Electronic Journal of Differential Equations, Vol. 2019 (2019), No. 110, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF A UNIQUE SOLUTION TO AN ELLIPTIC PARTIAL DIFFERENTIAL EQUATION

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Abstract. The purpose of this article is to prove the existence of a unique classical solution to the quasilinear elliptic equation $-\nabla\cdot(a(u)\nabla u) = f$ for $\mathbf{x}\in\Omega$, which satisfies the condition that $u(\mathbf{x}_0) = u_0$ at a given point $\mathbf{x}_0\in\Omega$, under the boundary condition $\mathbf{n}(\mathbf{x})\cdot\nabla u(\mathbf{x}) = 0$ for $\mathbf{x}\in\partial\Omega$ where $\mathbf{n}(\mathbf{x})$ is the outward unit normal vector and where $\frac{1}{|\Omega|}\int_\Omega f\,d\mathbf{x} = 0$. The domain $\Omega\subset\mathbb{R}^N$ is a bounded, connected, open set with a smooth boundary, and $N = 2$ or $N = 3$. The key to the proof lies in obtaining a priori estimates for the solution.

1. Introduction

In this article, we consider the existence of a unique, classical solution $u(\mathbf{x})$ to the quasilinear elliptic equation

$$-\nabla\cdot(a(u)\nabla u) = f \tag{1.1}$$

for $\mathbf{x}\in\Omega$, which satisfies the condition

$$u(\mathbf{x}_0) = u_0, \tag{1.2}$$

where $\mathbf{x}_0$ is a given point in the domain Ω, under the Neumann boundary condition

$$\mathbf{n}(\mathbf{x})\cdot\nabla u(\mathbf{x}) = 0 \tag{1.3}$$

for $\mathbf{x}\in\partial\Omega$, where $\mathbf{n}(\mathbf{x})$ is the outward unit normal vector, and $\frac{1}{|\Omega|}\int_\Omega f\,d\mathbf{x} = 0$. The domain $\Omega\subset\mathbb{R}^N$ is a bounded, connected, open set with a smooth boundary $\partial\Omega$, and $N = 2$ or $N = 3$.

The purpose of this article is to prove the existence of a unique classical solution u to (1.1)–(1.3). The proof of the existence theorem uses the method of successive approximations in which an iteration scheme, based on solving a linearized version of equation (1.1), will be defined and then convergence of the sequence of approximating solutions to a unique solution satisfying the quasilinear equation will be proven. The key to the proof lies in obtaining a priori estimates for u.

This article is organized as follows. The main result, Theorem 2.1, is presented and proven in the next section. The existence of a solution to the linearized equation used in the iteration scheme is proven in Appendix A. Appendix B presents lemmas supporting the proof of the theorem.

2010 *Mathematics Subject Classification.* 35A05.
Key words and phrases. Existence; uniqueness; quasilinear; elliptic.

Submitted February 24, 2018. Published September 26, 2019.

2. Existence theorem

We will be working with the Sobolev space $H^s(\Omega)$ (where $s \geq 0$ is an integer) of real-valued functions in $L^2(\Omega)$ whose distribution derivatives up to order s are in $L^2(\Omega)$. The norm is $\|u\|_s^2 = \sum_{0\leq|\alpha|\leq s}\int_\Omega |D^\alpha u|^2 d\mathbf{x}$. We are using the standard multi-index notation. We define $|F|_{0,\overline{G}_0} = \max\{|F(u_*)| : u_* \in \overline{G}_0\}$, where F is a function of u and where $\overline{G}_0 \subset \mathbb{R}$ is a closed, bounded interval. Also, we let both ∇u and Du denote the gradient of u. And $C^k(\Omega)$ is the set of real-valued functions having all derivatives of order $\leq k$ continuous in Ω (where k = integer ≥ 0 or $k = \infty$). The purpose of this paper is to prove the following theorem:

Theorem 2.1. *Let a be a smooth, positive function of u such that*

$$\Big|\frac{d^2a}{du^2}\Big|_{0,\overline{G}_0} \leq \frac{1}{a_0}\Big|\frac{da}{du}\Big|^2_{0,\overline{G}_0},$$

where the constant $a_0 = \min_{u_*\in\overline{G}_0} a(u_*)$ and $\overline{G}_0 \subset \mathbb{R}$ is a closed, bounded interval. Let $u(\mathbf{x}_0) = u_0$ be a given value of u, where $\mathbf{x}_0 \in \Omega$ is a given point and where the domain Ω is a bounded, connected, open set in $\mathbb{R}^N$, and $N = 2$ or $N = 3$. Let the boundary $\partial\Omega$ be C^∞. Let $f \in H^2(\Omega)$ and let $\frac{1}{|\Omega|}\int_\Omega f\, d\mathbf{x} = 0$.

There exists a constant C_0 which depends only on N, Ω such that if

$$\frac{1}{(\min_{u_*\in\overline{G}_0} a(u_*))^4}\Big|\frac{da}{du}\Big|^2_{0,\overline{G}_0}\|\nabla f\|_0^2 \leq C_0$$

then there exists a unique solution $u \in C^2(\Omega)$ to the equation

$$-\nabla\cdot(a(u)\nabla u) = f \tag{2.1}$$

which satisfies the condition

$$u(\mathbf{x}_0) = u_0 \tag{2.2}$$

under the Neumann boundary condition

$$\mathbf{n}(\mathbf{x})\cdot\nabla u(\mathbf{x}) = 0 \tag{2.3}$$

for $\mathbf{x} \in \partial\Omega$, where $\mathbf{n}(\mathbf{x})$ is the outward unit normal vector.

Proof. We begin by using the change of variables

$$v = \Big(\frac{a_0}{\|\nabla f\|_0}\Big)u, \quad b(v) = \frac{1}{a_0}a\Big(\frac{\|\nabla f\|_0}{a_0}v\Big), \quad g = \Big(\frac{1}{\|\nabla f\|_0}\Big)f \tag{2.4}$$

where the constant $a_0 = \min_{u_*\in\overline{G}_0} a(u_*)$ and $\overline{G}_0 \subset \mathbb{R}$ is a closed, bounded interval.

Under this change of variables equation (2.1) becomes

$$-\nabla\cdot(b(v)\nabla v) = g \tag{2.5}$$

And under this change of variables, (2.2), (2.3) become

$$v(\mathbf{x}_0) = v_0 = \frac{a_0}{\|\nabla f\|_0}u_0, \tag{2.6}$$

$$\mathbf{n}(\mathbf{x})\cdot\nabla v(\mathbf{x}) = 0 \tag{2.7}$$

for $\mathbf{x} \in \partial\Omega$.

We fix bounded intervals $\overline{G}_0 \subset \mathbb{R}$ and $\overline{G}_1 \subset \mathbb{R}$ by $\overline{G}_0 = \{u_* \in \mathbb{R} : |u_* - u_0|_{L^\infty(\Omega)} \leq \frac{R\|\nabla f\|_0}{a_0}\}$ and $\overline{G}_1 = \{v_* \in \mathbb{R} : |v_* - v_0|_{L^\infty(\Omega)} \leq R\}$, where R is a constant to be defined later. We will prove that $v(\mathbf{x}) \in \overline{G}_1$ for $\mathbf{x} \in \Omega$. It follows that $u(\mathbf{x}) \in \overline{G}_0$ for $\mathbf{x} \in \Omega$.

We will construct the solution of problem (2.5), (2.6), (2.7) through an iteration scheme. To define the iteration scheme, we will let the sequence of approximate solutions be $\{v^k\}$. Set the initial iterate $v^0 = v_0$. For $k = 0, 1, 2, \dots$, construct v^{k+1} from the previous iterate v^k by solving the linear equation

$$-\nabla \cdot (b(v^k)\nabla v^{k+1}) = g \tag{2.8}$$

which satisfies the condition

$$v^{k+1}(\mathbf{x}_0) = v_0 \tag{2.9}$$

under the Neumann boundary condition

$$\mathbf{n}(\mathbf{x}) \cdot \nabla v^{k+1}(\mathbf{x}) = 0 \tag{2.10}$$

for $\mathbf{x} \in \partial\Omega$.

The existence of a unique solution $v^{k+1} \in C^2(\Omega)$ to the linear equation (2.8) for fixed k which satisfies (2.9), (2.10) is proven in Appendix A. Lemmas supporting the proof are presented in Appendix B. We proceed now to prove convergence of the iterates as $k \to \infty$ to a unique, classical solution v of (2.5), (2.6), (2.7), which therefore produces a unique, classical solution $u = \frac{\|\nabla f\|_0}{a_0} v$ of (2.1), (2.2), (2.3).

We begin by proving the following proposition.

Proposition 2.2. *Assume that the hypotheses of Theorem 2.1 hold. Then there exist constants C_1, C_2, C_3, and R such that the following inequalities hold for $k = 1, 2, 3 \dots$:*

$$\|\nabla v^k\|_2^2 \le C_1, \tag{2.11}$$

$$\|v^k\|_4^2 \le C_2, \tag{2.12}$$

$$|v^k - v_0|_{L^\infty(\Omega)} \le R, \tag{2.13}$$

$$\|\nabla(v^{k+1} - v^k)\|_1^2 \le \frac{1}{5}\|\nabla(v^k - v^{k-1})\|_1^2, \tag{2.14}$$

$$\|v^{k+1} - v^k\|_2^2 \le \Big(\frac{1}{5}\Big)^k C_3, \tag{2.15}$$

where the constants C_1, C_3 depend on N and Ω, and where the constant C_2 depends on R, v_0, a_0, $\|\nabla f\|_0$, $\|\nabla f\|_1$, $\big|\frac{d(\ln(a(u)))}{du}\big|_{2,\overline{G}_0}$, $\big|\frac{d((a(u))^{-1})}{du}\big|_{1,\overline{G}_0}$, N, and Ω. And the constant R depends on N and Ω. From (2.13) it follows that $v^k(\mathbf{x}) \in \overline{G}_1$ for $\mathbf{x} \in \Omega$ and for $k = 1, 2, 3 \dots$.

Proof. The proof is by induction on k. We prove in Lemma 4.3 in Appendix B that if v^k satisfies (2.11) and (2.13), then v^{k+1} satisfies (2.11) and (2.12). See Lemma 4.3 in Appendix B for the detailed proof.

It only remains to prove inequalities (2.13) for v^{k+1}, (2.14) for $\nabla(v^{k+1} - v^k)$ and (2.15) for $v^{k+1} - v^k$.

In the estimates below, we will let C denote a generic constant whose value may change from one relation to the next.

Estimate for $|v^{k+1} - v_0|_{L^\infty(\Omega)}$: From Lemmas 4.2 and 4.3 in Appendix B, we obtain the inequality

$$\begin{aligned} |v^{k+1} - v_0|_{L^\infty(\Omega)} &\le C\|\nabla(v^{k+1} - v_0)\|_1 \\ &\le C\|\nabla v^{k+1}\|_2 \\ &\le C\sqrt{C_1} = R \end{aligned} \tag{2.16}$$

where the constants C and C_1 depend on Ω and N. Here we used the fact that $v^{k+1}(\mathbf{x}_0) = v_0$, where v_0 is a constant. And we used the fact that $|v^{k+1}-v_0|_{L^\infty(\Omega)} \leq C\|\nabla(v^{k+1} - v_0)\|_1$ by Lemma 4.2 in Appendix B. And we used that fact that $\|\nabla v^{k+1}\|_1^2 \leq \|\nabla v^{k+1}\|_2^2 \leq C_1$ by Lemma 4.3 in Appendix B. We define $R = C\sqrt{C_1}$. Therefore inequality (2.13) of Proposition 2.2 holds for v^{k+1}.

Estimate for $\|\nabla(v^{k+1} - v^k)\|_1^2$: From subtracting successive iterates of equation (2.8) we obtain the identity

$$\begin{aligned}
-\nabla\cdot(b(v^k)\nabla(v^{k+1}-v^k)) &= -\nabla\cdot(b(v^k)\nabla v^{k+1}) + \nabla\cdot(b(v^k)\nabla v^k)\\
&= g + \nabla\cdot((b(v^k)-b(v^{k-1}))\nabla v^k) + \nabla\cdot(b(v^{k-1})\nabla v^k)\\
&= g + \nabla\cdot((b(v^k)-b(v^{k-1}))\nabla v^k) - g\\
&= \nabla\cdot((b(v^k)-b(v^{k-1}))\nabla v^k)
\end{aligned}$$

From this equality, we obtain the equation

$$\begin{aligned}
\Delta(v^{k+1}-v^k) = &-\frac{1}{b(v^k)}\nabla b(v^k)\cdot\nabla(v^{k+1}-v^k)\\
&-\frac{1}{b(v^k)}\nabla\cdot((b(v^k)-b(v^{k-1}))\nabla v^k)
\end{aligned} \tag{2.17}$$

We will be using a standard regularity estimate for the equation $\Delta v = h$ when $\mathbf{n}\cdot\nabla v = 0$ on the boundary $\partial\Omega$ (see, e.g., Bourguignon and Brezis [1], Embid [4]), namely

$$\|\nabla v\|_1^2 \leq C\|h\|_0^2 \tag{2.18}$$

where the constant C depends on N and Ω.

By the boundary condition (2.10), $\nabla(v^{k+1}-v^k)\cdot\mathbf{n} = 0$ on the boundary $\partial\Omega$. Applying (2.18) to the equation (2.17) yields the inequality

$$\begin{aligned}
&\|\nabla(v^{k+1}-v^k)\|_1^2\\
&\leq C\|\frac{1}{b(v^k)}\nabla b(v^k)\cdot\nabla(v^{k+1}-v^k) + \frac{1}{b(v^k)}\nabla\cdot((b(v^k)-b(v^{k-1}))\nabla v^k)\|_0^2\\
&\leq C\Big|\frac{1}{b}\Big|_{0,\overline{G}_0}^2\Big|\frac{db}{dv}\Big|_{0,\overline{G}_0}^2|\nabla v^k|_{L^\infty(\Omega)}^2\|\nabla(v^{k+1}-v^k)\|_0^2\\
&\quad + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_0}^2\|\nabla(b(v^k)-b(v^{k-1}))\cdot\nabla v^k\|_0^2 \qquad (2.19)\\
&\quad + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_0}^2\|(b(v^k)-b(v^{k-1}))\Delta v^k\|_0^2\\
&\leq C\Big|\frac{1}{b}\Big|_{0,\overline{G}_0}^2\Big|\frac{db}{dv}\Big|_{0,\overline{G}_0}^2\|\nabla v^k\|_2^2\|\nabla(v^{k+1}-v^k)\|_0^2\\
&\quad + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_0}^2\|(b'(v^k)-b'(v^{k-1}))\nabla v^k\cdot\nabla v^k \qquad (2.20)\\
&\quad + b'(v^{k-1})\nabla(v^k-v^{k-1})\cdot\nabla v^k\|_0^2\\
&\quad + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_0}^2|b(v^k)-b(v^{k-1})|_{L^\infty(\Omega)}^2\|\Delta v^k\|_0^2\\
&\leq C\Big|\frac{1}{b}\Big|_{0,\overline{G}_0}^2\Big|\frac{db}{dv}\Big|_{0,\overline{G}_0}^2\|\nabla v^k\|_2^2\|\nabla(v^{k+1}-v^k)\|_0^2\\
&\quad + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_0}^2|b'(v^k)-b'(v^{k-1})|_{L^\infty(\Omega)}^2|\nabla v^k|_{L^\infty(\Omega)}^2\|\nabla v^k\|_0^2
\end{aligned}$$

$$
\begin{aligned}
&+ C\Big|\frac{1}{b}\Big|^2_{0,\overline{G}_0}\Big|\frac{db}{dv}\Big|^2_{0,\overline{G}_0}\|\nabla(v^k-v^{k-1})\|_0^2|\nabla v^k|^2_{L^\infty(\Omega)}\\
&+ C\Big|\frac{1}{b}\Big|^2_{0,\overline{G}_0}|b(v^k)-b(v^{k-1})|^2_{L^\infty(\Omega)}\|\nabla v^k\|_1^2\\
&\le C\Big|\frac{1}{b}\Big|^2_{0,\overline{G}_0}\Big|\frac{db}{dv}\Big|^2_{0,\overline{G}_0}\|\nabla v^k\|_2^2\|\nabla(v^{k+1}-v^k)\|_0^2\\
&+ C\Big|\frac{1}{b}\Big|^2_{0,\overline{G}_0}\Big|\frac{d^2b}{dv^2}\Big|^2_{0,\overline{G}_0}|v^k-v^{k-1}|^2_{L^\infty(\Omega)}\|\nabla v^k\|_2^2\|\nabla v^k\|_0^2\\
&+ C\Big|\frac{1}{b}\Big|^2_{0,\overline{G}_0}\Big|\frac{db}{dv}\Big|^2_{0,\overline{G}_0}\|\nabla(v^k-v^{k-1})\|_0^2\|\nabla v^k\|_2^2\\
&+ C\Big|\frac{1}{b}\Big|^2_{0,\overline{G}_0}\Big|\frac{db}{dv}\Big|^2_{0,\overline{G}_0}|v^k-v^{k-1}|^2_{L^\infty(\Omega)}\|\nabla v^k\|_1^2 \qquad (2.21)
\end{aligned}
$$

where C depends on N, Ω. Here we used the fact that $\|\Delta v^k\|_0^2 = \|\nabla\cdot(\nabla v^k)\|_0^2 \le C\|\nabla v^k\|_1^2$. And we used the facts that $\big|\frac{1}{b(v_*)}\big| \le \big|\frac{1}{b}\big|_{0,\overline{G}_1}$ and that $\big|\frac{db}{dv}(v_*)\big| \le \big|\frac{db}{dv}\big|_{0,\overline{G}_1}$ and that $\big|\frac{d^2b}{dv^2}(v_*)\big| \le \big|\frac{d^2b}{dv^2}\big|_{0,\overline{G}_1}$ for $v_*(\mathbf{x}) \in \overline{G}_1$, where $v^k(\mathbf{x}) \in \overline{G}_1$ and $v^{k-1}(\mathbf{x}) \in \overline{G}_1$ for $\mathbf{x} \in \Omega$, so $v_* = tv^k+(1-t)v^{k-1} \in \overline{G}_1$ for any $t \in [0,1]$. And we used the standard Sobolev space inequality $|f|_{L^\infty(\Omega)} \le C\|f\|_{s_0}$, where the constant C depends on N, Ω, and where $s_0 = [\frac{N}{2}]+1 = 2$ when $N=2$ or $N=3$ (see, e.g, Embid [4], Evans [5], Majda [8]).

Using the fact that $\|\nabla v^k\|_2^2 \le C_1$ by the induction hypothesis, we obtain from (2.21) the inequality

$$
\begin{aligned}
&\|\nabla(v^{k+1}-v^k)\|_1^2\\
&\le CC_1\Big|\frac{1}{b}\Big|^2_{0,\overline{G}_1}\Big|\frac{db}{dv}\Big|^2_{0,\overline{G}_1}\|\nabla(v^{k+1}-v^k)\|_0^2\\
&\quad + CC_1^2\Big|\frac{1}{b}\Big|^2_{0,\overline{G}_1}\Big|\frac{d^2b}{dv^2}\Big|^2_{0,\overline{G}_1}|v^k-v^{k-1}|^2_{L^\infty(\Omega)} + CC_1\Big|\frac{1}{b}\Big|^2_{0,\overline{G}_1}\Big|\frac{db}{dv}\Big|^2_{0,\overline{G}_1}\|\nabla(v^k-v^{k-1})\|_0^2\\
&\quad + CC_1\Big|\frac{1}{b}\Big|^2_{0,\overline{G}_1}\Big|\frac{db}{dv}\Big|^2_{0,\overline{G}_1}|v^k-v^{k-1}|^2_{L^\infty(\Omega)}\\
&\le CC_1\Big|\frac{1}{b}\Big|^2_{0,\overline{G}_1}\Big|\frac{db}{dv}\Big|^2_{0,\overline{G}_1}\|\nabla(v^{k+1}-v^k)\|_1^2 + CC_1^2\Big|\frac{1}{b}\Big|^2_{0,\overline{G}_1}\Big|\frac{d^2b}{dv^2}\Big|^2_{0,\overline{G}_1}\|\nabla(v^k-v^{k-1})\|_1^2\\
&\quad + CC_1\Big|\frac{1}{b}\Big|^2_{0,\overline{G}_1}\Big|\frac{db}{dv}\Big|^2_{0,\overline{G}_1}\|\nabla(v^k-v^{k-1})\|_1^2 + CC_1\Big|\frac{1}{b}\Big|^2_{0,\overline{G}_1}\Big|\frac{db}{dv}\Big|^2_{0,\overline{G}_1}\|\nabla(v^k-v^{k-1})\|_1^2\\
&\le CC_1\frac{\|\nabla f\|_0^2}{a_0^4}\Big|\frac{da}{du}\Big|^2_{0,\overline{G}_0}\|\nabla(v^{k+1}-v^k)\|_1^2 + CC_1^2\frac{\|\nabla f\|_0^4}{a_0^6}\Big|\frac{d^2a}{du^2}\Big|^2_{0,\overline{G}_0}\|\nabla(v^k-v^{k-1})\|_1^2\\
&\quad + CC_1\frac{\|\nabla f\|_0^2}{a_0^4}\Big|\frac{da}{du}\Big|^2_{0,\overline{G}_0}\|\nabla(v^k-v^{k-1})\|_1^2\\
&\le CC_1\frac{\|\nabla f\|_0^2}{a_0^4}\Big|\frac{da}{du}\Big|^2_{0,\overline{G}_0}\|\nabla(v^{k+1}-v^k)\|_1^2 + CC_1^2\frac{\|\nabla f\|_0^4}{a_0^8}\Big|\frac{da}{du}\Big|^4_{0,\overline{G}_0}\|\nabla(v^k-v^{k-1})\|_1^2\\
&\quad + CC_1\frac{\|\nabla f\|_0^2}{a_0^4}\Big|\frac{da}{du}\Big|^2_{0,\overline{G}_0}\|\nabla(v^k-v^{k-1})\|_1^2\\
&\le CC_1C_0\|\nabla(v^{k+1}-v^k)\|_1^2 + CC_1^2C_0^2\|\nabla(v^k-v^{k-1})\|_1^2 + CC_1C_0\|\nabla(v^k-v^{k-1})\|_1^2\\
&\le CC_1^2C_0\|\nabla(v^{k+1}-v^k)\|_1^2 + CC_1^2C_0\|\nabla(v^k-v^{k-1})\|_1^2
\end{aligned}
$$

$$
\begin{aligned}
&= C_4 C_0 \|\nabla(v^{k+1} - v^k)\|_1^2 + C_4 C_0 \|\nabla(v^k - v^{k-1})\|_1^2 \\
&\le \frac{1}{6}\|\nabla(v^{k+1} - v^k)\|_1^2 + \frac{1}{6}\|\nabla(v^k - v^{k-1})\|_1^2
\end{aligned}
\tag{2.22}
$$

where we define $C_4 = CC_1^2$. Here we used the fact that $\left|\frac{1}{b}\right|_{0,\overline{G}_1} = 1$, where $b(v) = \frac{1}{a_0} a(\frac{\|\nabla f\|_0}{a_0} v)$ and where $a_0 = \min_{u_* \in \overline{G}_0} a(u_*)$. And we used the facts that $\frac{1}{a_0^4}\left|\frac{da}{du}\right|_{1,\overline{G}_0}^2 \|\nabla f\|_0^2 \le C_0$, and that $\left|\frac{d^2 a}{du^2}\right|_{0,\overline{G}_0} \le \frac{1}{a_0}\left|\frac{da}{du}\right|_{0,\overline{G}_0}^2$ by assumption from the statement of the theorem. And we used the Sobolev space inequality $|v^k - v^{k-1}|_{L^\infty(\Omega)}^2 \le C\|\nabla(v^k - v^{k-1})\|_1^2$ by Lemma 4.2 in Appendix B, where we used the fact that $v^{k-1}(\mathbf{x}_0) = v^k(\mathbf{x}_0) = v_0$. We now define the constant C_0 to be sufficiently small so that $C_4 C_0 \le \max\{C_4, C_5\} C_0 \le \frac{1}{6}$ where C_5 is a constant which is defined in Lemma 4.3 in Appendix B, and where C, C_1, C_4, C_5, C_0 depend on N, Ω, and where we may assume that $C_1 > 1$ and $C_4 > 1$ and $C_5 > 1$ and $C_0 < 1$.

After re-arranging the terms in (2.22) we obtain

$$
\|\nabla(v^{k+1} - v^k)\|_1^2 \le \frac{1}{5}\|\nabla(v^k - v^{k-1})\|_1^2 \tag{2.23}
$$

Therefore inequality (2.14) of Proposition 2.2 holds for $\nabla(v^{k+1} - v^k)$.

Repeatedly applying (2.23) yields for $k \ge 1$:

$$
\begin{aligned}
\|\nabla(v^{k+1} - v^k)\|_1^2 &\le \big(\frac{1}{5}\big)^k \|\nabla(v^1 - v^0)\|_1^2 \\
&= \big(\frac{1}{5}\big)^k \|\nabla v^1\|_1^2 \\
&\le \big(\frac{1}{5}\big)^k C_1
\end{aligned}
\tag{2.24}
$$

where we used the fact that $\|\nabla v^1\|_1^2 \le \|\nabla v^1\|_2^2 \le C_1$ by Lemma 4.3 in Appendix B, where the constant C_1 depends on Ω and N.

By Lemma 4.2 in Appendix B,

$$
\|v^{k+1} - v^k\|_2^2 \le C\|\nabla(v^{k+1} - v^k)\|_1^2 \tag{2.25}
$$

where we used the fact that $v^{k+1}(\mathbf{x}_0) = v^k(\mathbf{x}_0) = v_0$. Here the constant C depends on N, Ω.

Therefore from (2.24)–(2.25), it follows that

$$
\begin{aligned}
\|v^{k+1} - v^k\|_2^2 &\le C\|\nabla(v^{k+1} - v^k)\|_1^2 \\
&\le \big(\frac{1}{5}\big)^k C C_1 \\
&= \big(\frac{1}{5}\big)^k C_3
\end{aligned}
\tag{2.26}
$$

where the constant C_3 depends on N, Ω.

Therefore inequality (2.15) of Proposition 2.2 holds for $v^{k+1} - v^k$. This completes the proof of Proposition 2.2. □

Now we complete the proof of Theorem 2.1. From inequality (2.26) it follows that $\|v^{k+1} - v^k\|_2 \to 0$ as $k \to \infty$. From the estimate (2.12) for $\|v^k\|_4$ and for $\|v^{k+1}\|_4$, and from the standard interpolation inequality $\|v^{k+1} - v^k\|_r \le C\|v^{k+1} - v^k\|_2^\beta \|v^{k+1} - v^k\|_4^{1-\beta}$, where $\beta = \frac{4-r}{2}$, and $2 < r < 4$, it follows that $\|v^{k+1} - v^k\|_r \to 0$ as $k \to \infty$ for $2 < r < 4$. Therefore there exists $v \in H^r(\Omega)$ such that $\|v^k - v\|_r \to 0$

as $k \to \infty$. The fact that $v \in H^4(\Omega)$ can be deduced using boundedness in high norm and a standard compactness argument (see, for example, Embid [4], Majda [8]). Sobolev's lemma implies that $v \in C^2(\Omega)$.

From Lemma 3.1 in Appendix A, $v^{k+1} \in C^2(\Omega)$ is a solution of the linear equation $-\nabla \cdot (b(v^k)\nabla v^{k+1}) = g$ for each $k \geq 0$, and v^{k+1} satisfies the condition that $v^{k+1}(\mathbf{x}_0) = v_0$, and v^{k+1} satisfies the boundary condition $\mathbf{n}(\mathbf{x}) \cdot \nabla v^{k+1}(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial\Omega$. It follows that v is a classical solution of the equation $-\nabla \cdot (b(v)\nabla v) = g$, and v satisfies the condition that $v(\mathbf{x}_0) = v_0$, and v satisfies the boundary condition $\mathbf{n}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial\Omega$. The uniqueness of the solution follows by a standard proof using estimates similar to the estimates used in the proof of the contraction inequality (2.14). Therefore, there exists a unique classical solution $u = \big(\frac{\|\nabla f\|_0}{a_0}\big)v$ of $-\nabla \cdot (a(u)\nabla u) = f$ which satisfies the condition that $u(\mathbf{x}_0) = u_0$, and satisfies the boundary condition $\mathbf{n}(\mathbf{x}) \cdot \nabla u(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial\Omega$. This completes the proof of the theorem. □

3. Appendix A: Existence for the linear equation

In this section, we present the proof of the existence of a unique, classical solution to the linear problem (2.8)–(2.10).

Lemma 3.1. *Let b be a smooth positive function of w. Let $w \in C^2(\Omega)$ and let $w(\mathbf{x}) \in \overline{G}_1 \subset \mathbb{R}$ for $\mathbf{x} \in \Omega$. Let $g \in H^2(\Omega)$ and let $\frac{1}{|\Omega|}\int_\Omega g\,d\mathbf{x} = 0$, where the domain $\Omega \subset \mathbb{R}^N$ is a bounded, connected, open set, with $N = 2$ or $N = 3$, and where the boundary $\partial\Omega$ is C^∞. Then there exists a unique solution $v \in C^2(\Omega)$ of the equation*

$$-\nabla \cdot (b(w)\nabla v) = g \tag{3.1}$$

which satisfies the condition

$$v(\mathbf{x}_0) = v_0 \tag{3.2}$$

where $\mathbf{x}_0 \in \Omega$ is a given point and v_0 is a given value, under the Neumann boundary condition

$$\mathbf{n}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) = 0 \tag{3.3}$$

for $\mathbf{x} \in \partial\Omega$.

Proof. We define the zero-mean function

$$\overline{v} = v - \frac{1}{|\Omega|}\int_\Omega v\,d\mathbf{x} \tag{3.4}$$

The existence of a unique zero-mean solution $\overline{v} \in C^2(\Omega)$ to equation (3.1) with a Neumann boundary condition $\mathbf{n}(\mathbf{x}) \cdot \nabla\overline{v}(\mathbf{x}) = 0$ on the boundary $\partial\Omega$, is a well-known result from the standard theory of elliptic equations (see, e.g., Embid [4], Evans [5], Gilbarg and Trudinger [6]).

It follows that the function v defined by

$$v(\mathbf{x}) = \overline{v}(\mathbf{x}) + v_0 - \overline{v}(\mathbf{x}_0) \tag{3.5}$$

is the unique solution to equation (3.1) which satisfies the condition (3.2) that $v(\mathbf{x}_0) = v_0$, and also satisfies the boundary condition (3.3) that $\mathbf{n}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial\Omega$. This completes the proof of the lemma. □

4. Appendix B: A priori estimates

In this section, we present lemmas supporting the proof of the theorem. We begin by listing several standard Sobolev space inequalities.

Lemma 4.1 (Standard Sobolev Space Inequalities)**.**
(a) Let $h(w)$ be a smooth function of w, where $w(\mathbf{x})$ is a continuous function and where $w(\mathbf{x}) \in \overline{G}_1 \subset \mathbb{R}$ for $\mathbf{x} \in \Omega \subset \mathbb{R}^N$ and $w \in H^{r+1}(\Omega) \cap L^\infty(\Omega)$. Then for $r \geq 0$,

$$\|\nabla(h(w))\|_r \leq C\Big|\frac{dh}{dw}\Big|_{r,\overline{G}_1}(1+|w|_{L^\infty(\Omega)})^r\|\nabla w\|_r, \tag{4.1}$$

where $|F|_{r,\overline{G}_1} = \max\{|\frac{d^jF}{dw^j}(w_*)| : w_* \in \overline{G}_1, 0 \leq j \leq r\}$, and where C depends on r, N, and Ω.

(b) If $f \in H^m(\Omega)$, $g \in H^n(\Omega)$, and $r = \min\{m, n, m+n-s_0\} \geq 0$, where $\Omega \subset \mathbb{R}^N$ and $s_0 = [\frac{N}{2}] + 1$, then $fg \in H^r(\Omega)$ and

$$\|fg\|_r \leq C\|f\|_m\|g\|_n \tag{4.2}$$

Here C is a constant which depends on m, n, N, Ω.

(c) If $f \in H^{s_0}(\Omega)$ where $\Omega \subset \mathbb{R}^N$ and $s_0 = [\frac{N}{2}] + 1$, then

$$|f|_{L^\infty(\Omega)} \leq C\|f\|_{s_0} \tag{4.3}$$

Here C is a constant which depends on N, Ω.

(d) If $f \in H^n(\Omega)$, where $\Omega \subset \mathbb{R}^N$, and $r = \beta m + (1-\beta)n$, with $0 \leq \beta \leq 1$ and $m < n$, then

$$\|f\|_r \leq C\|f\|_m^\beta\|f\|_n^{1-\beta} \tag{4.4}$$

Here C is a constant which depends on m, n, N, Ω.

These inequalities are well-known. Their proofs may be found, for example, in [7, 9]. These inequalities also appear in [2, 4].

Lemma 4.2. *Let f, g be $H^r(\Omega)$ functions on a bounded domain $\Omega \subset \mathbb{R}^N$, where $N = 2$ or $N = 3$ and $r \geq 2$. And let $f(\mathbf{x}_0) = g(\mathbf{x}_0)$ at a point $\mathbf{x}_0 \in \Omega$. Then $f - g$ satisfies the following inequalities:*

$$\|f-g\|_0^2 \leq C\|\nabla(f-g)\|_1^2, \tag{4.5}$$

$$\|f-g\|_r^2 \leq C\|\nabla(f-g)\|_{r-1}^2, \quad r \geq 2 \tag{4.6}$$

$$|f-g|_{L^\infty(\Omega)}^2 \leq C\|\nabla(f-g)\|_1^2 \tag{4.7}$$

$$\|f-g\|_1^2 \leq C\|\nabla(f-g)\|_1^2, \tag{4.8}$$

Here C is a constant which depends on N, Ω.

Proof. A proof of the inequality (4.5) appears in [3]. We now use inequality (4.5) to prove the remaining inequalities (4.6)–(4.8).

From (4.5) we obtain the following inequality for $r \geq 2$:

$$\begin{aligned}\|f-g\|_r^2 &\leq \|f-g\|_0^2 + C\|\nabla(f-g)\|_{r-1}^2 \\ &\leq C\|\nabla(f-g)\|_1^2 + C\|\nabla(f-g)\|_{r-1}^2 \quad \leq C\|\nabla(f-g)\|_{r-1}^2\end{aligned} \tag{4.9}$$

for $r \geq 2$. This completes the proof of (4.6).

From (4.6) with $r = 2$, we obtain

$$|f-g|_{L^\infty}^2 \leq C\|f-g\|_2^2 \leq C\|\nabla(f-g)\|_1^2 \tag{4.10}$$

where the constant C depends on N, Ω. Here we used the Sobolev lemma to obtain $|f-g|_{L^\infty(\Omega)} \le C\|f-g\|_{H^{s_0}(\Omega)}$, where $s_0 = [\frac{N}{2}]+1 = 2$ for $N=2$ or $N=3$ (see, e.g., [4, 8]). This completes the proof of (4.7).

The inequality (4.8) for $\|f-g\|_1^2$ follows immediately from applying (4.6) with $r=2$ and using the fact that $\|f-g\|_1^2 \le \|f-g\|_2^2$. □

Lemma 4.3. *Let b be a smooth positive function of w defined by*

$$b(w) = \frac{1}{a_0} a\Big(\frac{\|\nabla f\|_0}{a_0} w\Big),$$

where $a_0 = \min_{u_* \in \overline{G}_0} a(u_*)$ and $\overline{G}_0 \subset \mathbb{R}$ is a closed, bounded interval. Let $w \in C^2(\Omega)$ and $g \in H^2(\Omega)$, where $\frac{1}{|\Omega|}\int_\Omega g\,d\mathbf{x} = 0$ and where $g = \big(\frac{1}{\|\nabla f\|_0}\big) f$. The domain $\Omega \subset \mathbb{R}^N$ is a bounded, connected, open set, with $N=2$ or $N=3$, and the boundary $\partial\Omega$ is C^∞. Let $w(\mathbf{x}) \in \overline{G}_1$ for $\mathbf{x} \in \Omega$, where $\overline{G}_1 \subset \mathbb{R}$ is a closed, bounded interval, and let $|w-v_0|_{L^\infty(\Omega)} \le R$ where v_0 and R are constants. Let $\|\nabla w\|_2^2 \le C_1$, where the constant C_1 depends on N, Ω. Let $\big(\frac{\|\nabla f\|_0^2}{a_0^4}\big)\big|\frac{da}{du}\big|_{0,\overline{G}_0}^2 \le C_0$, where C_0 is sufficiently small so that $\max\{C_4, C_5\}C_0 \le \frac{1}{6}$, where the constants C_4, C_5 depend on N, Ω. And let $\Big|\frac{d^2 a}{du^2}\Big|_{0,\overline{G}_0} \le \frac{1}{a_0}\big|\frac{da}{du}\big|_{0,\overline{G}_0}^2$.

Let $v \in C^2(\Omega)$ be the solution from Lemma 3.1 of

$$-\nabla\cdot(b(w)\nabla v) = g \tag{4.11}$$

which satisfies the condition

$$v(\mathbf{x}_0) = v_0 \tag{4.12}$$

where $\mathbf{x}_0 \in \Omega$ is a given point and v_0 is a given constant, under the Neumann boundary condition

$$\mathbf{n}(\mathbf{x})\cdot\nabla v(\mathbf{x}) = 0 \tag{4.13}$$

for $\mathbf{x} \in \partial\Omega$.

Then ∇v and v satisfy the inequalities

$$\|\nabla v\|_2^2 \le C_1, \quad \|v\|_4^2 \le C_2 \tag{4.14}$$

where the constant C_1 depends on N and Ω and where the constant C_2 depends on R, v_0, a_0, $\|\nabla f\|_0$, $\|\nabla f\|_1$, $\big|\frac{d(\ln(a(u)))}{du}\big|_{2,\overline{G}_0}$, $\big|\frac{d((a(u))^{-1})}{du}\big|_{1,\overline{G}_0}$, N, and Ω.

Proof. In the estimates below, we will let C denote a generic constant whose value may change from one relation to the next.

Estimate for $\|\nabla v\|_2^2$: We write equation (4.11) equivalently as

$$\Delta v = -\nabla(\ln(b(w)))\cdot\nabla v - \frac{1}{b(w)} g \tag{4.15}$$

We will be using the standard regularity estimate for the equation $\Delta v = h$, when $\mathbf{n}\cdot\nabla v = 0$ on the boundary $\partial\Omega$ (see, e.g., Bourguignon and Brezis [1], Embid [4])

$$\|\nabla v\|_r^2 \le C\|h\|_{r-1}^2 \tag{4.16}$$

where $r \ge 1$ and where the constant C depends on r, N, and Ω.

In the following estimate, we will use the notation $f_\alpha = D^\alpha f$. Applying (4.16) to the equation (4.15) and letting $r=2$, and using the Sobolev space inequalities in Lemma 4.1, yields

$$\|\nabla v\|_2^2 \le C\|\nabla(\ln(b(w)))\cdot\nabla v + \frac{1}{b(w)} g\|_1^2$$

$$
\begin{aligned}
&= C \sum_{0\le|\alpha|\le1} \|D^\alpha\Big(\nabla(\ln(b(w)))\cdot\nabla v + \frac{1}{b(w)}g\Big)\|_0^2 \\
&\le C\|\nabla(\ln(b(w)))\cdot\nabla v\|_0^2 + C\sum_{|\alpha|=1}\|D^\alpha(\nabla(\ln(b(w)))\cdot\nabla v)\|_0^2 \\
&\quad + C\|\frac{1}{b(w)}g\|_0^2 + C\sum_{|\alpha|=1}\|D^\alpha\Big(\frac{1}{b(w)}g\Big)\|_0^2 \\
&\le C|\nabla(\ln(b(w)))|_{L^\infty(\Omega)}^2\|\nabla v\|_0^2 \\
&\quad + C\sum_{|\alpha|=1}\Big(\|(\nabla(\ln(b(w))))_\alpha\cdot\nabla v\|_0^2 + \|\nabla(\ln(b(w)))\cdot\nabla v_\alpha\|_0^2\Big) \\
&\quad + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2\|g\|_0^2 + C\sum_{|\alpha|=1}\Big(\|\frac{1}{b(w)}g_\alpha\|_0^2 + \|\Big(\frac{1}{b(w)}\Big)_\alpha g\|_0^2\Big) \\
&\le C\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2\Big|\frac{db}{dw}\Big|_{0,\overline{G}_1}^2|\nabla w|_{L^\infty(\Omega)}^2\|\nabla v\|_0^2 + C\sum_{|\alpha|=1}\|(\frac{1}{b(w)}\frac{db}{dw}\nabla w)_\alpha\|_0^2|\nabla v|_{L^\infty(\Omega)}^2 \\
&\quad + C\sum_{|\alpha|=1}\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2\Big|\frac{db}{dw}\Big|_{0,\overline{G}_1}^2|\nabla w|_{L^\infty(\Omega)}^2\|\nabla v_\alpha\|_0^2 + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2\|g\|_0^2 \\
&\quad + C\sum_{|\alpha|=1}\Big(\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2\|g_\alpha\|_0^2 + \Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^4\Big|\frac{db}{dw}\Big|_{0,\overline{G}_1}^2\|w_\alpha g\|_0^2\Big) \\
&\le C\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2\Big|\frac{db}{dw}\Big|_{0,\overline{G}_1}^2|\nabla w|_{L^\infty(\Omega)}^2\|\nabla v\|_0^2 \\
&\quad + C\sum_{|\alpha|=1}\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2\Big|\frac{db}{dw}\Big|_{0,\overline{G}_1}^2\|\nabla w_\alpha\|_0^2|\nabla v|_{L^\infty(\Omega)}^2 \\
&\quad + C\sum_{|\alpha|=1}\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^4\Big|\frac{db}{dw}\Big|_{0,\overline{G}_1}^4\|w_\alpha\|_0^2|\nabla w|_{L^\infty(\Omega)}^2|\nabla v|_{L^\infty(\Omega)}^2 \\
&\quad + C\sum_{|\alpha|=1}\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2\Big|\frac{d^2b}{dw^2}\Big|_{0,\overline{G}_1}^2\|w_\alpha\|_0^2|\nabla w|_{L^\infty(\Omega)}^2|\nabla v|_{L^\infty(\Omega)}^2 \\
&\quad + C\sum_{|\alpha|=1}\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2\Big|\frac{db}{dw}\Big|_{0,\overline{G}_1}^2|\nabla w|_{L^\infty(\Omega)}^2\|\nabla v_\alpha\|_0^2 + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2\|g\|_0^2 \\
&\quad + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2\|\nabla g\|_0^2 + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^4\Big|\frac{db}{dw}\Big|_{0,\overline{G}_1}^2|\nabla w|_{L^\infty(\Omega)}^2\|g\|_0^2 \\
&\le C\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2\Big|\frac{db}{dw}\Big|_{0,\overline{G}_1}^2\|\nabla w\|_2^2\|\nabla v\|_0^2 \\
&\quad + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2\Big|\frac{db}{dw}\Big|_{0,\overline{G}_1}^2\|\nabla w\|_1^2\|\nabla v\|_2^2 \\
&\quad + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^4\Big|\frac{db}{dw}\Big|_{0,\overline{G}_1}^4\|\nabla w\|_0^2\|\nabla w\|_2^2\|\nabla v\|_2^2 \\
&\quad + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2\Big|\frac{d^2b}{dw^2}\Big|_{0,\overline{G}_1}^2\|\nabla w\|_0^2\|\nabla w\|_2^2\|\nabla v\|_2^2 \\
&\quad + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2\Big|\frac{db}{dw}\Big|_{0,\overline{G}_1}^2\|\nabla w\|_2^2\|\nabla v\|_1^2 + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2\|\nabla g\|_0^2
\end{aligned}
$$

$$
\begin{aligned}
&+ C\Big|\frac{1}{b}\Big|^4_{0,\overline{G}_1}\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}\|\nabla w\|_2^2\|\nabla g\|_0^2\\
&\le CC_1\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}\|\nabla v\|_0^2 + CC_1\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}\|\nabla v\|_2^2 + CC_1^2\Big|\frac{db}{dw}\Big|^4_{0,\overline{G}_1}\|\nabla v\|_2^2\\
&+ CC_1^2\Big|\frac{d^2b}{dw^2}\Big|^2_{0,\overline{G}_1}\|\nabla v\|_2^2 + CC_1\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}\|\nabla v\|_1^2 + C\|\nabla g\|_0^2\\
&+ CC_1\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}\|\nabla g\|_0^2\\
&\le C\Big(C_1\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1} + C_1^2\Big|\frac{db}{dw}\Big|^4_{0,\overline{G}_1} + C_1^2\Big|\frac{d^2b}{dw^2}\Big|^2_{0,\overline{G}_1}\Big)\|\nabla v\|_2^2 + C + CC_1\Big|\frac{db}{dw}\Big|^2_{0,\overline{G}_1}\\
&\le C\Big(C_1\Big(\frac{\|\nabla f\|_0^2}{a_0^4}\Big)\Big|\frac{da}{du}\Big|^2_{0,\overline{G}_0} + C_1^2\Big(\frac{\|\nabla f\|_0^4}{a_0^8}\Big)\Big|\frac{da}{du}\Big|^4_{0,\overline{G}_0}\\
&+ C_1^2\Big(\frac{\|\nabla f\|_0^4}{a_0^6}\Big)\Big|\frac{d^2a}{du^2}\Big|^2_{0,\overline{G}_0}\Big)\|\nabla v\|_2^2 + C + CC_1\Big(\frac{\|\nabla f\|_0^2}{a_0^4}\Big)\Big|\frac{da}{du}\Big|^2_{0,\overline{G}_0}\\
&\le C\Big(C_1\Big(\frac{\|\nabla f\|_0^2}{a_0^4}\Big)\Big|\frac{da}{du}\Big|^2_{0,\overline{G}_0} + C_1^2\Big(\frac{\|\nabla f\|_0^4}{a_0^8}\Big)\Big|\frac{da}{du}\Big|^4_{0,\overline{G}_0}\Big)\|\nabla v\|_2^2\\
&+ C + CC_1\Big(\frac{\|\nabla f\|_0^2}{a_0^4}\Big)\Big|\frac{da}{du}\Big|^2_{0,\overline{G}_0}\\
&\le \big(CC_1C_0 + CC_1^2C_0^2\big)\|\nabla v\|_2^2 + C + CC_1C_0\\
&\le CC_1^2C_0\|\nabla v\|_2^2 + C + CC_1^2C_0\\
&= C_5C_0\|\nabla v\|_2^2 + C + C_5C_0\\
&\le \frac{1}{6}\|\nabla v\|_2^2 + C + \frac{1}{6} \qquad (4.17)
\end{aligned}
$$

where we define $C_5 = CC_1^2$. Here we used the facts that $b(w) = \frac{1}{a_0}a(\frac{\|\nabla f\|_0}{a_0}w)$ and that $\big(\frac{\|\nabla f\|_0^2}{a_0^4}\big)\big|\frac{da}{du}\big|^2_{0,\overline{G}_0} \le C_0$, where C_0 is sufficiently small so that $C_5C_0 \le \max\{C_4, C_5\}C_0 \le \frac{1}{6}$ where C_4 is a constant which was defined in the proof of Proposition 2.2, and where C_4, C_5, C_0 depend on N, Ω. And we may assume that $C_1 > 1$ and $C_4 > 1$ and $C_5 > 1$ and $C_0 < 1$. And we used the facts that $\big|\frac{1}{b(w_*)}\big| \le \big|\frac{1}{b}\big|_{0,\overline{G}_1}$ and that $\big|\frac{db}{dw}(w_*)\big| \le \big|\frac{db}{dw}\big|_{0,\overline{G}_1}$ and that $\big|\frac{d^2b}{dw^2}(w_*)\big| \le \big|\frac{d^2b}{dw^2}\big|_{0,\overline{G}_1}$ because $w_*(\mathbf{x}) \in \overline{G}_1$ for $\mathbf{x} \in \Omega$. And we used the facts that $\big|\frac{1}{b}\big|_{0,\overline{G}_1} = 1$ by the definition of the function b in (2.4), and that $\|\nabla g\|_0^2 = 1$. We also used the fact that $\|\nabla w\|_2^2 \le C_1$. And we used the fact that $\big|\frac{d^2a}{du^2}\big|_{0,\overline{G}_0} \le \frac{1}{a_0}\big|\frac{da}{du}\big|^2_{0,\overline{G}_0}$. We also used Poincaré's inequality to obtain $\|g\|_0^2 \le C\|\nabla g\|_0$ since g is a zero-mean function, where the constant C depends on N, Ω. And we used the Sobolev space inequality $|f|^2_{L^\infty(\Omega)} \le C\|f\|^2_{s_0}$, where $s_0 = [\frac{N}{2}] + 1 = 2$ when $N = 2$ or $N = 3$.

Re-arranging terms in (4.17) yields

$$
\|\nabla v\|_2^2 \le \frac{6}{5}C + \frac{1}{5} = C_1 \qquad (4.18)
$$

where the constants C, C_1 depend on N, Ω.

Estimate for $\|\nabla v\|_3^2$: If $r = 3$ in (4.16), then after using the Sobolev space inequalities in Lemma 4.1, we obtain from equation (4.15) the estimate

$$\begin{aligned}
\|\nabla v\|_3^2 &\le C\|\nabla(\ln(b(w)))\cdot\nabla v + \frac{1}{b(w)}g\|_2^2 \\
&\le C\|\nabla(\ln(b(w)))\|_2^2\|\nabla v\|_2^2 + C\|\frac{1}{b(w)}\|_2^2\|g\|_2^2 \\
&\le C\|\nabla(\ln(b(w)))\|_2^2\|\nabla v\|_2^2 + C\Big(\|\frac{1}{b(w)}\|_0^2 + C\|\nabla\Big(\frac{1}{b(w)}\Big)\|_1^2\Big)\|g\|_2^2 \\
&\le C\Big|\frac{d(\ln(b(w)))}{dw}\Big|_{2,\overline{G}_1}^2(1+|w|_{L^\infty(\Omega)})^4\|\nabla w\|_2^2\|\nabla v\|_2^2 \\
&\quad + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2|\Omega|\|g\|_2^2 + C\Big|\frac{d((b(w))^{-1})}{dw}\Big|_{1,\overline{G}_1}^2(1+|w|_{L^\infty(\Omega)})^2\|\nabla w\|_1^2\|g\|_2^2 \\
&\le CC_1^2\Big|\frac{d(\ln(b(w)))}{dw}\Big|_{2,\overline{G}_1}^2(1+|w-v_0|_{L^\infty(\Omega)}+|v_0|)^4 + C\Big|\frac{1}{b}\Big|_{0,\overline{G}_1}^2|\Omega|\|\nabla g\|_1^2 \\
&\quad + CC_1\Big|\frac{d((b(w))^{-1})}{dw}\Big|_{1,\overline{G}_1}^2(1+|w-v_0|_{L^\infty(\Omega)}+|v_0|)^2\|\nabla g\|_1^2 \\
&\le CC_1^2\Big|\frac{d(\ln(b(w)))}{dw}\Big|_{2,\overline{G}_1}^2(1+R+|v_0|)^4 + C|\Omega|\|\nabla g\|_1^2 \\
&\quad + CC_1\Big|\frac{d((b(w))^{-1})}{dw}\Big|_{1,\overline{G}_1}^2(1+R+|v_0|)^2\|\nabla g\|_1^2 \\
&= CC_1^2\Big|\frac{\|\nabla f\|_0}{a_0}\frac{1}{a(u)}\frac{da}{du}\Big|_{2,\overline{G}_0}^2(1+R+|v_0|)^4 + C|\Omega|\frac{\|\nabla f\|_1^2}{\|\nabla f\|_0^2} \\
&\quad + CC_1\Big|\|\nabla f\|_0\frac{-1}{a(u)^2}\frac{da}{du}\Big|_{1,\overline{G}_0}^2(1+R+|v_0|)^2\frac{\|\nabla f\|_1^2}{\|\nabla f\|_0^2} \\
&= CC_1^2\frac{\|\nabla f\|_0^2}{a_0^2}\Big|\frac{d(\ln(a(u)))}{du}\Big|_{2,\overline{G}_0}^2(1+R+|v_0|)^4 + \frac{C|\Omega|\|\nabla f\|_1^2}{\|\nabla f\|_0^2} \\
&\quad + CC_1\Big|\frac{d((a(u))^{-1})}{du}\Big|_{1,\overline{G}_0}^2(1+R+|v_0|)^2\|\nabla f\|_1^2
\end{aligned} \tag{4.19}$$

where C, C_1 are constants which depend on N, Ω. Here, we used the facts that $b(w) = \frac{1}{a_0}a(\frac{\|\nabla f\|_0}{a_0}w)$, and that $\big|\frac{1}{b}\big|_{0,\overline{G}_1} = 1$, and that $g = \frac{1}{\|\nabla f\|_0}f$ from the definition of the functions b, g in (2.4). And we used the inequality $\|\nabla v\|_2^2 \le C_1$ from (4.18). We also used the facts that $|w-v_0|_{L^\infty(\Omega)} \le R$ and that $\|\nabla w\|_2^2 \le C_1$. And we used Poincaré's inequality to obtain $\|g\|_2^2 \le \|g\|_0^2 + C\|\nabla g\|_1^2 \le C\|\nabla g\|_0^2 + C\|\nabla g\|_1^2 \le C\|\nabla g\|_1^2$, since g is a zero-mean function.

Estimate for $\|v\|_4^2$: From (4.19) and by Lemma 4.2, it follows that

$$\begin{aligned}
\|v\|_4^2 &= \|v - v_0 + v_0\|_4^2 \\
&\le 2\|v-v_0\|_4^2 + 2\|v_0\|_4^2 \\
&\le C\|\nabla(v-v_0)\|_3^2 + 2|v_0|^2|\Omega| \\
&= C\|\nabla v\|_3^2 + 2|v_0|^2|\Omega| \\
&\le CC_1^2\frac{\|\nabla f\|_0^2}{a_0^2}\Big|\frac{d(\ln(a(u)))}{du}\Big|_{2,\overline{G}_0}^2(1+R+|v_0|)^4 + \frac{C|\Omega|\|\nabla f\|_1^2}{\|\nabla f\|_0^2}
\end{aligned}$$

$$
\begin{aligned}
&+ CC_1 \Big| \frac{d((a(u))^{-1})}{du} \Big|^2_{1,\overline{G}_0} (1 + R + |v_0|)^2 \|\nabla f\|_1^2 + 2|v_0|^2|\Omega| \\
&= C_3
\end{aligned}
$$

where $\|v - v_0\|_4^2 \le C\|\nabla(v - v_0)\|_3^2$ by Lemma 4.2, and where we used the fact that $v(\mathbf{x}_0) = v_0$, where v_0 is a constant. This completes the proof. □

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