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GROWTH OF SOLUTIONS OF COMPLEX DIFFERENTIAL EQUATIONS IN A SECTOR OF THE UNIT DISC

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Abstract. In this article, we study the growth of solutions of homogeneous linear complex differential equation by using the concept of lower $[p,q]$-order and lower $[p,q]$-type in a sector of the unit disc instead of the whole unit disc, and we obtain similar results as in the case of the unit disc.

1. Introduction

In this article, we assume that readers are familiar with the fundamental results and the standard notations of Nevanlinna's theory in the complex plane and in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, see [5, 6, 7, 9, 15, 23].

For $k \geq 2$ consider the complex differential equation

$$f^{(k)}(z) + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = 0, \tag{1.1}$$

where coefficients A_j $(j = 0, 1, \dots, k-1)$ are analytic functions in the unit disc Δ. It is well-known that every solution of (1.1) is analytic in Δ, and there are exactly k linearly independent solutions of (1.1) (see e.g. [7]). The theory of complex differential equations in the unit disc has been developed since 1980's, see [13]. In 2000, Heittokangas [7] firstly investigated the growth and oscillation theory of equation (1.1) when the coefficients A_j $(j = 0, 1, \dots, k-1)$ are analytic functions in the unit disc Δ by introducing the definition of the function spaces. His results also gave some important tools for further investigations on the theory of meromorphic solutions of equations (1.1). In 1994, Wu [17, 18] used the Nevanlinna theory in an angle to study the order of growth of solutions of the second-order linear differential equation in an angular region. Later Xu and Yi [22], Wu [19], Wu and Li [20], Zhang [24] generalized some results of [17, 18] to the case of linear higher order differential equations in angular domains by using the concepts of iterated $p-$order and the spread relation. Recently, Wu in [21] developed a new investigation related to linear differential equations with analytic coefficients in a sector of the unit disc

$$\Omega_{\alpha,\beta} = \{z \in \mathbb{C} : \alpha < \arg z < \beta,\ |z| < 1\},$$

and obtained some results about the order of growth of solutions of the differential equation

$$A_k(z) f^{(k)}(z) + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = 0, \tag{1.2}$$

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where coefficients A_j $(j = 0, 1, \dots, k)$ are analytic functions in the sector $\Omega_{\alpha,\beta}$. After that, Long [11, 12], Zemirni and Belaïdi [25] obtained different results concerning the growth of solutions of (1.1) and (1.2) by using the concepts of iterated p-order and $[p, q]$-order in the sector $\Omega_{\alpha,\beta}$. In this article, we continue to investigate this new problem and study the growth of solutions of equation (1.1) when the coefficients A_j $(j = 0, 1, \dots, k-1)$ are analytic functions of $[p, q]$-order in the sector $\Omega_{\alpha,\beta}$. Before stating our main results, we give some notation and basic definitions of meromorphic functions in the unit disc Δ and in a sector $\Omega_{\alpha,\beta}$ of the unit disc. The order of a meromorphic function f in Δ is defined by

$$\rho(f) = \limsup_{r\to 1^-} \frac{\log T(r,f)}{\log \frac{1}{1-r}},$$

where $T(r, f)$ is the Nevanlinna characteristic function of f. If f is analytic function in Δ, then

$$\rho_M(f) = \limsup_{r\to 1^-} \frac{\log\log M(r,f)}{\log \frac{1}{1-r}},$$

where $M(r, f) = \max_{|z|=r,\, z\in\Delta} |f(z)|$ is the maximum modulus function.

Remark 1.1. The following two statements hold [15, p. 205].

(a) If f is an analytic function in Δ, then

$$\rho(f) \le \rho_M(f) \le \rho(f) + 1.$$

(b) There exist analytic functions f in Δ which satisfy $\rho_M(f) \ne \rho(f)$. For example, let $\mu > 1$ be a constant, and set

$$h(z) = \exp\{(1-z)^{-\mu}\},$$

where we choose the principal branch of the logarithm. Then $\rho(h) = \mu - 1$ and $\rho_M(h) = \mu$, see [4].

In contrast, the possibility that occurs in (b) cannot occur in the whole plane $\mathbb{C}$, because if $\rho(f)$ and $\rho_M(f)$ denote the order of an entire function f in the plane $\mathbb{C}$ (defined by the Nevanlinna characteristic and the maximum modulus, respectively), then it is well-know that

$$\rho(f) = \limsup_{r\to +\infty} \frac{\log T(r,f)}{\log r} = \rho_M(f) = \limsup_{r\to +\infty} \frac{\log\log M(r,f)}{\log r}.$$

The meromorphic function f in the unit disc can be divided into the following three classes:

1. bounded type if $T(r, f) = O(1)$ as $r \to 1^-$;
2. rational or non-admissible type if $T(r, f) = O(\frac{1}{1-r})$ and f does not belong to (1);
3. admissible in Δ if
$$\limsup_{r\to 1^-} \frac{T(r,f)}{\log \frac{1}{1-r}} = \infty.$$

Definition 1.2 ([2, 3])**.** Let $p \ge q \ge 1$ be integers. Let f be a meromorphic function in Δ, the $[p, q]$-order of f is defined by

$$\rho_{[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_p^+ T(r,f)}{\log_q \frac{1}{1-r}},$$

where $\log_1^+ r := \log^+ r = \max(0, \log r)$, $\log_{p+1}^+ r := \log^+(\log_p^+ r)$, $p \in \mathbb{N}$. For an analytic function f in Δ, we also define

$$\rho_{M,[p,q]}(f) = \limsup_{r \to 1^-} \frac{\log_{p+1}^+ M(r,f)}{\log_q \frac{1}{1-r}}.$$

It is easy to see that $0 \le \rho_{[p,q]}(f) \le +\infty$. If f is non-admissible, then $\rho_{[p,q]}(f) = 0$ for any $p \ge q \ge 1$. By Definition 1.2, $\rho_{[1,1]}(f) = \rho(f)$ is the order of f in Δ, $\rho_{[2,1]}(f) = \rho_2(f)$ is the hyper-order of f in Δ and $\rho_{[p,1]}(f) = \rho_p(f)$ is the p-iterated order of f in Δ.

Proposition 1.3 ([2]). *Let $p \ge q \ge 1$ be integers, and let f be an analytic function in Δ of $[p,q]$-order. The following two statements hold:*

(i) *If $p = q$, then $\rho_{[p,q]}(f) \le \rho_{M,[p,q]}(f) \le \rho_{[p,q]}(f) + 1$.*
(ii) *If $p > q$, then $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f)$.*

Proposition 1.4 ([8]). *Let $p \ge q \ge 1$ be integers, and let f be an analytic function in Δ of $[p,q]$-order. The following two statements hold:*

(i) *If $p = q$, then $\mu_{[p,q]}(f) \le \mu_{M,[p,q]}(f) \le \mu_{[p,q]}(f) + 1$.*
(ii) *If $p > q$, then $\mu_{[p,q]}(f) = \mu_{M,[p,q]}(f)$.*

In what follows, we give some notation and definitions of a meromorphic function in a sector in unit disc. Throughout this paper, Ω denotes the sector $\Omega_{\alpha,\beta}$ $(0 \le \alpha < \beta \le 2\pi)$ of the unit disc, and for any given $\varepsilon \in (0, \frac{\beta-\alpha}{2})$, Ω_ε denotes the sector

$$\Omega_{\alpha,\beta,\varepsilon} = \{z \in \mathbb{C} : \alpha + \varepsilon < \arg z < \beta - \varepsilon,\ |z| < 1\}.$$

Wu [21] used the Ahlfors-Shimizu characteristic function to measure the order of growth of a meromorphic function f in Ω. We recall the definition of the Ahlfors-Shimizu characteristic function, see [5, 6]. Let f be a meromorphic function in Ω, set

$$\begin{aligned}\Omega(r) &= \Omega \cap \{z \in \mathbb{C} : 0 < |z| < r < 1\}\\ &= \{z \in \mathbb{C} : \alpha < \arg z < \beta,\ 0 < |z| < r < 1\}.\end{aligned}$$

Then, the Ahlfors-Shimizu characteristic function is defined by

$$T_0(r, \Omega, f) = \int_0^r \frac{S(t, \Omega, f)}{t} dt,$$

where

$$S(r, \Omega, f) = \frac{1}{\pi} \iint_{\Omega(r)} \Big(\frac{|f'(z)|}{1 + |f(z)|^2}\Big)^2 d\sigma, \quad z = re^{i\theta},\ d\sigma = r\,dr\,d\theta.$$

It follows by Hayman [6], Goldberg and Ostrovskii [5] that

$$T_0(r, \mathbb{C}, f) = T(r,f) + O(1), \quad 0 < r < 1.$$

The meromorphic function f in a sector Ω of the unit disc can be divided into the following three classes:

(1) bounded type if $T_0(r, \Omega, f) = O(1)$ as $r \to 1^-$;
(2) rational or non-admissible type if $T_0(r, \Omega, f) = O(\frac{1}{1-r})$ and f does not belong to (1);

(3) admissible in Ω if

$$\limsup_{r\to 1^-}\frac{T_0(r,\Omega,f)}{\log\frac{1}{1-r}}=\infty.$$

Now, we introduce the concept of $[p,q]$-order and $[p,q]$-type of meromorphic functions in a sector Ω.

Definition 1.5 ([12, 25])**.** Let $p\ge q\ge 1$ be integers. Let f be a meromorphic function in Ω, the $[p,q]$-order of f is defined by

$$\rho_{[p,q],\Omega}(f)=\limsup_{r\to 1^-}\frac{\log_p^+ T_0(r,\Omega,f)}{\log_q\frac{1}{1-r}}.$$

It is clear that $0\le\rho_{[p,q],\Omega}(f)\le+\infty$. If f is non-admissible in Ω, then $\rho_{[p,q],\Omega}(f)=0$. By Definition 1.5, $\rho_{[1,1],\Omega}(f)=\rho_\Omega(f)$ is the order of f in Ω, see [21], $\rho_{[p,1],\Omega}(f)=\rho_{p,\Omega}(f)$ is the iterated p-order of f in Ω, see [11, 24].

Definition 1.6 ([25])**.** Let $p\ge q\ge 1$ be integers and f be a meromorphic function in Ω with $[p,q]$-order $0<\rho_{[p,q],\Omega}(f)<+\infty$. Then, the $[p,q]$-type of f is defined by

$$\tau_{[p,q],\Omega}(f)=\limsup_{r\to 1^-}\frac{\log_{p-1}^+ T_0(r,\Omega,f)}{(\log_{q-1}\frac{1}{1-r})^{\rho_{[p,q],\Omega}(f)}}.$$

Now, we introduce the concept of lower $[p,q]$-order and lower $[p,q]$-type of a meromorphic function in a sector Ω.

Definition 1.7. Let $p\ge q\ge 1$ be integers. Let f be a meromorphic function in Ω, the lower $[p,q]$-order of f is defined by

$$\mu_{[p,q],\Omega}(f)=\liminf_{r\to 1^-}\frac{\log_p^+ T_0(r,\Omega,f)}{\log_q\frac{1}{1-r}}.$$

It is clear that $0\le\mu_{[p,q],\Omega}(f)\le+\infty$. If f is non-admissible in Ω, then $\mu_{[p,q],\Omega}(f)=0$. By Definition 1.7, $\mu_{[1,1],\Omega}(f)=\mu_\Omega(f)$ is the lower order of f in Ω and $\mu_{[p,1],\Omega}(f)=\mu_{p,\Omega}(f)$ is the lower iterated p-order of f in Ω.

Definition 1.8. Let $p\ge q\ge 1$ be integers and f be a meromorphic function in Ω with lower $[p,q]$-order $0<\mu_{[p,q],\Omega}(f)<+\infty$. Then, the lower $[p,q]$-type of f is defined by

$$\underline{\tau}_{[p,q],\Omega}(f)=\liminf_{r\to 1^-}\frac{\log_{p-1}^+ T_0(r,\Omega,f)}{(\log_{q-1}\frac{1}{1-r})^{\mu_{[p,q],\Omega}(f)}}.$$

2. MAIN RESULTS

Several authors [2, 3, 8, 10, 16] have investigated the growth of solutions of the equation (1.1) by using the concepts of $[p,q]$-order in the unit disc Δ. Long [12] studied the growth of solutions of equation (1.2) in a sector of the unit disc with analytic coefficients of finite $[p,q]$-order, and obtained the following results.

Theorem 2.1 ([12])**.** *Let $p\ge q\ge 1$ be integers and $\varepsilon\in(0,\frac{\beta-\alpha}{2})$. Let E be a set of complex numbers satisfying $\overline{\text{dens}}\{|z|=r: z\in E\subset\Omega\}>0$, and let $A_0(z),A_1(z),\dots,A_k(z)$ be analytic functions in Ω such that for some real constants satisfying $0\le\gamma<\lambda$, we have*

$$T_0(r,\Omega_\varepsilon,A_0(z))\ge\exp_p\{\lambda\log_q\big(\frac{1}{1-|z|}\big)\},$$

$$T_0(r,\Omega,A_j(z)) \le \exp_p\big\{\gamma\log_q\big(\frac{1}{1-|z|}\big)\big\},\quad j=1,2,\dots,k$$

as $|z|=r\to 1^-$ *for* $z\in E$*. Then every nontrivial solution* f *of* (1.2) *satisfies* $\rho_{[p,q],\Omega}(f)=+\infty$ *and*

$$\rho_{[p+1,q],\Omega}(f)\ge\lambda.$$

Theorem 2.2 ([12])**.** *Let* $p\ge q\ge 1$ *be integers and* $\varepsilon\in(0,\frac{\beta-\alpha}{2})$*. Let* $A_0(z)$*,* $A_1(z)$*, …,* $A_k(z)$ *be analytic functions in* Ω*. If*

$$\max_{1\le j\le k}\{\rho_{[p,q],\Omega}(A_j)\}<\rho_{[p,q],\Omega_\varepsilon}(A_0),$$

then every nontrivial solution of (1.2) *satisfies*

$$\rho_{[p+1,q],\Omega}(f)\ge\rho_{[p,q],\Omega_\varepsilon}(A_0).$$

Remark 2.3. In the Theorems 2.1 and 2.2, we note that if $A_k(z)=1$, then all the solutions of (1.2) are analytic functions. But if $A_k(z)$ is a non-constant analytic function, then obviously the solution f of (1.2) can be meromorphic function. The hypotheses in Theorems 2.1 and 2.2 do not provide that a solution is meromorphic in Ω, so it is a priori assumed that f is meromorphic.

Very recently, Zemirni and Belaïdi [25] continued the study of the growth of solutions of equation (1.1) instead of equation (1.2) in a sector of the unit disc with analytic coefficients of finite $[p,q]$-order, and obtained the following results.

Theorem 2.4 ([25])**.** *Let* $p\ge q\ge 1$ *be integers and* $\varepsilon\in(0,\frac{\beta-\alpha}{2})$*. Let* $A_0(z)$*,* $A_1(z)$*, …,* $A_{k-1}(z)$ *be analytic functions in* Ω*. If*

$$\max_{1\le j\le k-1}\{\rho_{[p,q],\Omega}(A_j)\}<\rho_{[p,q],\Omega_\varepsilon}(A_0),$$

then every nontrivial solution of (1.1) *satisfies* $\rho_{[p,q],\Omega}(f)=+\infty$ *and*

$$\rho_{[p,q],\Omega_\varepsilon}(A_0)\le\rho_{[p+1,q],\Omega}(f),\quad \rho_{[p+1,q],\Omega_\varepsilon}(f)\le\rho_{[p,q],\Omega}(A_0)+1.$$

Furthermore, if $p>q$*, then*

$$\rho_{[p,q],\Omega_\varepsilon}(A_0)\le\rho_{[p+1,q],\Omega}(f),\quad \rho_{[p+1,q],\Omega_\varepsilon}(f)\le\rho_{[p,q],\Omega}(A_0).$$

Theorem 2.5 ([25])**.** *Let* $p\ge q\ge 1$ *be integers and* $\varepsilon\in(0,\frac{\beta-\alpha}{2})$*. Let* $A_0(z)$*,* $A_1(z)$*, …,* $A_{k-1}(z)$ *be analytic functions in* Ω*. Suppose that*

$$\max_{1\le j\le k-1}\{\rho_{[p,q],\Omega}(A_j)\}\le\rho_{[p,q],\Omega_\varepsilon}(A_0)=\rho\ (0<\rho<+\infty),$$

and

$$\begin{aligned}\max_{1\le j\le k-1}\{\tau_{[p,q],\Omega}(A_j):\rho_{[p,q],\Omega}(A_j)=\rho_{[p,q],\Omega_\varepsilon}(A_0)\}\\ <\tau_{[p,q],\Omega_\varepsilon}(A_0)=\tau\quad(0<\tau<+\infty).\end{aligned}$$

Then, every nontrivial solution of (1.1) *satisfies* $\rho_{[p,q],\Omega}(f)=+\infty$ *and*

$$\rho_{[p,q],\Omega_\varepsilon}(A_0)\le\rho_{[p+1,q],\Omega}(f),\quad \rho_{[p+1,q],\Omega_\varepsilon}(f)\le\rho_{[p,q],\Omega}(A_0)+1.$$

Furthermore, if $p>q$*, then*

$$\rho_{[p,q],\Omega_\varepsilon}(A_0)\le\rho_{[p+1,q],\Omega}(f),\quad \rho_{[p+1,q],\Omega_\varepsilon}(f)\le\rho_{[p,q],\Omega}(A_0).$$

Thus, the following questions arise naturally: (i) Whether the results similar to Theorem 2.4 can be obtained in Ω if $A_0(z)$ dominates the other coefficients in the sense of lower $[p,q]$-order?

(ii) If we use the lower $[p,q]$-type of $A_0(z)$ to dominate the other coefficients, what can be said about $\mu_{[p+1,q],\Omega}(f)$, similar to Theorem 2.5?

In this article, we give some answers to the above questions. In fact, by using the concept of lower $[p,q]$-type, we obtain some results which indicate growth estimate of every non-trivial analytic solution of equation (1.1) by the growth estimate of the coefficient $A_0(z)$. We mainly obtain the following results.

Theorem 2.6. *Let $p \geq q \geq 1$ be integers and $\varepsilon \in (0, \frac{\beta-\alpha}{2})$. Let $A_0(z)$, $A_1(z)$, $\dots$, $A_{k-1}(z)$ be analytic functions in Ω. If*

$$\max_{1\leq j\leq k-1}\{\rho_{[p,q],\Omega}(A_j)\} < \mu_{[p,q],\Omega_\varepsilon}(A_0),$$

then every nontrivial solution of (1.1) *satisfies*

$$\rho_{[p,q],\Omega}(f) = \mu_{[p,q],\Omega}(f) = +\infty,$$
$$\mu_{[p,q],\Omega_\varepsilon}(A_0) \leq \mu_{[p+1,q],\Omega}(f) \leq \rho_{[p+1,q],\Omega}(f),$$
$$\mu_{[p+1,q],\Omega_\varepsilon}(f) \leq \mu_{[p,q],\Omega}(A_0) + 1.$$

Furthermore, if $p > q$, then

$$\mu_{[p,q],\Omega_\varepsilon}(A_0) \leq \mu_{[p+1,q],\Omega}(f) \leq \rho_{[p+1,q],\Omega}(f),$$
$$\mu_{[p+1,q],\Omega_\varepsilon}(f) \leq \mu_{[p,q],\Omega}(A_0).$$

Remark 2.7. Theorem 2.6 is similar to [16, Theorem 2.2 (i)] in the unit disc Δ.

Corollary 2.8. *Let $p \geq q \geq 1$ be integers and $\varepsilon \in (0, \frac{\beta-\alpha}{2})$. Let $A_0(z)$, $A_1(z)$, $\dots$, $A_{k-1}(z)$ be analytic functions in Ω. If*

$$\max_{1\leq j\leq k-1}\{\rho_{[p,q],\Omega}(A_j)\} < \mu_{[p,q],\Omega_\varepsilon}(A_0) = \rho_{[p,q],\Omega_\varepsilon}(A_0),$$

then every nontrivial solution of (1.1) *satisfies $\rho_{[p,q],\Omega}(f) = \mu_{[p,q],\Omega}(f) = +\infty$ and*

$$\mu_{[p,q],\Omega_\varepsilon}(A_0) \leq \mu_{[p+1,q],\Omega}(f) \leq \rho_{[p+1,q],\Omega}(f),$$
$$\mu_{[p+1,q],\Omega_\varepsilon}(f) \leq \rho_{[p+1,q],\Omega_\varepsilon}(f) \leq \mu_{[p,q],\Omega}(A_0) + 1.$$

Furthermore, if $p > q$, then

$$\mu_{[p,q],\Omega_\varepsilon}(A_0) \leq \mu_{[p+1,q],\Omega}(f) \leq \rho_{[p+1,q],\Omega}(f),$$
$$\mu_{[p+1,q],\Omega_\varepsilon}(f) \leq \rho_{[p+1,q],\Omega_\varepsilon}(f) \leq \mu_{[p,q],\Omega}(A_0).$$

Theorem 2.9. *Let $p \geq q \geq 1$ be integers and $\varepsilon \in (0, \frac{\beta-\alpha}{2})$. Let $A_0(z)$, $A_1(z)$, $\dots$, $A_{k-1}(z)$ be analytic functions in Ω such that*

$$0 < \mu = \mu_{[p,q],\Omega_\varepsilon}(A_0) \leq \rho_{[p,q],\Omega_\varepsilon}(A_0) < +\infty.$$

Suppose that

$$\max_{1\leq j\leq k-1}\{\rho_{[p,q],\Omega}(A_j)\} \leq \mu_{[p,q],\Omega_\varepsilon}(A_0)$$

and

$$\max_{1\leq j\leq k-1}\{\tau_{[p,q],\Omega}(A_j) : \rho_{[p,q],\Omega}(A_j) = \mu_{[p,q],\Omega_\varepsilon}(A_0)\} < \underline{\tau}_{[p,q],\Omega_\varepsilon}(A_0) < +\infty.$$

Then, every nontrivial solution of (1.1) *satisfies*

$$\rho_{[p,q],\Omega}(f) = \mu_{[p,q],\Omega}(f) = +\infty,$$

$$\begin{gathered}\mu_{[p,q],\Omega_\varepsilon}(A_0) \le \mu_{[p+1,q],\Omega}(f) \le \rho_{[p+1,q],\Omega}(f),\\ \mu_{[p+1,q],\Omega_\varepsilon}(f) \le \mu_{[p,q],\Omega}(A_0)+1.\end{gathered}$$

Furthermore, if $p > q$*, then*

$$\begin{gathered}\mu_{[p,q],\Omega_\varepsilon}(A_0) \le \mu_{[p+1,q],\Omega}(f) \le \rho_{[p+1,q],\Omega}(f),\\ \mu_{[p+1,q],\Omega_\varepsilon}(f) \le \mu_{[p,q],\Omega}(A_0).\end{gathered}$$

Remark 2.10. Theorem 2.9 is similar to [8, Theorem 2.1] in the unit disc Δ.

Remark 2.11. We note that in Theorems 2.6 and 2.9, the growth estimate of the solution f is expressed by the growth estimate of dominant coefficient A_0 in the terms of lower $[p,q]$-order on both sides.

3. Auxiliary lemmas

Lemma 3.1 ([14]). *Let*

$$u(z)=\frac{(ze^{-i\theta_0})^{\pi/\delta}+2(ze^{-i\theta_0})^{\pi/(2\delta)}-1}{(ze^{-i\theta_0})^{\pi/\delta}-2(ze^{-i\theta_0})^{\pi/(2\delta)}-1}, \tag{3.1}$$

where $0 \le \theta_0 = \frac{\alpha+\beta}{2} < 2\pi$, $0 < \delta = \frac{\beta-\alpha}{2} < \pi$*. Then* $u(z)$ *is a conformal map of angular domain* Ω $(0 < \beta-\alpha < 2\pi)$ *onto the unit disc* Δ*. Moreover, for any positive number* ε *satisfying* $0<\varepsilon<\delta$*, the transformation* (3.1) *satisfies*

$$\begin{gathered}u\Big(\{z:\frac{1}{2}<|z|<r\}\cap\{z:|\arg z-\theta_0|<\delta-\varepsilon\}\Big)\\ \subset\{u:|u|<1-\frac{\varepsilon}{2^{\frac{\pi}{2\delta}+1}\delta}(1-r)\},\\ u^{-1}(\{u:|u|<\varrho\})\subset\Big(\{z:|z|<1-\frac{\delta}{8\pi}(1-\varrho)\}\cap\{z:|\arg z-\theta_0|<\delta\}\Big),\end{gathered}$$

where $\varrho<1$ *is a constant. The inverse transformation of* (3.1) *is*

$$z(u)=e^{i\theta_0}\Big(\frac{-(1+u)+\sqrt{2(1+u^2)}}{1-u}\Big)^{2\delta/\pi}. \tag{3.2}$$

Lemma 3.2 ([21]). *Let* f *be a meromorphic function in* Ω*, where* $0<\beta-\alpha<2\pi$*. For any given* $\varepsilon\in(0,\frac{\beta-\alpha}{2})$*, set* $\delta=\frac{\beta-\alpha}{2}$ *and* $b=\frac{\varepsilon}{2^{\pi/(2\delta)+1}\delta}$*. Then*

$$T_0(\varrho,\mathbb{C},f(z(u)))\le\frac{16\pi}{\delta}T_0\Big(1-\frac{\delta}{8\pi}(1-\varrho),\Omega,f(z)\Big)+O(1), \tag{3.3}$$

$$T_0(r,\Omega_\varepsilon,f(z))\le\frac{2}{b}T_0(1-b(1-r),\mathbb{C},f(z(u)))+O(1), \tag{3.4}$$

where $z(u)$ *is the inverse transformation of* (3.1).

Remark 3.3. By applying the formula $T(r,f)=T_0(r,\mathbb{C},f)+O(1)$ $(0<r<1)$, Lemma 3.2, the definition of $[p,q]$-order and lower $[p,q]$-order, we immediately obtain that

$$\begin{gathered}\rho_{[p,q],\Omega_\varepsilon}(f(z))\le\rho_{[p,q]}(f(z(u)))\le\rho_{[p,q],\Omega}(f(z)),\\ \mu_{[p,q],\Omega_\varepsilon}(f(z))\le\mu_{[p,q]}(f(z(u)))\le\mu_{[p,q],\Omega}(f(z)).\end{gathered}$$

Lemma 3.4 ([21])**.** *Let f be a meromorphic function in Ω, where $0<\beta-\alpha<2\pi$ and $z(u)$ be the inverse transformation of* (3.1)*. Set $F(u)=f(z(u))$ and $\psi(u)=f^{(\ell)}(z(u))$ Then*

$$\psi(u)=\sum_{j=1}^{\ell}\alpha_j F^{(j)}(u), \tag{3.5}$$

where the coefficients α_j are polynomials (with numerical coefficients) in the variables $V(u)(=\frac{1}{z'(u)}), V'(u), V''(u), \dots$. Moreover, we have

$$T(\varrho,\alpha_j)=O\Big(\log\frac{1}{1-\varrho}\Big),\quad j=1,2,\dots,\ell. \tag{3.6}$$

For convenience of the readers, we give the statement and the proof of Lemma 3.5 [25, Lemma 3.4] with more precision.

Lemma 3.5. *Suppose $f\not\equiv 0$ is a solution of* (1.1) *in Ω. Then $F(u)=f(z(u))$ is a solution of*

$$F^{(k)}(u)+B_{k-1}(u)F^{(k-1)}(u)+\dots+B_0(u)F(u)=0 \tag{3.7}$$

in Δ, where

$$B_0(u)=\frac{1}{\alpha_k}A_0(z(u)) \tag{3.8}$$

and for $j=1,2,\dots,k-1$,

$$B_j(u)=\frac{\alpha_j}{\alpha_k}+\frac{\alpha_j}{\alpha_k}\sum_{n=j}^{k-1}A_n(z(u)). \tag{3.9}$$

Consequently,

$$T(\varrho,B_0)\le T(r,A_0(z(u)))+O\Big(\log\frac{1}{1-\varrho}\Big), \tag{3.10}$$

$$T(\varrho,B_j)\le\sum_{n=j}^{k-1}T(r,A_n(z(u)))+O\Big(\log\frac{1}{1-\varrho}\Big). \tag{3.11}$$

Proof. Suppose that $f\not\equiv 0$ is a solution of (1.1) in the sector Ω. By Lemma 3.4, we have

$$\begin{aligned}
&f^{(k)}(z(u))+\sum_{n=1}^{k-1}A_n(z(u))f^{(n)}(z(u))+A_0(z(u))f(z(u))\\
&=\sum_{j=1}^{k}\alpha_jF^{(j)}(u)+\sum_{n=1}^{k-1}A_n(z(u))\sum_{j=1}^{n}\alpha_jF^{(j)}(u)+A_0(z(u))f(z(u))\\
&=\sum_{j=1}^{k}\alpha_jF^{(j)}(u)+\sum_{j=1}^{k-1}\Big(\alpha_j\sum_{n=j}^{k-1}A_n(z(u))\Big)F^{(j)}(u)+A_0(z(u))f(z(u))\\
&=\alpha_kF^{(k)}(u)+\sum_{j=1}^{k-1}\Big(\alpha_j\sum_{n=j}^{k-1}A_n(z(u))+\alpha_j\Big)F^{(j)}(u)+A_0(z(u))F(u).
\end{aligned}$$

It follows that $F(u)=f(z(u))$ is a solution of

$$F^{(k)}(u)+B_{k-1}(u)F^{(k-1)}(u)+\dots+B_0(u)F(u)=0,$$

where $B_0(u) = \frac{1}{\alpha_k} A_0(z(u))$ and

$$B_j(u) = \frac{\alpha_j}{\alpha_k} + \frac{\alpha_j}{\alpha_k} \sum_{n=j}^{k-1} A_n(z(u)), \quad j = 1, 2, \dots, k-1.$$

From the proof of Lemma 3.4, we can get that [21, p. 63]

$$\begin{aligned} \alpha_k = V^k(u) &= \Big(\frac{1}{z'(u)}\Big)^k \\ &= \Big(\frac{\omega}{e^{i\theta_0}} \Big(\frac{1-u}{-(1+u)+\sqrt{2(1+u^2)}}\Big)^{\frac{1}{\omega}-1} \frac{(1-u)^2\sqrt{1+u^2}}{\sqrt{2}(1+u)-2\sqrt{1+u^2}}\Big)^k, \end{aligned}$$

which is analytic in Δ, where $\theta_0 = \frac{\alpha+\beta}{2}$ and $\omega = \frac{\pi}{\beta-\alpha}$. Since $\alpha_k = V^k(u) \neq 0$ in Δ, then $B_0(u) = \frac{1}{\alpha_k} A_0(z(u))$ and

$$B_j(u) = \frac{\alpha_j}{\alpha_k} + \frac{\alpha_j}{\alpha_k} \sum_{n=j}^{k-1} A_n(z(u)), \quad j = 1, 2, \dots, k-1$$

are also analytic in Δ. Because

$$T(\varrho, \alpha_j) = O\Big(\log \frac{1}{1-\varrho}\Big), \quad j = 1, 2, \dots, k,$$

it follows from this and the properties of Nevanlinna's characteristic function that

$$\begin{aligned} T(\varrho, B_0) \le T\big(\varrho, \frac{1}{\alpha_k}\big) + T(\varrho, A_0(z(u))) &= T(\varrho, \alpha_k) + T(\varrho, A_0(z(u))) + O(1) \\ &= T(\varrho, A_0(z(u))) + O\Big(\log \frac{1}{1-\varrho}\Big), \end{aligned}$$

and for $j = 1, 2, \dots, k-1$,

$$\begin{aligned} T(\varrho, B_j) &\le T\Big(\varrho, \frac{\alpha_j}{\alpha_k}\Big) + \sum_{n=j}^{k-1} T(\varrho, A_n(z(u))) + O(1) \\ &\le T(\varrho, \alpha_j) + T\Big(\varrho, \frac{1}{\alpha_k}\Big) + \sum_{n=j}^{k-1} T(\varrho, A_n(z(u))) + O(1) \\ &= T(\varrho, \alpha_j) + T(\varrho, \alpha_k) + \sum_{n=j}^{k-1} T(\varrho, A_n(z(u))) + O(1) \\ &= \sum_{n=j}^{k-1} T(\varrho, A_n(z(u))) + O\Big(\log \frac{1}{1-\varrho}\Big). \end{aligned}$$

□

Lemma 3.6 ([16]). *Let $p \ge q \ge 1$ be integers. If $B_0(u), B_1(u), \dots, B_{k-1}(u)$ are analytic functions of $[p,q]$-order in the unit disc Δ, then every solution $F \not\equiv 0$ of* (3.7) *satisfies*

$$\mu_{[p+1,q]}(F) = \mu_{M,[p+1,q]}(F) \le \max_{1\le j\le k-1} \{\mu_{M,[p,q]}(B_0), \rho_{M,[p,q]}(B_j)\}.$$

Lemma 3.7. *Let $p \geq q \geq 1$ be integers. If $A_0(z), \dots, A_{k-1}(z)$ are analytic functions of $[p,q]$-order in sector Ω satisfying $\max_{1\leq j\leq k-1}\{\rho_{[p,q],\Omega}(A_j)\} < \mu_{[p,q],\Omega_\varepsilon}(A_0)$, then for any given $\varepsilon \in (0, \frac{\beta-\alpha}{2})$, every solution $f \not\equiv 0$ of (1.1) satisfies*

$$\mu_{[p+1,q],\Omega_\varepsilon}(f) \leq \mu_{[p,q],\Omega}(A_0) + 1.$$

Furthermore, if $p > q$ then

$$\mu_{[p+1,q],\Omega_\varepsilon}(f) \leq \mu_{[p,q],\Omega}(A_0).$$

Proof. Let $f \not\equiv 0$ be a solution of equation (1.1). Then by Lemma 3.5, $F(u) = f(z(u))$ is a solution of equation (3.7) and by using Remark 3.3, Proposition 1.3, Proposition 1.4 and Lemma 3.6, we obtain

$$\begin{aligned}
\mu_{[p+1,q],\Omega_\varepsilon}(f) &\leq \mu_{[p+1,q]}(F) = \mu_{M,[p+1,q]}(F) \\
&\leq \max_{1\leq j\leq k-1}\{\mu_{M,[p,q]}(B_0), \rho_{M,[p,q]}(B_j)\} \\
&\leq \max_{1\leq j\leq k-1}\{\mu_{[p,q]}(B_0), \rho_{[p,q]}(B_j)\} + 1 \\
&\leq \max_{1\leq j\leq k-1}\{\mu_{[p,q],\Omega}(A_0), \rho_{[p,q],\Omega}(A_j)\} + 1 \\
&\leq \max_{1\leq j\leq k-1}\{\mu_{[p,q],\Omega}(A_0), \mu_{[p,q],\Omega_\varepsilon}(A_0)\} + 1 \\
&= \mu_{[p,q],\Omega}(A_0) + 1.
\end{aligned}$$

If $p > q$, we obtain

$$\begin{aligned}
\mu_{[p+1,q],\Omega_\varepsilon}(f) &\leq \mu_{[p+1,q]}(F) = \mu_{M,[p+1,q]}(F) \\
&\leq \max_{1\leq j\leq k-1}\{\mu_{M,[p,q]}(B_0), \rho_{M,[p,q]}(B_j)\} \\
&= \max_{1\leq j\leq k-1}\{\mu_{[p,q]}(B_0), \rho_{[p,q]}(B_j)\} \\
&\leq \max_{1\leq j\leq k-1}\{\mu_{[p,q],\Omega}(A_0), \rho_{[p,q],\Omega}(A_j)\} \\
&\leq \max_{1\leq j\leq k-1}\{\mu_{[p,q],\Omega}(A_0), \mu_{[p,q],\Omega_\varepsilon}(A_0)\} \\
&= \mu_{[p,q],\Omega}(A_0).
\end{aligned}$$

□

Lemma 3.8 ([7, 15])**.** *Let f be a meromorphic function in the unit disc Δ and let $k \in \mathbb{N}$. Then*

$$m\big(r, \frac{f^{(k)}}{f}\big) = S(r,f),$$

where

$$S(r,f) = O\Big(\log^+ T(r,f) + \log\big(\frac{1}{1-r}\big)\Big),$$

possibly outside a set $F \subset [0,1)$ with $\int_F \frac{dr}{1-r} < \infty$.

Lemma 3.9 ([1, 7])**.** *Let $g : (0,1) \to \mathbb{R}$ and $h : (0,1) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ holds outside of an exceptional set $E \subset [0,1)$ for which $\int_E \frac{dr}{1-r} < \infty$. Then there exists a constant $d \in (0,1)$ such that if $s(r) = 1 - d(1-r)$, then $g(r) \leq h(s(r))$ for all $r \in [0,1)$.*

Lemma 3.10 ([25])**.** *Let $p \geq q \geq 1$ be integers. If $A_0(z), \dots, A_{k-1}(z)$ are analytic functions of $[p,q]$-order in sector Ω satisfying $\max_{0\leq j\leq k-1}\{\rho_{[p,q],\Omega}(A_j)\} \leq \eta$, then for any given $\varepsilon \in (0, \frac{\beta-\alpha}{2})$, every solution $f \not\equiv 0$ of* (1.1) *satisfies*

$$\rho_{[p+1,q],\Omega_\varepsilon}(f) \leq \eta + 1.$$

Furthermore, if $p > q$ then $\rho_{[p+1,q],\Omega_\varepsilon}(f) \leq \eta$.

4. Proof of main results

Proof of Theorem 2.6. Suppose that $f \not\equiv 0$ is a solution of (1.1) in the sector Ω. From Lemma 3.5, the function $F(u) = f(z((u))$ is a solution of (3.7), where $z(u)$ is defined by (3.2). Then, by Lemma 3.2 and the properties of characteristic function of Nevanlinna, we have

$$\begin{aligned} T(\varrho, B_0(u)) &= T\Big(\varrho, \frac{1}{\alpha_k}A_0(z(u))\Big) \\ &\geq T(\varrho, A_0(z(u))) - T(\varrho, \alpha_k) \\ &= T_0(\varrho, \mathbb{C}, A_0(z(u))) + O(1) - T(\varrho, \alpha_k) \\ &\geq \frac{b}{2}T_0(1 - \frac{1-\varrho}{b}, \Omega_\varepsilon, A_0(z)) + O(1) - T(\varrho, \alpha_k). \end{aligned} \tag{4.1}$$

By (3.3), (3.11) and the formula $T(r,f) = T_0(r,\mathbb{C},f) + O(1)$ $(0 < r < 1)$, for $j = 1, 2, \dots, k-1$ we have

$$\begin{aligned} T(\varrho, B_j(u)) &\leq \sum_{n=j}^{k-1} T(\varrho, A_n(z(u))) + O\Big(\log\frac{1}{1-\varrho}\Big) \\ &= \sum_{n=j}^{k-1} T_0(\varrho, \mathbb{C}, A_n(z(u))) + O(1) + O\Big(\log\frac{1}{1-\varrho}\Big) \\ &\leq \frac{16\pi}{\delta}\sum_{n=j}^{k-1} T_0\Big(1 - \frac{\delta}{8\pi}(1-\varrho), \Omega, A_n(z)\Big) + O\Big(\log\frac{1}{1-\varrho}\Big). \end{aligned} \tag{4.2}$$

Set

$$\eta = \max_{1\leq j\leq k-1}\{\rho_{[p,q],\Omega}(A_j)\} < \mu_{[p,q],\Omega_\varepsilon}(A_0) = \mu.$$

Then, for any given $\epsilon (0 < 2\epsilon < \mu - \eta)$ and $r \to 1^-$, for $j = 1, 2, \dots, k-1$ we have

$$T_0(r, \Omega, A_j(z)) \leq \exp_p\big\{(\eta + \epsilon)\log_q\frac{1}{1-r}\big\}. \tag{4.3}$$

By the definition of lower $[p,q]$ order

$$T_0(r, \Omega_\varepsilon, A_0(z)) \geq \exp_p\big\{(\mu - \epsilon)\log_q\frac{1}{1-r}\big\}. \tag{4.4}$$

Now, as $|u| = \varrho \to 1^-$, it follows from (4.1), (4.2), (4.3) and (4.4) that

$$\begin{aligned} T(\varrho, B_0) &\geq \frac{b}{2}T_0\Big(1 - \frac{1-\varrho}{b}, \Omega_\varepsilon, A_0(z)\Big) + O(1) - T(\varrho, \alpha_k) \\ &\geq \frac{b}{2}\exp_p\big\{(\mu-\epsilon)\log_q\big(\frac{b}{1-\varrho}\big)\big\} + O(1) - T(\varrho, \alpha_k) \\ &= O\Big(\exp_p\big\{(\mu-\epsilon)\log_q\big(\frac{1}{1-\varrho}\big)\big\}\Big) - T(\varrho, \alpha_k) \end{aligned} \tag{4.5}$$

and for $j = 1, 2, \dots, k-1$,

$$\begin{aligned} T(\varrho, B_j) &\leq \frac{16\pi}{\delta}(k-j)\exp_p\big\{(\eta+\epsilon)\log_q\big(\frac{8\pi}{\delta(1-\varrho)}\big)\big\} + O\big(\log\frac{1}{1-\varrho}\big) \\ &= O\Big(\exp_p\big\{(\eta+\epsilon)\log_q\big(\frac{1}{1-\varrho}\big)\big\} + \log\frac{1}{1-\varrho}\Big). \end{aligned} \tag{4.6}$$

By (3.7), we can write

$$\begin{aligned} T(\varrho, B_0) &= m(\varrho, B_0) \\ &\leq \sum_{j=1}^{k-1} m(\varrho, B_j) + \sum_{j=1}^{k} m\big(\varrho, \frac{F^{(j)}}{F}\big) + O(1) \\ &= \sum_{j=1}^{k-1} T(\varrho, B_j) + \sum_{j=1}^{k} m\big(\varrho, \frac{F^{(j)}}{F}\big) + O(1). \end{aligned} \tag{4.7}$$

It follows from (4.5), (4.6), (4.7) and Lemma 3.8 that

$$\begin{aligned} &O\Big(\exp_p\Big\{(\mu-\epsilon)\log_q\big(\frac{1}{1-\varrho}\big)\Big\}\Big) \\ &\leq O\Big(\exp_p\Big\{(\eta+\epsilon)\log_q\big(\frac{1}{1-\varrho}\big)\Big\}\Big) \\ &\quad + O\big(\log\frac{1}{1-\varrho}\big) + T(\varrho, \alpha_k) + O\Big(\log^+ T(\varrho, F) + \log\frac{1}{1-\varrho}\Big) \end{aligned} \tag{4.8}$$

holds for all u satisfying $|u| = \varrho \notin E$ as $\varrho \to 1^-$ and $E \subset (0,1)$ is a set with $\int_E \frac{d\varrho}{1-\varrho} < +\infty$. By using Lemma 3.9 and (4.8), for all u satisfying $|u| = \varrho$, as $\varrho \to 1^-$, we obtain

$$\begin{aligned} &\exp_p\Big\{(\mu-\epsilon)\log_q\big(\frac{1}{1-\varrho}\big)\Big\} \\ &\leq O\Big(\exp_p\Big\{(\eta+\epsilon)\log_q\big(\frac{1}{1-\varrho}\big)\Big\}\Big) \\ &\quad + O\big(\log\frac{1}{d(1-\varrho)}\big) + O\big(\log^+ T(1-d(1-\varrho), F)\big). \end{aligned} \tag{4.9}$$

Thus, from (4.9) we obtain $\sigma_{[p,q]}(F) = \mu_{[p+1,q]}(F) = +\infty$ and

$$\sigma_{[p+1,q]}(F) \geq \mu_{[p+1,q]}(F) \geq \mu.$$

Then, by Remark 3.3, we obtain

$$\begin{gathered} \rho_{[p,q],\Omega}(f(z)) = \mu_{[p,q]}(f(z)) = +\infty, \\ \rho_{[p+1,q],\Omega}(f(z)) \geq \mu_{[p+1,q],\Omega}(f(z)) \geq \mu. \end{gathered}$$

On the other hand, by Lemma 3.7 we have $\mu_{[p+1,q],\Omega_\varepsilon}(f) \leq \mu_{[p,q],\Omega}(A_0) + 1$, and if $p > q$, we have $\mu_{[p+1,q],\Omega_\varepsilon}(f) \leq \mu_{[p,q],\Omega}(A_0)$. □

Proof of Corollary 2.8. By using Theorem 2.6 and Lemma 3.10, we easily obtain Corollary 2.8. □

Proof of the Theorem 2.9. Suppose that $f \not\equiv 0$ is a solution of (1.1) in the sector Ω. From Lemma 3.5, the function $F(u) = f(z((u))$ is a solution of (3.7), where $z(u)$ is defined by (3.2). If $\rho_{[p,q],\Omega}(A_j) < \mu_{[p,q],\Omega_\varepsilon}(A_0) = \mu$ for all $j = 1, \dots, k-1$, then Theorem 2.9 reduces to Theorem 2.6. Thus, we assume that at least one of

A_j $(j=1,\dots,k-1)$ satisfies $\rho_{[p,q],\Omega}(A_j)=\mu_{[p,q],\Omega_\varepsilon}(A_0)=\mu$. So, there exists a set $I\subseteq\{1,\dots,k-1\}$ such that for $j\in I$ we have $\rho_{[p,q],\Omega}(A_j)=\mu_{[p,q],\Omega_\varepsilon}(A_0)=\mu$ and

$$\tau_1=\max_{j\in I}\{\tau_{[p,q],\Omega}(A_j):\rho_{[p,q],\Omega}(A_j)=\mu_{[p,q],\Omega_\varepsilon}(A_0)\}<\underline{\tau}_{[p,q],\Omega_\varepsilon}(A_0)=\tau<+\infty$$

and for $j\in\{1,\dots,k-1\}\backslash I$, we have

$$b=\max_{j\in\{1,\dots,k-1\}\backslash I}\big\{\rho_{[p,q],\Omega}(A_j)\big\}<\mu_{[p,q],\Omega_\varepsilon}(A_0)=\mu.$$

Then, for any given ϵ $(0<2\epsilon<\min\{\mu-b,\tau-\tau_1\})$ and for $r\to 1^-$, and $j\in\{1,\dots,k-1\}\backslash I$, we have

$$T_0(r,\Omega,A_j(z))\le\exp_p\big\{(b+\epsilon)\log_q\frac{1}{1-r}\big\}\le\exp_p\big\{(\mu-\epsilon)\log_q\frac{1}{1-r}\big\}\tag{4.10}$$

and for $j\in I$, we obtain

$$T_0(r,\Omega,A_j(z))\le\exp_{p-1}\Big\{(\tau_1+\epsilon)\Big(\log_{q-1}\frac{1}{1-r}\Big)^{\mu}\Big\}.\tag{4.11}$$

By the definition of lower $[p,q]$-type, for $r\to 1^-$ we have

$$T_0(r,\Omega_\varepsilon,A_0(z))\ge\exp_{p-1}\Big\{(\tau-\epsilon)\Big(\log_{q-1}\frac{1}{1-r}\Big)^{\mu}\Big\}.\tag{4.12}$$

Then, by (4.1) and (4.12), as $|u|=\varrho\to 1^-$,

$$\begin{aligned}T(\varrho,B_0(u))&=T\Big(\varrho,\frac{1}{\alpha_k}A_0(z(u))\Big)\\&\ge\frac{b}{2}T_0\Big(1-\frac{1-\varrho}{b},\Omega_\varepsilon,A_0(z)\Big)+O(1)-T(\varrho,\alpha_k)\\&\ge\frac{b}{2}\exp_{p-1}\Big\{(\tau-\epsilon)\Big(\log_{q-1}\frac{b}{1-\varrho}\Big)^{\mu}\Big\}+O(1)-T(\varrho,\alpha_k)\\&=O\Big(\exp_{p-1}\Big\{(\tau-\epsilon)(\log_{q-1}\frac{1}{1-\varrho})^{\mu}\Big\}\Big)-T(\varrho,\alpha_k).\end{aligned}\tag{4.13}$$

Also, by (4.2), (4.10) and (4.11), for $j=1,2,\dots,k-1$,

$$\begin{aligned}T(\varrho,B_j)&\le\frac{16\pi}{\delta}\sum_{n=j}^{k-1}T_0\Big(1-\frac{\delta}{8\pi}(1-\varrho),\Omega,A_n(z)\Big)+O\Big(\log\frac{1}{1-\varrho}\Big)\\&\le O\Big(\exp_p\Big\{(\mu-\epsilon)\log_q\frac{8\pi}{\delta(1-\varrho)}\Big\}\Big)\\&\quad+O\Big(\exp_{p-1}\Big\{(\tau_1+\epsilon)\Big(\log_{q-1}\frac{8\pi}{\delta(1-\varrho)}\Big)^{\mu}\Big\}\Big)+O\Big(\log\frac{1}{1-\varrho}\Big)\\&=O\Big(\exp_{p-1}\Big\{(\tau_1+\epsilon)(\log_{q-1}\frac{1}{1-\varrho})^{\mu}\Big\}+\log\frac{1}{1-\varrho}\Big).\end{aligned}\tag{4.14}$$

It follows from (4.7), (4.13), (4.14) and Lemma 3.8 that

$$\begin{aligned}&O\Big(\exp_{p-1}\Big\{(\tau-\epsilon)\Big(\log_{q-1}\frac{1}{1-\varrho}\Big)^{\mu}\Big\}\Big)\\&\le O\Big(\exp_{p-1}\Big\{(\tau_1+\epsilon)\Big(\log_{q-1}\frac{1}{1-\varrho}\Big)^{\mu}\Big\}\Big)\\&\quad+O\Big(\log\frac{1}{1-\varrho}\Big)+T(\varrho,\alpha_k)+O\Big(\log^+T(\varrho,F)+\log\frac{1}{1-\varrho}\Big)\end{aligned}\tag{4.15}$$

holds for all u satisfying $|u| = \varrho \notin E$ as $\varrho \to 1^-$, where $E \subset (0,1)$ is a set with $\int_E \frac{d\varrho}{1-\varrho} < +\infty$. By using Lemma 3.9 and (4.15), for all u satisfying $|u| = \varrho \to 1^-$, we obtain

$$\begin{aligned}
&\exp_{p-1}\Big\{(\tau-\epsilon)\Big(\log_{q-1}\frac{1}{1-\varrho}\Big)^{\mu}\Big\}\\
&\le O\Big(\exp_{p-1}\Big\{(\tau_1+\epsilon)\Big(\log_{q-1}\frac{1}{d(1-\varrho)}\Big)^{\mu}\Big\}\Big) + O\Big(\log\frac{1}{d(1-\varrho)}\Big) \qquad (4.16)\\
&\quad + O\Big(\log^+ T(1-d(1-\varrho),F)\Big).
\end{aligned}$$

Thus, from this inequality we obtain $\rho_{[p,q]}(F) = \mu_{[p,q]}(F) = +\infty$ and

$$\rho_{[p+1,q]}(F) \ge \mu_{[p+1,q]}(F) \ge \mu.$$

Then, by Remark 3.3, we obtain

$$\rho_{[p,q],\Omega}(f(z)) = \mu_{[p,q]}(f(z)) = +\infty, \quad \rho_{[p+1,q],\Omega}(f(z)) \ge \mu_{[p+1,q],\Omega}(f(z)) \ge \mu.$$

On the other hand, by Lemma 3.7 we have $\mu_{[p+1,q],\Omega_\varepsilon}(f) \le \mu_{[p,q],\Omega}(A_0) + 1$, and if $p > q$, we have $\mu_{[p+1,q],\Omega_\varepsilon}(f) \le \mu_{[p,q],\Omega}(A_0)$. □

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