1,p}(\Omega)$ for every $k>0$;*

(2) *u_n converges almost everywhere in Ω to some measurable function u which is finite almost everywhere, and such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$;*
(3) *u_n is bounded in $L^{r_1,\infty}(\Omega)$ for some $r_1 > 0$, and u belongs to the same $L^{r_1,\infty}(\Omega)$;*
(4) *there exists $\tau > 0$ such that $|\nabla u_n|^\tau$ is bounded in $L^{r_2}(\Omega)$, for some $r_2 > 1$, and $|\nabla u|^\tau$ belongs to the same $L^{r_2}(\Omega)$.*

Then, up to a subsequence, ∇u_n converges almost everywhere in Ω to ∇u, the weak gradient of u. Furthermore, u satisfies

$$\begin{gathered}-\operatorname{div}\Big(\frac{a(x,u,\nabla u)}{(1+|u|)^{\theta(p-1)}}\Big) = f - \operatorname{div}(g), \quad x \in \Omega,\\ u = 0, \quad x \in \partial\Omega.\end{gathered}$$

The proof of this proposition is essentially the same as the proof of [1, Theorem 4.1].

3. Approximate problem

In this section, we consider a priori estimate of $|\nabla u|^{p-1}$ in $L^{N',\infty}(\Omega)$. For the convenience of the reader, we consider the following approximate problem firstly,

$$\begin{gathered}-\operatorname{div}\Big(\frac{a(x,u_n,\nabla u_n)+K_n(x,u_n)}{(1+|u_n|)^{\theta(p-1)}}\Big) = \mu_n, \quad x \in \Omega,\\ u_n = 0, \quad x \in \partial\Omega,\end{gathered}\tag{3.1}$$

where

$$K_n(x,s) = K(x,T_n(s)),\tag{3.2}$$

$$G_n(x,s) = T_n(G(x,s)),\tag{3.3}$$

and $\mu_n = f_n - \operatorname{div}(g) \in W^{-1,p'}(\Omega)$ is a sequence such that

$$\mu_n \overset{*}{\to} \mu, \quad]\text{in } \mathfrak{M}_0(\Omega), \quad \|\mu_n\|_{L^1(\Omega)} \le \mu(\Omega).\tag{3.4}$$

f_n is a sequence of $L^{p'}(\Omega)$ functions that converges to f weakly in $L^1(\Omega)$.

Obviously, by (1.8), $K_n(x,s)$ satisfies

$$\begin{gathered}|K_n(x,s)| \le |K(x,s)| \le c_0(x)|s|^\gamma + c_1(x),\\ |K_n(x,s)| \le c_0(x)n^\gamma + c_1(x),\ \gamma = p-1.\end{gathered}\tag{3.5}$$

Thus, taking into account hypotheses (3.2)–(3.5), $\mu_n \in W_0^{-1,p'}(\Omega) \cap L^\infty(\Omega)$, there exists at least one solution $u_n \in W_0^{1,p}(\Omega)$ to problem (3.1) in the sense, for each $\varphi \in W^{1,p}(\Omega)$,

$$\int_\Omega \Big(\frac{a(x,u_n,\nabla u_n)+K_n(x,u_n)}{(1+|u_n|)^{\theta(p-1)}}\Big)\cdot\nabla\varphi = \int_\Omega \mu_n\varphi.\tag{3.6}$$

3.1. **Logarithmic estimate.** The following estimates will be useful in the proof of Theorem 3.2. Similar estimate also appears in [9, 10, 11].

Theorem 3.1. *Let $u_n \in W_0^{1,p}(\Omega)$ be a weak solution to* (3.1) *in the sense of* (3.6)*. Then there exists C, such that*

$$\|\ln\big(A^{\kappa'/\kappa} + |u_n|\big)\|_{W_0^{1,p}(\Omega)} \le C,\tag{3.7}$$

where $A \ge 1$ will be made explicit in (3.14) *below,* $\kappa = 1 + (p-1)(1-\theta)$ *and* $1/\kappa' + 1/\kappa = 1$.

Proof. Define

$$\varphi(s) = \int_0^s \frac{1}{\big(A^{\kappa'/\kappa} + |t|\big)^\kappa} dt. \tag{3.8}$$

It can be easily seen that

$$|\varphi(s)| \le \frac{1}{(p-1)(1-\theta)A}.$$

Note that $\varphi(s)$ is a Lipschitz function such that $\varphi(0) = 0$, this fact, combined with $u_n \in W^{1,p}(\Omega)$, shows that $\varphi(u_n) \in W^{1,p}(\Omega)$. Therefore, choose $\varphi(u_n)$ as a test function in (3.6), we find that

$$\begin{aligned} &\int_\Omega \frac{a(x,u_n,\nabla u_n)}{(1+|u_n|)^{\theta(p-1)}} \cdot \nabla u_n \varphi'(u_n) + \int_\Omega \frac{K_n(x,u_n)}{(1+|u_n|)^{\theta(p-1)}} \cdot \nabla u_n \varphi'(u_n) \\ &= \int_\Omega \mu_n \varphi(u_n). \end{aligned} \tag{3.9}$$

Using (1.7), we have

$$\begin{aligned} &\int_\Omega \frac{a(x,u_n,\nabla u_n)}{(1+|u_n|)^{\theta(p-1)}} \cdot \nabla u_n \varphi'(u_n) \\ &\ge \alpha_1 \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta(p-1)}} \frac{1}{\big(A^{\kappa'/\kappa} + |u_n|\big)^\kappa} \\ &\ge \alpha_1 \int_\Omega \big|\nabla \ln\big(A^{\kappa'/\kappa} + |u_n|\big)\big|^p \\ &= \alpha_1 \|\nabla \ln\big(A^{\kappa'/\kappa} + |u_n|\big)\|^p_{L^p(\Omega)}, \end{aligned} \tag{3.10}$$

where α_1 appears in (1.7). Taking into account (3.5), we obtain

$$\begin{aligned} &\int_\Omega \frac{K_n(x,u_n)}{(1+|u_n|)^{\theta(p-1)}} \cdot \nabla u_n \varphi'(u_n) \\ &\le \int_\Omega \frac{c_0(x)|u_n|^{\theta(p-1)}}{(1+|u_n|)^{\theta(p-1)}} \frac{|u_n|^{(1-\theta)(p-1)}}{(A^{\kappa'/\kappa} + |u_n|)^{(1-\theta)(p-1)}} \frac{\nabla u_n}{A^{\kappa'/\kappa} + |u_n|} + \int_\Omega \frac{c_1(x)\nabla u_n}{\big(A^{\kappa'/\kappa} + |u_n|\big)^{\kappa'}} \\ &\le \int_\Omega \frac{c_0(x)\nabla u_n}{A^{\kappa'/\kappa} + |u_n|} + \int_\Omega \frac{c_1(x)\nabla u_n}{A^{\kappa'/\kappa} + |u_n|} \\ &\le \|c_0(x)\|_{L^{p'}(\Omega)} \|\nabla \ln\big(A^{\kappa'/\kappa} + |u_n|\big)\|_{L^p(\Omega)} \\ &\quad + \|c_1(x)\|_{L^{p'}(\Omega)} \|\nabla \ln\big(A^{\kappa'/\kappa} + |u_n|\big)\|_{L^p(\Omega)} \\ &\le \frac{3^{p'/p}}{p'\alpha_1^{p'/p}} \|c_0(x)\|^{p'}_{L^{p'}(\Omega)} + \frac{\alpha_1}{3p} \|\nabla \ln\big(A^{\kappa'/\kappa} + |u_n|\big)\|^p_{L^p(\Omega)} \\ &\quad + \frac{3^{p'/p}}{p'\alpha_1^{p'/p}} \|c_1(x)\|^{p'}_{L^{p'}(\Omega)} + \frac{\alpha_1}{3p} \|\nabla \ln\big(A^{\kappa'/\kappa} + |u_n|\big)\|^p_{L^p(\Omega)} \\ &= \frac{3^{p'/p}}{p'\alpha_1^{p'/p}} \|c_0(x)\|^{p'}_{L^{p'}(\Omega)} + \frac{3^{p'/p}}{p'\alpha_1^{p'/p}} \|c_1(x)\|^{p'}_{L^{p'}(\Omega)} + \frac{2\alpha_1}{3p} \|\nabla \ln\big(A^{\kappa'/\kappa} + |u_n|\big)\|^p_{L^p(\Omega)}. \end{aligned}$$

It is well-known that
$$\|c_0(x)\|_{L^{p'}(\Omega)} \le \frac{Np}{(p-1)(N-p)t}|\Omega|^{\frac{(p-1)(N-p)t}{Np}}\|c_0(x)\|_{L^{\frac{N}{p-1},r}(\Omega)},$$
where $r \ge \frac{N}{p-1}$ appears in (1.8) and t satisfies
$$\frac{1}{p'} = \frac{1}{t} + \frac{1}{r}.$$
Consequently,
$$\begin{aligned}
&\int_\Omega \frac{K_n(x,u_n)}{(1+|u_n|)^{\theta(p-1)}} \cdot \nabla u_n \varphi'(u_n) \\
&\le \frac{3^{p'/p}}{p'\alpha_1^{p'/p}}\Big(\frac{Np}{(p-1)(N-p)t}\Big)^{p'}|\Omega|^{\frac{(n-p)t}{N}}\|c_0(x)\|^{p'}_{L^{\frac{N}{p-1},r}(\Omega)} \\
&\quad + \frac{3^{p'/p}}{p'\alpha_1^{p'/p}}\|c_1(x)\|^{p'}_{L^{p'}(\Omega)} + \frac{2\alpha_1}{3p}\|\nabla \ln\big(A^{\kappa'/\kappa}+|u_n|\big)\|^p_{L^p(\Omega)}.
\end{aligned} \tag{3.11}$$
Now we consider the right-hand side of (3.9). Obviously, (3.4) leads to
$$\int_\Omega \mu_n\varphi(u_n) \le \frac{1}{(p-1)(1-\theta)A}\|\mu_n\|_{L^1(\Omega)} \le \frac{1}{(p-1)(1-\theta)A}\mu(\Omega), \tag{3.12}$$
Combining with (3.10)-(3.12), we have
$$\begin{aligned}
&\|\nabla \ln\big(A^{\kappa'/\kappa}+|u_n|\big)\|^p_{L^p(\Omega)} \\
&\le \frac{3^{p'/p}}{\alpha_1^{p'/p}}\Big(\frac{Np}{(p-1)(N-p)t}\Big)^{p'}|\Omega|^{\frac{(n-p)t}{N}}\|c_0(x)\|^{p'}_{L^{\frac{N}{p-1},r}(\Omega)} \\
&\quad + \frac{3^{p'/p}}{\alpha_1^{p'/p+1}}\|c_1(x)\|^{p'}_{L^{p'}(\Omega)} + \frac{p'\mu(\Omega)}{\alpha_1(p-1)(1-\theta)A},
\end{aligned} \tag{3.13}$$
Define
$$A = 1 + \frac{p'\mu(\Omega)}{\alpha_1(p-1)(1-\theta)}. \tag{3.14}$$
This fact, and (3.13) show that
$$\begin{aligned}
&\|\nabla \ln\big(A^{\kappa'/\kappa}+|u_n|\big)\|^p_{L^p(\Omega)} \\
&\le \frac{3^{p'/p}}{\alpha_1^{p'/p}}\Big(\frac{Np}{(p-1)(N-p)t}\Big)^{p'}|\Omega|^{\frac{(n-p)t}{N}}\|c_0(x)\|^{p'}_{L^{\frac{N}{p-1},r}(\Omega)} \\
&\quad + \frac{3^{p'/p}}{\alpha_1^{p'/p+1}}\|c_1(x)\|^{p'}_{L^{p'}(\Omega)} + 1,
\end{aligned} \tag{3.15}$$
which leads to (3.7). $\square$

Secondly, we consider the approximate problem
$$\begin{gathered}
-\operatorname{div}\Big(\frac{a(x,u_n,\nabla u_n)}{(1+|u_n|)^{\theta(p-1)}}\Big) + H_n(x,u_n,\nabla u_n) = \mu_n, \quad x\in\Omega, \\
u_n = 0, \quad x \in \partial\Omega,
\end{gathered} \tag{3.16}$$
where
$$H_n(x,s,\xi) = T_n(H(x,s,\xi)), \tag{3.17}$$

and $\mu_n = f_n - \operatorname{div}(g) \in W^{-1,p'}(\Omega)$ satisfies (3.4). By (1.9),

$$
\begin{gathered}
|H_n(x,s,\xi)| \le |H(x,s,\xi)| \le \frac{b_0(x)|\xi|^\lambda}{(1+|s|)^{\theta(p-1)}} + \frac{b_1(x)}{(1+|s|)^{\theta(p-1)}}, \\
|H_n(x,s,\xi)| \le n.
\end{gathered} \tag{3.18}
$$

For problem (3.16), through a similar argument as above, we can show that (3.7) holds. In this case, we also choose $\varphi(u_n)$ as a test function in (3.16) and use the following estimates:

$$
\begin{aligned}
&\int_\Omega H_n(x,u_n,\nabla u_n)\varphi(u_n) \\
&\le \int_\Omega \frac{b_0(x)|\nabla u_n|^{p-1}}{(1+|u|)^{\theta(p-1)}}\varphi(u_n) + \int_\Omega \frac{b_1(x)}{(1+|u|)^{\theta(p-1)}}\varphi(u_n) \\
&\le \frac{1}{(p-1)(1-\theta)A}\int_\Omega b_0(x)\frac{|\nabla u_n|^{p-1}}{(1+|u|)^{\theta(p-1)}} + \frac{1}{(p-1)(1-\theta)A}\int_\Omega b_1(x) \\
&\le \frac{1}{(p-1)(1-\theta)A}\Big\{\|b_0(x)\|_{L^{N,1}(\Omega)}\|\frac{|\nabla u_n|^{p-1}}{(1+|u|)^{\theta(p-1)}}\|_{L^{N',\infty}(\Omega)} + \|b_1(x)\|_{L^1(\Omega)}\Big\},
\end{aligned}
$$

here we use (2.2). Thus

$$
\|\nabla \ln\big(A^{\kappa'/\kappa} + |u_n|\big)\|^p_{L^p(\Omega)} \le \frac{p'M_1}{\alpha_1(p-1)(1-\theta)A}, \tag{3.19}
$$

where

$$
M_1 = \|b_0(x)\|_{L^{N,1}(\Omega)}\|\frac{|\nabla u_n|^{p-1}}{(1+|u|)^{\theta(p-1)}}\|_{L^{N',\infty}(\Omega)} + \|b_1\|_{L^1(\Omega)} + \mu(\Omega).
$$

We define

$$
A = 1 + \frac{p'M_1}{\alpha_1(p-1)(1-\theta)}.
$$

This and (3.19) show that

$$
\|\nabla \ln\big(A^{\kappa'/\kappa} + |u_n|\big)\|^p_{L^p(\Omega)} \le 1,
$$

which leads to (3.7) again.

Similarly, we can show that (3.7) holds for the more general equation

$$
\begin{gathered}
\begin{aligned}
&-\operatorname{div}\Big(\frac{a(x,u_n,\nabla u_n) + K_n(x,u_n)}{(1+|u_n|)^{\theta(p-1)}}\Big) + H_n(x,u_n,\nabla u_n) + G_n(x,u_n) \\
&= \mu_n, \quad x \in \Omega,
\end{aligned} \\
u_n = 0, \quad x \in \partial\Omega,
\end{gathered} \tag{3.20}
$$

where $K_n(x,s)$ satisfies (3.2), $H_n(x,u_n,\nabla u_n)$ satisfies (3.18), and $G_n(x,u_n) = T_n(G(x,s))$ satisfies

$$
G_n(x,s)s \ge 0, \quad |G_n(x,s)| \le |G(x,s)| \le d_0(x)|s|^t + d_1(x), \quad |G_n(x,s)| \le n.
$$

The details of this case be omitted.

3.2. **Estimate of $|\nabla u_n|^{p-1}$ and $|u_n|^{p-1}$.** The main result of this subsection is

Theorem 3.2. *Let $u_n \in W^{1,p}(\Omega)$ be a weak solution to* (3.1) *in the sense of* (3.6). *Let $\alpha > 1$, then there exists $C > 0$, such that*

$$\| |\nabla u_n|^{p-1} \|_{L^{N',\infty}(\Omega)} \le C, \tag{3.21}$$

$$\| |u_n|^{(p-1)(1-\theta)} \|_{L^{\frac{N}{N-p},\infty}(\Omega)} \le C. \tag{3.22}$$

Proof. Here we will only give the proof when assumption (1) in Theorem 1.4 is satisfied. The details of the proofs under the assumption (2) and (3) exactly the same, and will be omitted. The proof goes in two steps.

Step one. To establish the bound of the second term of the left-hand side of (3.21), we define

$$\psi(s) = \int_0^s \frac{1}{(1+|t|)^\theta} dt. \tag{3.23}$$

Using $T_\varrho(\psi(u_n))\chi_{E_2(\varrho)}$ as a test function in (3.6), where ϱ is a positive constant which will be specified later, $\chi_{E_2(\varrho)}$ is the characteristic function of $E_2(\varrho)$, where $E_2(\varrho) = \{x \in \Omega : |u_n| \ge \varrho\}$. We find

$$\int_{E_2(\varrho)} \Big(\frac{a(x, u_n, \nabla u_n) + K_n(x, u_n)}{(1+|u_n|)^{\theta(p-1)}} \Big) \cdot \frac{\nabla u_n}{(1+|u_n|)^\theta} = \int_{E_2(\varrho)} \mu_n \psi(u_n). \tag{3.24}$$

Now we evaluate the various integrals in (3.24). From the ellipticity condition (1.7), we have

$$\int_{E_2(\varrho)} \frac{a(x, u_n, \nabla u_n)}{(1+|u_n|)^{\theta(p-1)}} \cdot \frac{\nabla u_n}{(1+|u_n|)^\theta} \ge \alpha_1 \int_{E_2(\varrho)} \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta p}}, \tag{3.25}$$

here we use that $\nabla u_n = \chi_{E_1(\varrho)} \nabla u_n$.

Taking into account the growth assumption (3.5) on K_n, Young inequality, and Hölder inequality, we derive that

$$\begin{aligned}
&\int_{E_2(\varrho)} \frac{K_n(x, u_n)}{(1+|u_n|)^{\theta(p-1)}} \cdot \frac{\nabla u_n}{(1+|u_n|)^\theta} \\
&\le \int_{E_2(\varrho)} \frac{c_0(x)|u_n|^{p-1}|\nabla u_n|}{(1+|u_n|)^{\theta p}} + \int_{E_2(\varrho)} \frac{c_1(x)|\nabla u_n|}{(1+|u_n|)^{\theta p}} \\
&= \int_{E_2(\varrho)} c_0(x) \frac{|u_n|^{p-1}}{(1+|u_n|)^{\theta(p-1)}} \frac{|\nabla u_n|}{(1+|u_n|)^\theta} + \int_{E_2(\varrho)} \frac{c_1(x)|\nabla u_n|}{(1+|u_n|)^{\theta p}}.
\end{aligned} \tag{3.26}$$

For the first term of the right hand side of (3.26), the Hölder inequality and generalized Sobolev inequality yield

$$\begin{aligned}
&\int_{E_2(\varrho)} c_0(x) \frac{|u_n|^{p-1}}{(1+|u_n|)^{\theta(p-1)}} \frac{\nabla u_n}{(1+|u_n|)^\theta} \\
&\le \|c_0(x)\|_{L^{\frac{N}{p-1},r}(E_2(\varrho))} \Big\| \frac{|u_n|}{(1+|u_n|)^\theta} \Big\|^{p-1}_{L^{p^*,t}(E_2(\varrho))} \Big\| \frac{|\nabla u_n|}{(1+|u_n|)^\theta} \Big\|_{L^p(E_2(\varrho))} \\
&\le C \|c_0(x)\|_{L^{\frac{N}{p-1},r}(E_2(\varrho))} \Big\| \Big|\nabla \Big(\frac{|u_n|}{(1+|u_n|)^\theta} \Big) \Big| \Big\|^{p-1}_{L^p(E_2(\varrho))} \Big\| \frac{|\nabla u_n|}{(1+|u_n|)^\theta} \Big\|_{L^p(E_2(\varrho))},
\end{aligned} \tag{3.27}$$

where t satisfies

$$\frac{1}{r} + \frac{p-1}{t} + \frac{1}{p} = 1.$$

From
$$|\Big(\frac{|s|}{(1+|s|)^\theta}\Big)'| \le \frac{1+\theta}{(1+|s|)^\theta},$$
we have
$$\big\||\nabla\Big(\frac{|u_n|}{(1+|u_n|)^\theta}\Big)|\big\|_{L^p(E_2(\varrho))} \le (1+\theta)\big\|\frac{|\nabla u_n|}{(1+|u_n|)^\theta}\big\|_{L^p(E_2(\varrho))}.$$
This fact, combining the Sobolev's inequality and (3.27), implies that
$$\begin{aligned} &\int_{E_2(\varrho)} c_0(x)\frac{|u_n|^{(p-1)}}{(1+|u_n|)^{\theta(p-1)^2}}\frac{\nabla u_n}{(1+|u_n|)^\theta} \\ &\le C\|c_0(x)\|_{L^{\frac{N}{p-1},r}(E_2(\varrho))}\big\|\frac{|\nabla u_n|}{(1+|u_n|)^\theta}\big\|^p_{L^p(E_2(\varrho))}. \end{aligned} \tag{3.28}$$
For the second term of the right hand side of (3.26), using Hölder's inequality with exponents p and p' and Young's inequality, we derive that
$$\begin{aligned} &\int_{E_2(\varrho)} \frac{c_1(x)\nabla u_n}{(1+|u_n|)^{\theta p}} \\ &\le \|c_1(x)\|_{L^{p'}(E_2(\varrho))}\big\|\frac{|\nabla u_n|}{(1+|u_n|)^\theta}\big\|_{L^p(E_2(\varrho))} \\ &\le \frac{1}{p'}\big(\frac{1}{\alpha_1}\big)^{p'/p}\|c_1(x)\|^{p'}_{L^{p'}(E_2(\varrho))} + \frac{\alpha_1}{p}\big\|\frac{|\nabla u_n|}{(1+|u_n|)^\theta}\big\|^p_{L^p(E_2(\varrho))}. \end{aligned} \tag{3.29}$$
Combining with (3.28)–(3.29), we conclude that
$$\begin{aligned} &\int_{E_2(\varrho)} \frac{K_n(x,u_n)}{(1+|u_n|)^{\theta(p-1)}}\cdot\frac{\nabla u_n}{(1+|u_n|)^\theta} \\ &\le C\|c_0(x)\|_{L^{\frac{N}{p-1},r}(E_2(\varrho))}\big\|\frac{|\nabla u_n|}{(1+|u_n|)^\theta}\big\|^p_{L^p(E_2(\varrho))} \\ &\quad + \frac{1}{p'}\big(\frac{1}{\alpha_1}\big)^{p'/p}\|c_1(x)\|^{p'}_{L^{p'}(E_2(\varrho))} + \frac{\alpha_1}{p}\big\|\frac{|\nabla u_n|}{(1+|u_n|)^\theta}\big\|^p_{L^p(E_2(\varrho)}. \end{aligned} \tag{3.30}$$
One easily sees that
$$\int_{E_2(\varrho)} \mu_n T_\varrho(\psi((u_n))) \le \varrho\|\mu_n\|_{L^1(\Omega)} \le \varrho\mu(\Omega), \tag{3.31}$$
Combining (3.25)–(3.31), we have
$$\begin{aligned} &\int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^{\theta p}} \\ &\le C\|c_0(x)\|_{L^{\frac{N}{p-1},r}(E_2(\varrho))}\big\|\frac{|\nabla u_n|}{(1+|u_n|)^\theta}\big\|^p_{L^p(E_2(\varrho)} + \varrho\mu(\Omega) + L_1, \end{aligned} \tag{3.32}$$
where
$$L_1 = \frac{1}{\alpha_1}\big(\frac{3}{\alpha_1}\big)^{p'/p}\|c_1(x)\|^{p'}_{L^{p'}(\Omega)}.$$

Step two. In this step, we consider the first integral term of the right hand of (3.32). To this end, thanks to (3.7), more princely (3.15), we have

$$\begin{aligned}\int_\Omega \big[\ln\Big(1+\frac{|u_n|}{A^{\kappa'/\kappa}}\Big)\big]^p &\le C\int_\Omega \big|\nabla \ln\Big(1+\frac{|u_n|}{A^{\kappa'/\kappa}}\Big)\big|^p\\ &= C\int_\Omega \big|\nabla \ln\Big(A^{\kappa'/\kappa}+|u_n|\Big)\big|^p \le M_1,\end{aligned}\tag{3.33}$$

where

$$\begin{aligned}M_1 =& C\frac{3^{p'/p}}{p'\alpha_1^{p'/p}}\Big(\frac{Np}{(p-1)(N-p)t}\Big)^{p'}|\Omega|^{\frac{(n-p)t}{N}}\|c_0(x)\|^{p'}_{L^{\frac{N}{p-1},r}(\Omega)}\\ &+ C\frac{3^{p'/p}}{\alpha_1^{p'/p+1}}\|c_1(x)\|^{p'}_{L^{p'}(\Omega)}+C.\end{aligned}$$

Consequently, for any $\eta>0$, we arrive at

$$\begin{aligned}\operatorname{meas}\{E_2(\eta A^{\kappa'/\kappa})\} &= \operatorname{meas}\{|u_n|\ge \eta A^{\kappa'/\kappa}\}\\ &= \frac{1}{[\ln(1+\eta)]^p}\int_{|u_n|\ge\eta A^{\kappa'/\kappa}}[\ln(1+\eta)]^p\\ &\le \frac{1}{[\ln(1+\eta)]^p}\int_{|u_n|\ge\eta A^{\kappa'/\kappa}}\big[\ln\big(1+\frac{u_n}{A^{\kappa'/\kappa}}\big)\big]^p\\ &\le \frac{1}{[\ln(1+\eta)]^p}\int_\Omega\big[\ln\big(1+\frac{u_n}{A^{\kappa'/\kappa}}\big)\big]^p\\ &\le \frac{M_1}{[\ln(1+\eta)]^p}.\end{aligned}$$

Equivalently,

$$\operatorname{meas}\big\{|u_n|\ge \exp\{\vartheta M_1^{1/p}\}A^{\kappa'/\kappa}\big\}\le\frac{1}{\vartheta^p}.$$

Therefore, there exists ϑ_0, which is independent on n, such that

$$C\|c_0(x)\|_{L^{N/(p-1),r}(E_2(\varrho_0))}<\frac12,$$

where

$$\varrho_0=\exp\{\vartheta_0 M_1^{1/p}\}A^{\kappa'/\kappa}.$$

With this choice of ϱ_0, we rewrite (3.32) as

$$\int_{E_2(\varrho_0)}\frac{|\nabla u_n|^p}{(1+|u_n|)^\beta}\le 2\mu(\Omega)\varrho_0+L_2,\tag{3.34}$$

where $L_2=2L_1$.

Therefore, for each $k\ge\varrho_0$, (3.34) leads to

$$\int_{E_2(k)}|\nabla T_k(\psi(u_n))|^p\le 2\mu(\Omega)k+L_2,\tag{3.35}$$

where ψ defined as in (3.23). By Proposition 2.7 with $\sigma = 1$, we arrive at

$$\begin{aligned} &\| \, |\nabla\psi(u_n)|^{p-1}\|_{L^{\frac{N}{N-1},\infty}(\Omega)} \\ &= \|\frac{|\nabla u_n|^{p-1}}{(1+|u_n|)^{\theta(p-1)}}\|_{L^{\frac{N}{N-1},\infty}(\Omega)} \\ &\le C[2\mu(\Omega) + |\Omega|^{\frac{1}{p}-\frac{1}{N}} L^{\frac{p-1}{p}}] \\ &\le C\frac{p'}{\alpha_1}\mu(\Omega) + C|\Omega|^{\frac{1}{p}-\frac{1}{N}}\big[\frac{1}{\alpha_1}\big(\frac{3}{\alpha_1}\big)^{p'/p}\|c_1(x)\|^{p'}_{L^{p'}(\Omega)}\big]^{\frac{p-1}{p}}. \end{aligned} \tag{3.36}$$

Therefore, (3.36), combined with the uniform boundedness of u_n, leads to

$$\begin{aligned} &\||\nabla u_n|^{p-1}\|_{L^{\frac{N}{N-1},\infty}(\Omega)} \\ &\le C\frac{p'}{\alpha_1}\mu(\Omega) + C|\Omega|^{\frac{1}{p}-\frac{1}{N}}\Big[\frac{1}{\alpha_1}\big(\frac{3}{\alpha_1}\big)^{p'/p}\|c_1(x)\|^{p'}_{L^{p'}(\Omega)}\Big]^{\frac{p-1}{p}}. \end{aligned} \tag{3.37}$$

These estimates show that (3.21) holds.

Using (3.35) and Proposition 2.7 with $\sigma = 1$ again, we derive that

$$\||\psi(u_n)|^{p-1}\|_{L^{\frac{p^*}{p},\infty}(\Omega)} \le C\mu(\Omega) + CL_2^{1-\frac{1}{p}}, \tag{3.38}$$

which, combined with the fact that $\psi(u_n)$ behaves like $|u_n|^{(p-1)(1-\theta)}$, implies that

$$\||u_n|^{(p-1)(1-\theta)}\|_{L^{\frac{p^*}{p},\infty}(\Omega)} \le C\mu(\Omega) + CL_2^{1-\frac{1}{p}}. \tag{3.39}$$

This is (3.22). □

4. Proof of Theorem 1.4

In this section, combining the results of Proposition 2.7 and 2.8, we prove Theorem 1.4. To do this, we will show that the terms $H_n(x, u_n, \nabla u_n)$ and $G_n(x, u_n)$, which appears in (3.1), converge strongly in $L^1(\Omega)$. This following arguments similar to these used in [6, 25, 26].

From (1.9), (3.36) and the Hölder inequality (2.2), we find that

$$\begin{aligned} &\|H_n(x, u_n, \nabla u_n)\|_{L^1(\Omega)} \\ &= \int_\Omega |H_n(x, u_n, \nabla u_n)| \\ &\le \int_\Omega \frac{b_0(x)|\nabla u_n|^{p-1}}{(1+|u_n|)^{\theta(p-1)}} + \int_\Omega \frac{b_1(x)}{(1+|u_n|)^{\theta(p-1)}} \\ &\le \|b_0(x)\|_{L^{N,1}(\Omega)}\|\frac{|\nabla u_n|^{p-1}}{(1+|u|)^{\theta(p-1)}}\|_{L^{\frac{N}{N-1},\infty}(\Omega)} + \|b_1\|_{L^1(\Omega)} \\ &\le \|b_0(x)\|_{L^{N,1}(\Omega)}\big\{\|b_1\|_{L^1(\Omega)} + \mu(\Omega) + \|c_1(x)\|_{L^{p'}(\Omega)}\big\} + \|b_1\|_{L^1(\Omega)}. \end{aligned} \tag{4.1}$$

Moreover, Gathering (1.10), (3.39) and Hölder's inequality (2.2) leads to

$$\begin{aligned} \|G_n(x,s)\|_{L^1(\Omega)} &\le \int_\Omega d_0(x)|u_n|^t + \int_\Omega d_1(x) \\ &\le \|d_0(x)\|_{L^{z',1}(\Omega)}\| \, |u_n|^{t(1-\theta)}\|_{L^{z,\infty}(\Omega)} + \|d_1(x)\|_{L^1(\Omega)} \\ &\le C, \end{aligned} \tag{4.2}$$

where z appears in (1.11).

Therefore, the solution u_n of (3.1) satisfies

$$\begin{gathered} -\operatorname{div}\Big(\frac{a(x,u_n,\nabla u_n)+K_n(x,u_n)}{(1+|u_n|)^{\theta(p-1)}}\Big)=f_n-\Phi_n-\operatorname{div}(g),\quad x\in\Omega,\\ u_n=0,\quad x\in\partial\Omega, \end{gathered}\tag{4.3}$$

where $\Phi_n=H_n(x,u_n,\nabla u_n)+G_n(x,u_n)$ is bounded in $L^1(\Omega)$.

Using $T_k(u_n)$ as test function in (4.3), we easily obtain that, for any $n\geq 1$, there exists some $\widetilde{M}$ and $\widetilde{L}$ such that

$$\int_\Omega |\nabla T_k(u_n)|^p\leq \widetilde{M}k^{1+\theta(p-1)}+\widetilde{L}.\tag{4.4}$$

The above estimate and the growth condition (3.5) on $K_n(x,u_n)$ allow us to use standard techniques (see for example [1, 4]) to say that a subsequence, still denoted by u_n, which is almost everywhere convergent in Ω, to a measurable function u such that $T_k(u)$ belongs to $W^{1,p}(\Omega)$ This fact, together with (3.37) and Proposition 2.8, leads to

$$\begin{gathered} u_n\to u,\quad \text{almost everywhere in } \Omega,\\ \nabla u_n\to\nabla u,\quad \text{almost everywhere in } \Omega,\\ \nabla T_k(u_n)\to\nabla T_k(u),\quad \text{in } (L^p(\Omega))^N \text{ weakly}. \end{gathered}\tag{4.5}$$

This implies that

$$\frac{a(x,u_n,\nabla u_n)}{(1+|u_n|)^{\theta(p-1)}}\to\frac{a(x,u,\nabla u)}{(1+|u|)^{\theta(p-1)}}\quad \text{almost everywhere in } \Omega.\tag{4.6}$$

From (3.17), we deduce that

$$H_n(x,u_n,\nabla u_n)\to H(x,u,\nabla u)\quad \text{almost everywhere in } \Omega.\tag{4.7}$$

Moreover, using the growth condition (3.18), Proposition 2.7, we derive that $H_n(x,u_n,\nabla u_n)$ is equi-integrable. Therefore the Vitali Theorem implies that

$$H_n(x,u_n,\nabla u_n)\to H(x,u,\nabla u) \text{ in } L^1(\Omega)\quad \text{strongly}.\tag{4.8}$$

Similar arguments show that

$$G_n(x,u_n)\to G(x,u)\quad \text{in } L^1(\Omega) \text{ strongly}.\tag{4.9}$$

Therefore, (4.8) and (4.9) lead to

$$\Phi_n=H_n(x,u_n,\nabla u_n)+G_n(x,u_n)\to H(x,u,\nabla u)+G(x,u)\tag{4.10}$$

in $L^1(\Omega)$ strongly.

Note that u_n is a weak solution of (3.1); this fact together with stability result Proposition (2.8), leads to Theorem 1.4 in the case (1).

For fixed $k>0$ and any $\omega\in W_0^{1,p}(\Omega)\bigcap L^\infty(\Omega)$, take $T_k(u-\omega)$ as a test function in (3.1), we find

$$\begin{aligned} &\int_\Omega\Big(\frac{a(x,u_n,\nabla u_n)+K(x,u_n)}{(1+|u_n|)^{\theta(p-1)}}\Big)\cdot\nabla T_k(u_n-\omega)\\ &+\int_\Omega H(x,u_n,\nabla u_n)T_k(u_n-\omega)+\int_\Omega G(x,u_n)T_k(u_n-\omega)\\ &=\langle f_n,T_k(u_n-\omega)\rangle+\langle g,\nabla T_k(u_n-\omega)\rangle. \end{aligned}\tag{4.11}$$

In view of f_n is strongly covergent in (at least) $L^1(\Omega)$, while $T_k(u_n-\omega)$ converges both weakly$*$ in $L^\infty(\Omega)$ and almost everywhere to $T_k(u-\omega)$. Thus

$$\langle f_n, T_k(u_n-\omega)\rangle + \langle g, \nabla T_k(u_n-\omega)\rangle \xrightarrow{n\to\infty} \langle f, T_k(u-\omega)\rangle + \langle g, \nabla T_k(u-\omega)\rangle.$$

For the first term of the right-hand side of (4.11), rewrite it as

$$\begin{aligned}
&\int_\Omega \Big(\frac{a(x,u_n,\nabla u_n)+K(x,u_n)}{(1+|u_n|)^{\theta(p-1)}}\Big)\cdot \nabla T_k(u_n-\omega)\\
&= \int_{\{|u_n-\omega|\le k\}} \Big(\frac{a(x,u_n,\nabla u_n)+K(x,u_n)}{(1+|u_n|)^{\theta(p-1)}}\Big)\cdot \nabla u_n\\
&\quad - \int_{\{|u_n-\omega|\le k\}} \Big(\frac{a(x,u_n,\nabla u_n)+K(x,u_n)}{(1+|u_n|)^{\theta(p-1)}}\Big)\cdot \nabla \omega\\
&= \int_{\{|u_n-\omega|\le k\}} \Big(\frac{a(x,u_n,\nabla u_n)+K(x,u_n)}{(1+|u_n|)^{\theta(p-1)}}\Big)\cdot \nabla u_n\\
&\quad - \int_{\{|u_n-\omega|\le k\}} \Big(\frac{a(x,T_M(u_n),\nabla T_M(u_n))+K(x,T_M(u_n))}{(1+|T_M(u_n)|)^{\theta(p-1)}}\Big)\cdot \nabla \omega
\end{aligned} \tag{4.12}$$

where $M = k + \|\omega\|_{L^\infty(\Omega)}$. Taking (1.7) into account, we have

$$\|\frac{a(x,T_M(u_n),\nabla T_M(u_n))+K(x,T_M(u_n))}{(1+|T_M(u_n)|)^{\theta(p-1)}}\|_{L^{p'}(\Omega)} \le C,$$

which combined with (4.6), implies that, in $\big(L^{p'}(\Omega)\big)^N$,

$$\frac{a(x,T_M(u),\nabla T_M(u))+K(x,T_M(u))}{(1+|T_M(u)|)^{\theta(p-1)}} \to \frac{a(x,u,\nabla u)+K(x,u)}{(1+|u|)^{\theta(p-1)}}.$$

Therefore

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\{|u_n-\omega|\le k\}} \Big(\frac{a(x,T_M(u_n),\nabla T_M(u_n))+K(x,T_M(u_n))}{(1+|T_M(u_n)|)^{\theta(p-1)}}\Big)\cdot \nabla \omega\\
&= \int_{\{|u_n-\omega|\le k\}} \frac{a(x,u,\nabla u)+K(x,u)}{(1+|u|)^{\theta(p-1)}}\cdot \nabla\omega.
\end{aligned} \tag{4.13}$$

On the other hand, by Fatou's lemma, we arrive at

$$\begin{aligned}
&\int_{\{|u_n-\omega|\le k\}} \Big(\frac{a(x,u,\nabla u)+K(x,u)}{(1+|u_n|)^{\theta(p-1)}}\Big)\cdot \nabla u\\
&\le \liminf_{n\to\infty}\int_{\{|u_n-\omega|\le k\}} \Big(\frac{a(x,u_n,\nabla u_n)+K(x,u_n)}{(1+|u_n|)^{\theta(p-1)}}\Big)\cdot \nabla u_n.
\end{aligned} \tag{4.14}$$

Equaitons (4.12) and (4.13) together with(4.14) lead to Theorem 1.4.

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