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INDIRECT BOUNDARY OBSERVABILITY OF SEMI-DISCRETE COUPLED WAVE EQUATIONS

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Abstract. This work concerns the indirect observability properties for the finite-difference space semi-discretization of the 1-d coupled wave equations with homogeneous Dirichlet boundary conditions. We assume that only one of the two components of the unknown is observed. As for a single wave equation, as well as for the direct (complete) observability of the coupled wave equations, we prove the lack of the numerical observability. However, we show that a uniform observability holds in the subspace of solutions in which the initial conditions of the observed component is generated by the low frequencies. Our main proofs use a two-level energy method at the discrete level and a Fourier decomposition of the solutions.

1. Introduction

This article deals with the boundary observability properties for the finite-difference approximation of the 1-d coupled wave equations and where we assume that only one of the two components of the unknown is observed. To clarify our aim, we will introduce first the problem of boundary observability in the continuous setting.

Thus, let us fix $T > 0$ and let us consider the linear system

$$\begin{gathered} u_{tt} - u_{xx} + \alpha v = 0 \quad \text{for } (x,t) \in (0,L) \times (0,T) \\ v_{tt} - v_{xx} + \alpha u = 0 \quad \text{for } (x,t) \in (0,L) \times (0,T) \\ u(0,t) = u(L,t) = 0 \quad \text{for } t \in (0,T) \\ v(0,t) = v(L,t) = 0 \quad \text{for } t \in (0,T) \\ u(0) = u^0, \quad u'(0) = u^1 \quad \text{for } x \in (0,L) \\ v(0) = v^0, \quad v'(0) = v^1 \quad \text{for } x \in (0,L), \end{gathered} \tag{1.1}$$

where $\alpha \in \mathbb{R}$ is the coupling constant and $(u^0, u^1, v^0, v^1) \in H_0^1(0,L) \times L^2(0,L) \times H_0^1(0,L) \times L^2(0,L)$ are the initial conditions. Here the subscript t stands for the partial derivative with respect to time variable while subscript x stands for the space variable.

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It can be shown that for T sufficiently large, more precisely for $T > 2L$, that these solutions satisfy the following *(complete) observability inequality* (see [8, 7], where in the latter this inequality has been established for a set of parameters larger than a single parameter α)

$$\begin{aligned} &E(u;0) + E(v;0) + \alpha\|uv\|^2_{L^2((0,T)\times(0,L))} \\ &\leq C(T)\int_0^T (|u_x(L,t)|^2 + |v_x(L,t)|^2)dt, \end{aligned} \tag{1.2}$$

where E is the energy of the solution of a single wave equation, defined, for a generic u, by the formula

$$E(u;t) = \frac{1}{2}\int_0^L (|u_t(x,t)|^2 + |u_x(x,t)|^2)dx. \tag{1.3}$$

We remark that in (1.2) one observes the L^2-norm of the derivatives of u and v on the extreme point of the boundary $x = L$, and get back information on the initial state of solution. Then, an interesting and difficult problem is to get back the energy of both components by using just the observation of a single component, say u, of the solution on $x = L$. More precisely, for system (1.1) this is equivalent to the estimate

$$E(u;0) + \widetilde{E}(v;0) \leq C(T)\int_0^T |u_x(1,t)|^2 dt, \tag{1.4}$$

where $\widetilde{E}$ is the *partial weakened energy* defined by

$$\widetilde{E}(v;t) = \frac{1}{2}\int_0^L \Big(|(-\partial_x^2)^{-1/2}v_t(x,t)|^2 + |v(x,t)|^2\Big)dx. \tag{1.5}$$

Here $(-\partial_x^2)^{-1/2}$ stands for the square root of the inverse of the Laplace operator with Dirichlet boundary conditions. The above estimate (1.4) is known as *indirect observability inequality*.

To our knowledge, this notion of indirect observability was introduced for the first time in the context of coupled wave equations in [1], to obtain an *exact indirect controllability* result, in which one wants to derive back the full coupled system to equilibrium by controlling only one component of the system. The author in this paper used a two level energy method and proved estimate (1.4) for small parameter $|\alpha|$ and a sufficiently large time $T > 0$.

In this work we analyze the analogue of the observability inequality (1.4) for space semi-discretization applied to the coupled wave equations (1.1) in a uniform meshes. For this purpose, let us introduce the space finite-difference scheme of equation (1.1). Let $N \in \mathbb{N}^*$ and we set $h = \frac{L}{N+1}$. We discretize $[0, L]$ by a uniform computational grid defined by $x_j = jh$, $j = 0, \dots, N+1$. Then the semi-discrete approximation of (1.1) reads

$$\begin{gathered} u_j'' + (-\partial_h^2 \vec{u}h)_j + \alpha v_j = 0 \quad \text{for } j = 1, \dots, N,\ t \in (0,T) \\ v_j'' + (-\partial_h^2 \vec{v}h)_j + \alpha u_j = 0 \quad \text{for } j = 1, \dots, N,\ t \in (0,T) \\ u_0(t) = 0,\ u_{N+1}(t) = 0 \quad \text{for } 0 < t < T \\ v_0(t) = 0,\ v_{N+1}(t) = 0 \quad \text{for } 0 < t < T \\ u_j(0) = u_j^0,\ u_j'(0) = u_j^1 \quad \text{for } j = 1, \dots, N \\ v_j(0) = v_j^0,\ v_j'(0) = v_j^1 \quad \text{for } j = 1, \dots, N, \end{gathered} \tag{1.6}$$

where $\vec{u}h(t) = (u_1(t), \dots, u_N(t))$, $\vec{v}h(t) = (v_1(t), \dots, v_N(t))$ and

$$(-\partial_h^2 \vec{u}h)_j = -\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}, \quad j = 1, \dots, N.$$

Here the superscript $'$ denotes partial differentiation with respect to time. The functions $u_j(t)$ and $v_j(t)$ are approximations of the solutions $u(x,t)$ and $v(x,t)$ of (1.1) in the grid point (x_j, t), provided that $(u_j^0, u_j^1, v_j^0, v_j^1)_{1\le j\le N}$ approximates the initial datum (u^0, u^1, v^0, v^1).

For each solution $(\vec{u}h, \vec{v}h)$ of system (1.6), we associate the following *discrete natural and weakened energies*, respectively,

$$E_h(\vec{u}h; t) = \frac{1}{2}\|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h} + \frac{1}{2}\|(-\partial_h^2)^{1/2}\vec{u}h(t)\|^2_{\mathbb{R}^N,h}, \tag{1.7}$$

$$\widetilde{E}_h(\vec{v}h; t) = \frac{1}{2}\|(-\partial_h^2)^{-1/2}\vec{v}h'(t)\|^2_{\mathbb{R}^N,h} + \frac{1}{2}\|\vec{v}h(t)\|^2_{\mathbb{R}^N,h}, \tag{1.8}$$

where we have used the notation

$$\|\vec{u}\|^2_{\mathbb{R}^N,h} = \langle \vec{u}, \vec{u}\rangle_{\mathbb{R}^N,h}, \quad \text{with} \quad \langle \vec{u}, \vec{v}\rangle_{\mathbb{R}^N,h} = h\sum_{j=1}^N u_j v_j$$

for every vectors $\vec{u} = (u_1, \dots, u_N)$ and $\vec{v} = (v_1, \dots, v_N)$ of $\mathbb{R}^N$.

Of course the discrete energies (1.7) and (1.8) are a discretization of the continuous ones defined by (1.3) and (1.5). However, they define the *total*, (natural and weakened), energies of system (1.6):

$$E_{T,h}(t) = E_h(\vec{u}h; t) + E_h(\vec{v}h; t) + \alpha\langle \vec{u}h(t), \vec{v}h(t)\rangle_{\mathbb{R}^N,h}, \tag{1.9}$$

$$\widetilde{E}_{T,h}(t) = \widetilde{E}_h(\vec{u}h; t) + \widetilde{E}_h(\vec{v}h; t) + \alpha\langle (-\partial_h^2)^{-1}\vec{u}h(t), \vec{v}h(t)\rangle_{\mathbb{R}^N,h}, \tag{1.10}$$

which are conserved along time, see Lemma 4.4, that is

$$E_{T,h}(t) = E_{T,h}(0), \quad \text{and} \quad \widetilde{E}_{T,h}(t) = \widetilde{E}_{T,h}(0), \quad \forall t \in [0,T]. \tag{1.11}$$

Our aim is to study the indirect observability property of the discrete equation (1.6). More precisely, we are concern with the following discrete version of (1.4),

$$E_h(\vec{u}h; 0) + \widetilde{E}_h(\vec{v}h; 0) \le C(T,\alpha)\int_0^T \big|\frac{u_N(t)}{h}\big|^2 dt \tag{1.12}$$

for large time T and a sufficiently small $|\alpha|$.

It is well known by now that in general estimates like equation (1.12) are not uniform for standard numerical discretization in uniform meshes, and that the observability constant $C = C(h)$ may diverge as $h \to 0$. Indeed, as it is explained in [5] (see also [3, 9, 10]), in general the semi-discrete dynamics generates high-frequency modes that do not exist at the continuous level. This high-frequency oscillations propagate with arbitrary small velocity and that cannot be observed uniformly with respect to the mesh size h.

By now, as witnessed in the bibliography of the review paper [11], there is a large number of publications on the uniform observability of discrete systems. For instance, in paper [5] the authors consider the problem of the boundary observability for a finite-difference and finite elements space semi-discretization of a single wave equation, and they proved that the observability inequality is not uniform with respect to the mesh size. However, they have shown that filtering the high frequency modes leads to a uniform bound for the observability constant.

The same approach was used in [3] dealing with coupled wave equations like (1.1) and analogously to [5], a uniform discrete version of inequality (1.2) in filtered space, namely the space generated by the low frequency eigenvalues of the discrete operator $(-\partial_h^2)$, has been obtained.

Our contribution in this paper is the analysis of the discrete inequality (1.12) in uniform meshes. The proof of our results are based on the Fourier decomposition of solutions and take advantages of the proof of observability estimate (1.4) proposed by Alabau-Boussouira [1] at the continuous level. However, our paper is also inspired on that of Infante and Zuazua [5]. To our knowledge, this problem of uniform indirect observability for a coupled wave equations was not considered before.

Now a description of the content of the paper can be given: In Section 2, we give the main results of this paper which are the lack of uniform discrete observability and a uniform observability result for solutions with filtered initial datums. At this stage, however, it is worth mentioning that the filtered mechanism is applied only to one of the two component of the solution, namely to the observed one. In Section 3, we establish the proof of the lack of uniform observability while the observability in filtered space is shown in Section 4.

2. Main Results

In this section we present the main results of this paper. The first result asserts the lack of uniform observability of the semi-discrete system (1.6), while the second one shows that a uniform bound holds in the subspace of solutions in which the initial conditions of the observed component is generated by the low frequencies.

Our result on the absence of uniform observability is given by the following theorem.

Theorem 2.1. *For each $T > 0$, we have*

$$\sup_{(\vec{u}h,\vec{v}h)\ \textit{solution of}\ (1.6)} \Big[\frac{E_h(\vec{u}h;0) + \widetilde{E}_h(\vec{v}h;0)}{\int_0^T |\frac{u_N(t)}{h}|^2 dt}\Big] \to \infty, \quad \textit{as } h \to 0. \tag{2.1}$$

As mentioned in the introduction, this lack of uniform observability is because of the high frequency modes generated by the discrete dynamic (1.6). Then, in order to get a uniform bound for the observability constant one has to filter out these spurious frequency modes. However, as we shall see, we need to rule just the high oscillations of the observed component of the solution.

Moreover, before giving a precise definition of this filtered space, we need to recall that the eigenvalues and eigenvectors of the matrix $(-\partial_h^2)$ can be given explicitly by

$$\lambda_k(h) = \frac{4}{h^2}\sin^2\Big(\frac{k\pi h}{2L}\Big) \quad k = 1, \dots, N$$
$$\varphi_{k,j} = \sqrt{\frac{2}{L}}\sin\Big(\frac{k\pi x_j}{L}\Big) \quad j,\, k = 1, \dots, N,$$

and that the set formed by this eigenvectors $\vec{\varphi}k := (\varphi_{k,j})_{1\le j\le N}$ is an orthonormal basis in the discrete space $(\mathbb{R}^N, \|\cdot\|_{\mathbb{R}^N,h})$, we refer to [6, pp. 458] (see also [4]) for the proof of these facts. Therefore, any vector $\vec{u} \in \mathbb{R}^N$ may be expressed as

$$\vec{u} = \sum_{k=1}^{N} \widehat{u}_k \vec{\varphi}k, \quad \text{with} \quad \widehat{u}_k = \langle \vec{u}, \vec{\varphi}k \rangle_{\mathbb{R}^N, h}.$$

Let $0 < \gamma < 4$. Then, as in [3, 5], we introduce the following filtered space

$$\mathcal{G}_h = \big\{ \vec{u} = \sum_{\lambda_k h^2 < \gamma} a_k \vec{\varphi}k;\ a_k \in \mathbb{R} \big\}. \tag{2.2}$$

We are ready to state our result on the uniform indirect observability of (1.6).

Theorem 2.2. *Assume that $0 < \gamma < 4$. Then for $|\alpha|$ sufficiently small, there exists $T(\alpha, \gamma) > 0$ such that for all $T > T(\alpha, \gamma)$, there exist $C(T, \alpha, \gamma)$ such that the following estimate holds as $h \to 0$,*

$$E_h(\vec{u}h; 0) + \widetilde{E}_h(\vec{v}h; 0) \le C(T, \alpha, \gamma) \int_0^T \big| \frac{u_N(t)}{h} \big|^2 dt \tag{2.3}$$

for every solution of (1.6) with initial datum $(\vec{u}h^0, \vec{u}h^1, \vec{v}h^0, \vec{v}h^1)$ in the class $\mathcal{S}_h := \mathcal{G}_h \times \mathcal{G}_h \times \mathbb{R}^N \times \mathbb{R}^N$.

3. Proof of Theorem 2.1

The main tool is a spectral decomposition of the solution of the observed system (1.6) given in Lemma 3.1 bellow. To begin with, we exapnad the initial data (u^0, u^1, v^0, v^1) in Fourier sequences with respect to the eigenfunctions $(\vec{\varphi}k)_{1 \le k \le N}$,

$$\vec{u}h^0 = \sum_{k=1}^{N} \widehat{u_k^0} \vec{\varphi}k, \quad \vec{u}h^1 = \sum_{k=1}^{N} \widehat{u_k^1} \vec{\varphi}k, \tag{3.1}$$

$$\vec{v}h^0 = \sum_{k=1}^{N} \widehat{v_k^0} \vec{\varphi}k, \quad \vec{v}h^1 = \sum_{k=1}^{N} \widehat{v_k^1} \vec{\varphi}k. \tag{3.2}$$

Then, we claim the following result.

Lemma 3.1. *Assume that $|\alpha| \le \big(\frac{\pi}{L}\big)^2$. Given $\vec{u}h^0$, $\vec{u}h^1$, $\vec{v}h^0$, $\vec{v}h^1$ arbitrary scalars, the problem (1.6) has a unique analytic solution $(\vec{u}h, \vec{v}h) : \mathbb{R}_+ \to \mathbb{R}^{2N}$ given by the spectral decomposition*

$$\begin{aligned} \vec{u}h(t) = \sum_{k=1}^{N} \Big[&\frac{\widehat{u_k^0} + \widehat{v_k^0}}{2} \cos\Big(\sqrt{\mu_k^+(h)t}\Big) + \frac{\widehat{u_k^1} + \widehat{v_k^1}}{2\sqrt{\mu_k^+(h)}} \sin\Big(\sqrt{\mu_k^+(h)t}\Big) \\ &+ \frac{\widehat{u_k^0} - \widehat{v_k^0}}{2} \cos\Big(\sqrt{\mu_k^-(h)t}\Big) + \frac{\widehat{u_k^1} - \widehat{v_k^1}}{2\sqrt{\mu_k^-(h)}} \sin\Big(\sqrt{\mu_k^-(h)t}\Big) \Big] \vec{\varphi}k, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \vec{v}h(t) = \sum_{k=1}^{N} \Big[&\frac{\widehat{u_k^0} + \widehat{v_k^0}}{2} \cos\Big(\sqrt{\mu_k^+(h)t}\Big) + \frac{\widehat{u_k^1} + \widehat{v_k^1}}{2\sqrt{\mu_k^+(h)}} \sin\Big(\sqrt{\mu_k^+(h)t}\Big) \\ &- \frac{\widehat{u_k^0} - \widehat{v_k^0}}{2} \cos\Big(\sqrt{\mu_k^-(h)t}\Big) - \frac{\widehat{u_k^1} - \widehat{v_k^1}}{2\sqrt{\mu_k^-(h)}} \sin\Big(\sqrt{\mu_k^-(h)t}\Big) \Big] \vec{\varphi}k, \end{aligned} \tag{3.4}$$

where $\widehat{u_k^0}, \widehat{u_k^1}, \widehat{v_k^0}, \widehat{v_k^1}$ *are the Fourier coefficients given in* (3.1)-(3.2)*, and the eigenvalues* $\mu_k^{\pm}(h)$ *are defined by*

$$\mu_k^{\pm}(h) = \lambda_k(h) \pm \alpha = \frac{4}{h^2}\sin^2\Big(\frac{k\pi h}{2L}\Big) \pm \alpha, \quad k = 1, \dots, N.$$

Proof. The proof is straightforward. Indeed, taking $\vec{w}h^+ = \vec{u}h + \vec{v}h$ and $\vec{w}h^- = \vec{u}h - \vec{v}h$, it follows that

$$\begin{gathered}
(w_j^+)'' + (-\partial_h^2 \vec{w}h^+)_j + \alpha w_j^+ = 0 \quad \text{for } j = 1, \dots, N,\ t \in (0, T) \\
w_0^+(t) = 0,\ w_{N+1}^+(t) = 0 \quad \text{for } 0 < t < T \\
w_j^+(0) = u_j^0 + v_j^0 \quad \text{for } j = 1, \dots, N \\
(w_j^+)'(0) = u_j^1 + v_j^1 \quad \text{for } j = 1, \dots, N,
\end{gathered} \tag{3.5}$$

and

$$\begin{gathered}
(w_j^-)'' + (-\partial_h^2 \vec{w}h^-)_j - \alpha w_j^- = 0 \quad \text{for } j = 1, \dots, N,\ t \in (0, T) \\
w_0^-(t) = 0,\ w_{N+1}^-(t) = 0 \quad \text{for } 0 < t < T \\
w_j^-(0) = u_j^0 - v_j^0 \quad \text{for } j = 1, \dots, N \\
(w_j^-)'(0) = u_j^1 - v_j^1 \quad \text{for } j = 1, \dots, N.
\end{gathered} \tag{3.6}$$

However, it is easy to see that the solutions of decoupled systems (3.5)-(3.6) are given by Fourier sequences development

$$\begin{aligned}
\vec{w}h^+(t) &= \sum_{k=1}^{N} \Big[\Big(\widehat{u_k^0} + \widehat{v_k^0}\Big) \cos\Big(\sqrt{\mu_k^+(h)}t\Big) + \frac{\widehat{u_k^1} + \widehat{v_k^1}}{\sqrt{\mu_k^+(h)}} \sin\Big(\sqrt{\mu_k^+(h)}t\Big)\Big] \vec{\varphi}k, \\
\vec{w}h^-(t) &= \sum_{k=1}^{N} \Big[\Big(\widehat{u_k^0} - \widehat{v_k^0}\Big) \cos\Big(\sqrt{\mu_k^-(h)}t\Big) + \frac{\widehat{u_k^1} - \widehat{v_k^1}}{\sqrt{\mu_k^-(h)}} \sin\Big(\sqrt{\mu_k^-(h)}t\Big)\Big] \vec{\varphi}k,
\end{aligned}$$

and we recover equations (3.3)-(3.4) by remarking that $\vec{u}h = \frac{\vec{w}h^+ + \vec{w}h^-}{2}$ and $\vec{v}h = \frac{\vec{w}h^+ - \vec{w}h^-}{2}$. This completes the proof. □

Remark 3.2. Throughout this paper, whenever the eigenvalues $\sqrt{\mu_k^{\pm}(h)}$ are mentioned, condition $|\alpha| \le \alpha_0 := (\frac{\pi}{L})^2$ is directly taken into consideration since otherwise $\sqrt{\mu_k^{\pm}(h)}$ is not well defined.

Remark 3.3. Having in mind the relation $e^{ix} = \cos(x) + i\sin(x)$, we can write the solution $(\vec{u}h, \vec{v}h)$ given by (3.3)-(3.4) in the following equivalent form

$$\begin{aligned}
\vec{u}h(t) &= \sum_{1 \le |k| \le N} \frac{a_k e^{i\sqrt{\mu_k^+(h)}t} + b_k e^{i\sqrt{\mu_k^-(h)}t}}{2} \vec{\varphi}k, \\
\vec{v}h(t) &= \sum_{1 \le |k| \le N} \frac{a_k e^{i\sqrt{\mu_k^+(h)}t} - b_k e^{i\sqrt{\mu_k^-(h)}t}}{2} \vec{\varphi}k,
\end{aligned}$$

where $\sqrt{\mu_k^{\pm}(h)} = -\sqrt{\mu_{-k}^{\pm}(h)}$ for $k < 0$, and a_k, b_k are suitable coefficients that can be computed explicitly in terms of the Fourier coefficients $\widehat{u_k^0}, \widehat{u_k^1}, \widehat{v_k^0}, \widehat{v_k^1}$.

Proof of Theorem 2.1. Let $(\vec{u}h, \vec{v}h)$ be the solution of equation (1.6) associated to the N-th eigenvector given by

$$\vec{u}h = \frac{e^{i\sqrt{\mu_N^+(h)}t} + e^{i\sqrt{\mu_N^-(h)}t}}{2}\vec{\varphi}N \quad \text{and} \quad \vec{v}h = \frac{e^{i\sqrt{\mu_N^+(h)}t} - e^{i\sqrt{\mu_N^-(h)}t}}{2}\vec{\varphi}N.$$

In view of Remark 3.3 and according to Lemma 3.1 the couple $(\vec{u}h, \vec{v}h)$ is indeed a solution of the discrete coupled wave equations (1.6). For this solution, we compute separately each of the three terms $E_h(\vec{u}h;0)$, $\widetilde{E}_h(\vec{v}h;0)$ and $\int_0^T \big|\frac{u_N(t)}{h}\big|^2 dt$ appearing in equation (2.1).

Computation of $E_h(\vec{u}h;0)$. We have

$$\begin{aligned} E_h(\vec{u}h;0) &= \frac{h}{2}\sum_{j=1}^{N}|u_j'(0)|^2 + \frac{h}{2}\sum_{j=0}^{N}\big|\frac{u_{j+1}(0)-u_j(0)}{h}\big|^2 \\ &= \frac{|\sqrt{\mu_N^+(h)}+\sqrt{\mu_N^-(h)}|^2}{8} h\sum_{j=1}^{N}|\varphi_{N,j}|^2 + \frac{h}{2}\sum_{j=0}^{N}\big|\frac{\varphi_{N,j+1}-\varphi_{N,j}}{h}\big|^2. \end{aligned} \tag{3.7}$$

Moreover, the eigenvector $\vec{\varphi}N$ satisfy the following identity (see [5])

$$h\sum_{j=0}^{N}\big|\frac{\varphi_{N,j+1}-\varphi_{N,j}}{h}\big|^2 = \lambda_N(h)h\sum_{j=1}^{N}|\varphi_{N,j}|^2. \tag{3.8}$$

Inserting this last equation into (3.7), we obtain

$$E_h(\vec{u}h;0) = \Big[\frac{|\sqrt{\mu_N^+(h)}+\sqrt{\mu_N^-(h)}|^2}{8\lambda_N(h)} + \frac{1}{2}\Big] h\sum_{j=0}^{N}\Big|\frac{\varphi_{N,j+1}-\varphi_{N,j}}{h}\Big|^2,$$

and in view of the identity, see for instance [3, 5],

$$h\sum_{j=0}^{N}\Big|\frac{\varphi_{N,j+1}-\varphi_{N,j}}{h}\Big|^2 = \frac{2L}{4-\lambda_N(h)h^2}\big|\frac{\varphi_{N,N}}{h}\big|^2, \tag{3.9}$$

we can write

$$E_h(\vec{u}h;0) = \Big[\frac{|\sqrt{\mu_N^+(h)}+\sqrt{\mu_N^-(h)}|^2}{4\lambda_N(h)} + 1\Big]\frac{L}{4-\lambda_N(h)h^2}\Big|\frac{\varphi_{N,N}}{h}\Big|^2. \tag{3.10}$$

Computation of $\widetilde{E}_h(\vec{v}h;0)$. We have

$$\begin{aligned} \widetilde{E}_h(\vec{v}h;0) &= \frac{1}{2}\big\|(-\partial_h^2)^{-1/2}\vec{v}h'(0)\big\|_{\mathbb{R}^N,h}^2 + \frac{h}{2}\sum_{j=1}^{N}|v_j(0)|^2 \\ &= \frac{|\sqrt{\mu_N^+(h)}-\sqrt{\mu_N^-(h)}|^2}{8}\big\|(-\partial_h^2)^{-1/2}\vec{\varphi}N\big\|_{\mathbb{R}^N,h}^2. \end{aligned}$$

Remarking that $(-\partial_h^2)^{-1/2}\vec{\varphi}N = \frac{1}{\lambda_N(h)}(-\partial_h^2)^{1/2}\vec{\varphi}N$ and using the identity

$$\|(-\partial_h^2)^{1/2}\vec{\varphi}N\|_{\mathbb{R}^N,h}^2 = h\sum_{j=0}^{N}\Big|\frac{\varphi_{N,j+1}-\varphi_{N,j}}{h}\Big|^2,$$

together with equation (3.9), we can write

$$\widetilde{E}_h(\vec{v}h;0)=\frac{\big|\sqrt{\mu_N^+(h)}-\sqrt{\mu_N^-(h)}\big|^2}{4\lambda_N^2(h)}\frac{L}{4-\lambda_N(h)h^2}\Big|\frac{\varphi_{N,N}}{h}\Big|^2. \tag{3.11}$$

Computation of $\int_0^T\Big|\frac{u_N(t)}{h}\Big|^2dt$. We have

$$\int_0^T\Big|\frac{u_N(t)}{h}\Big|^2dt=\int_0^T\Big|\frac{e^{i\sqrt{\mu_N^+(h)}t}+e^{i\sqrt{\mu_N^-(h)}t}}{2}\Big|^2dt\,\Big|\frac{\varphi_{N,N}}{h}\Big|^2,$$

and

$$\int_0^T\Big|\frac{e^{i\sqrt{\mu_N^+(h)}t}+e^{i\sqrt{\mu_N^-(h)}t}}{2}\Big|^2dt=\frac{T}{2}+\frac{\sin[\big(\sqrt{\mu_N^+(h)}-\sqrt{\mu_N^-(h)}\big)T]}{2\big(\sqrt{\mu_N^+(h)}-\sqrt{\mu_N^-(h)}\big)}.$$

Therefore, we obtain

$$\int_0^T\Big|\frac{u_N(t)}{h}\Big|^2dt=\Big[\frac{T}{2}+\frac{\sin[\big(\sqrt{\mu_N^+(h)}-\sqrt{\mu_N^-(h)}\big)T]}{2\big(\sqrt{\mu_N^+(h)}-\sqrt{\mu_N^-(h)}\big)}\Big]\Big|\frac{\varphi_{N,N}}{h}\Big|^2. \tag{3.12}$$

Next, combining (3.10), (3.11) and (3.12) we deduce that

$$\frac{E_h(\vec{u}h;0)+\widetilde{E}_h(\vec{v}h;0)}{\int_0^T\big|\frac{u_N(t)}{h}\big|^2dt}=\frac{C(T,h)}{4-\lambda_N(h)h^2}, \tag{3.13}$$

with

$$C(T,h)=\frac{L\Big[\frac{\big|\sqrt{\mu_N^+(h)}+\sqrt{\mu_N^-(h)}\big|^2}{4\lambda_N(h)}+1+\frac{\big|\sqrt{\mu_N^+(h)}-\sqrt{\mu_N^-(h)}\big|^2}{4\lambda_N^2(h)}\Big]}{\frac{T}{2}+\frac{\sin\big[\big(\sqrt{\mu_N^+(h)}-\sqrt{\mu_N^-(h)}\big)T\big]}{2\big(\sqrt{\mu_N^+(h)}-\sqrt{\mu_N^-(h)}\big)}}.$$

After straightforward calculations, we obtain

$$C(T,h)\to\frac{2L}{T}\quad\text{and}\quad\lambda_N(h)h^2\to4,\quad\text{as}\quad h\to0. \tag{3.14}$$

Indeed, we have

$$\begin{aligned}\frac{\big|\sqrt{\mu_N^+(h)}+\sqrt{\mu_N^-(h)}\big|^2}{4\lambda_N(h)}&=\frac{\mu_N^+(h)+\mu_N^-(h)+2\sqrt{\mu_N^+(h)\mu_N^-(h)}}{4\lambda_N(h)}\\&=\frac{2\lambda_N(h)+2\sqrt{\lambda_N^2(h)-\alpha^2}}{4\lambda_N(h)}\\&=\frac12+\frac12\sqrt{1-\frac{\alpha^2}{\lambda_N^2(h)}}\to1,\quad\text{as } h\to0,\end{aligned} \tag{3.15}$$

$$\frac{\big|\sqrt{\mu_N^+(h)}-\sqrt{\mu_N^-(h)}\big|^2}{4\lambda_N^2(h)}=\frac{4\alpha^2}{4\lambda_N^2(h)\big|\sqrt{\mu_N^+(h)}+\sqrt{\mu_N^-(h)}\big|^2}\to0,\quad\text{as } h\to0, \tag{3.16}$$

$$\frac{\sin\big[\big(\sqrt{\mu_N^+(h)}-\sqrt{\mu_N^-(h)}\big)T\big]}{2\big(\sqrt{\mu_N^+(h)}-\sqrt{\mu_N^-(h)}\big)}=\frac{\sin\big[\frac{2\alpha T}{\sqrt{\mu_N^+(h)}+\sqrt{\mu_N^-(h)}}\big]}{\frac{2}{T}\frac{2\alpha T}{\sqrt{\mu_N^+(h)}+\sqrt{\mu_N^-(h)}}}\to\frac{T}{2},\quad\text{as } h\to0, \tag{3.17}$$

and

$$\lambda_N(h)h^2 = 4\sin^2\Big(\frac{N\pi h}{2L}\Big) = 4\sin^2\Big(\frac{\pi}{2} - \frac{h\pi}{2L}\Big) = 4\cos^2\Big(\frac{h\pi}{2L}\Big) \to 4, \quad \text{as } h \to 0. \tag{3.18}$$

In view of (3.15), (3.16), (3.17) and (3.18) we immediately get (3.14). Hence, from (3.14) and (3.13), it follows that

$$\frac{E_h(\vec{u}h;0) + \widetilde{E}_h(\vec{v}h;0)}{\int_0^T \big|\frac{u_N(t)}{h}\big|^2 dt} \to \infty, \quad \text{as } h \to 0,$$

and the proof is complete. □

4. Proof of theorem 2.2

We prove the theorem using a discrete two-level energy method. However, the presentation of the proof is in four subsections. Subsection 4.1 devoted to presenting and proving a discrete version of the Poincaré inequality, uniform Poincaré inequality, which will be useful for what follows. In Subsection 4.2, we establish some technical estimates. Subsection 4.3 deals with the observability of a finite-difference space semi-discretization of the non homogeneous single wave equation, and shows how filtering the high frequency modes of the discrete initial data can be used to get a uniform bound for the observability constant. Results of aforementioned Subsections 4.1-4.3 are used in Subsection 4.4 to finish the proof of Theorem 2.2.

4.1. **Uniform Poincaré inequality.** In this subsection, we shall show the following inequality.

Theorem 4.1. *For any $\vec{u} = (u_1, \dots, u_N) \in \mathbb{R}^N$, we have*

$$h\sum_{j=1}^{N} |u_j|^2 \le \frac{h}{\alpha_0}\sum_{j=0}^{N}\Big|\frac{u_{j+1}-u_j}{h}\Big|^2, \tag{4.1}$$

where $u_0 := u_{N+1} := 0$, and α_0 has already been introduced in Remark 3.2.

Proof. We expand the vector $\vec{u}$ on the basis $\vec{\varphi}k$ of eigenfunctions of $-\partial_h^2$ as

$$\vec{u} = \sum_{k=1}^{N} \widehat{u}_k \vec{\varphi}k,$$

with $\widehat{u}_k = \langle \vec{u}, \vec{\varphi}k\rangle_{\mathbb{R}^N}$. Therefore

$$\begin{aligned} h\sum_{j=0}^{N}\Big|\frac{u_{j+1}-u_j}{h}\Big|^2 &= h\sum_{j=0}^{N}\Big|\sum_{k=1}^{N}\frac{\widehat{u}_k}{h}(\varphi_{k,j+1}-\varphi_{k,j})\Big|^2 \\ &= h\sum_{j=0}^{N}\sum_{k=1}^{N}\widehat{u}_k^2\Big|\frac{\varphi_{k,j+1}-\varphi_{k,j}}{h}\Big|^2 \\ &\quad + h\sum_{j=0}^{N}\sum_{\substack{k,k'=1\\ k\neq k'}}^{N}\frac{\widehat{u}_k\widehat{u}_{k'}}{h^2}(\varphi_{k,j+1}-\varphi_{k,j})(\varphi_{k',j+1}-\varphi_{k',j}). \end{aligned}$$

Moreover, the eigenvectors $\vec{\varphi}k$ satisfy the following identity (see [5, 4])

$$\sum_{j=0}^{N}(\varphi_{k,j+1}-\varphi_{k,j})(\varphi_{k',j+1}-\varphi_{k',j})=0 \tag{4.2}$$

for all $k\neq k'$. Hence, it follows that

$$h\sum_{j=0}^{N}\Big|\frac{u_{j+1}-u_j}{h}\Big|^2=\sum_{k=1}^{N}\widehat{u}_k^2\Big(h\sum_{j=0}^{N}\Big|\frac{\varphi_{k,j+1}-\varphi_{k,j}}{h}\Big|^2\Big), \tag{4.3}$$

and according to (3.8), we can write

$$h\sum_{j=0}^{N}\Big|\frac{u_{j+1}-u_j}{h}\Big|^2=\sum_{k=1}^{N}\widehat{u}_k^2\lambda_k(h)h\sum_{j=1}^{N}|\varphi_{k,j}|^2. \tag{4.4}$$

Using the fact that $\lambda_k(h)\geq\alpha_0$ for all $k=1,\dots,N$, we estimate the right-hand side of identity (4.4) as

$$\sum_{k=1}^{N}\widehat{u}_k^2\lambda_k(h)h\sum_{j=1}^{N}|\varphi_{k,j}|^2\geq\alpha_0 h\sum_{j=1}^{N}\sum_{k=1}^{N}\widehat{u}_k^2|\varphi_{k,j}|^2=\alpha_0 h\sum_{j=1}^{N}|u_j|^2. \tag{4.5}$$

Using (4.4) and (4.5), we immediately obtain the desired inequality (4.1). □

Remark 4.2. Inequality (4.1) is the discrete analogue of the well-known Poincaré's inequality in $H_0^1(0,L)$, that reads

$$\|u\|_{L^2(0,L)}\leq C\|u\|_{H_0^1(0,L)}$$

for every function $u\in H_0^1(0,L)$.

4.2. **Some elementary technical estimates.** Some basic but important estimates and properties of solutions $(\vec{u}h,\vec{v}h)$ are summarized in the next lemmas.

Lemma 4.3. *For all* $0<|\alpha|<\frac{\sqrt{\alpha_0}}{2}$,

$$\int_0^T E_h(\vec{u}h;t)\,dt\geq\frac{C_1'T}{2(1+|\alpha|T)}(E_h(\vec{u}h;0)-\widetilde{E}_h(\vec{v}h;0)), \tag{4.6}$$

where the constant C_1' *will be explicitly given in the course of the proof.*

Proof. We will split the proof into four steps.

Step 1. First estimates of the terms:

$$\int_0^T\|\vec{v}h(t)\|_{\mathbb{R}^N,h}^2dt,\ \int_0^T\|(-\partial_h^2)^{-1/2}\vec{v}h'(t)\|_{\mathbb{R}^N,h}^2dt,\quad\widetilde{E}_h(\vec{v}h;T)+\widetilde{E}_h(\vec{v}h;0).$$

We take the sum of the inner product of (1.6)-1 and (1.6)-2 with $\vec{v}h(t)$ and $-\vec{u}h(t)$, respectively, in $(\mathbb{R}^N,\|\cdot\|_{\mathbb{R}^N,h})$ to obtain

$$\begin{aligned}&\langle\vec{u}h''(t)-\partial_h^2\vec{u}h(t)+\alpha\vec{v}h(t),\vec{v}h(t)\rangle_{\mathbb{R}^N,h}\\&-\langle\vec{v}h''(t)-\partial_h^2\vec{v}h(t)+\alpha\vec{u}h(t),\vec{u}h(t)\rangle_{\mathbb{R}^N,h}=0.\end{aligned}$$

Hence, integrating this last equation over $t\in(0,T)$ and using the symmetry of the matrix $-\partial_h^2$, yield

$$\int_0^T(\langle\vec{u}h''(t),\vec{v}h(t)\rangle_{\mathbb{R}^N,h}-\langle\vec{v}h''(t),\vec{u}h(t)\rangle_{\mathbb{R}^N,h}+\alpha\|\vec{v}h(t)\|_{\mathbb{R}^N,h}^2-\alpha\|\vec{u}h(t)\|_{\mathbb{R}^N,h}^2)dt=0,$$

and in view of the two identities

$$\int_0^T \langle \vec{u}h''(t), \vec{v}h(t)\rangle_{\mathbb{R}^N,h} dt = [\langle \vec{u}h'(t), \vec{v}h(t)\rangle_{\mathbb{R}^N,h}]_0^T - \int_0^T \langle \vec{u}h'(t), \vec{v}h'(t)\rangle_{\mathbb{R}^N,h} dt,$$
$$\int_0^T \langle \vec{v}h''(t), \vec{u}h(t)\rangle_{\mathbb{R}^N,h} dt = [\langle \vec{v}h'(t), \vec{u}h(t)\rangle_{\mathbb{R}^N,h}]_0^T - \int_0^T \langle \vec{v}h'(t), \vec{u}h'(t)\rangle_{\mathbb{R}^N,h} dt,$$

it follows that

$$\alpha \int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt = [X_h(t)]_0^T + \alpha \int_0^T \|[\vec{u}h(t)\|^2_{\mathbb{R}^N,h} dt, \tag{4.7}$$

with

$$X_h(t) := \langle \vec{v}h'(t), \vec{u}h(t)\rangle_{\mathbb{R}^N,h} - \langle \vec{u}h'(t), \vec{v}h(t)\rangle_{\mathbb{R}^N,h}.$$

On the other hand,

$$\begin{aligned} |\langle \vec{v}h'(t), \vec{u}h(t)\rangle_{\mathbb{R}^N,h}| &= \big|\langle (-\partial_h^2)^{-1/2}\vec{v}h'(t), (-\partial_h^2)^{1/2}\vec{u}h(t)\rangle_{\mathbb{R}^N,h}\big| \\ &\le \frac{\varepsilon_1 \|(-\partial_h^2)^{-1/2}\vec{v}h'(t)\|^2_{\mathbb{R}^N,h}}{2} + \frac{\|(-\partial_h^2)^{1/2}\vec{u}h(t)\|^2_{\mathbb{R}^N,h}}{2\varepsilon_1}, \end{aligned}$$

$$\big|\langle \vec{u}h'(t), \vec{v}h(t)\rangle_{\mathbb{R}^N,h}\big| \le \frac{\|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h}}{2\varepsilon_1} + \frac{\varepsilon_1 \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h}}{2}$$

for all $\varepsilon_1 > 0$. In view of these two last inequalities, we can estimate the term $[X_h(t)]_0^T$ as

$$|[X_h(t)]_0^T| \le \frac{1}{\varepsilon_1}(E_h(\vec{u}h;T) + E_h(\vec{u}h;0)) + \varepsilon_1(\widetilde{E}_h(\vec{v}h;T) + \widetilde{E}_h(\vec{v}h;0)). \tag{4.8}$$

Using (4.7) and (4.8), we obtain

$$\begin{aligned} \int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt &\le \int_0^T \|\vec{u}h(t)\|^2_{\mathbb{R}^N,h} dt + \frac{1}{\varepsilon_1|\alpha|}(E_h(\vec{u}h;T) + E_h(\vec{u}h;0)) \\ &\quad + \frac{\varepsilon_1}{|\alpha|}(\widetilde{E}_h(\vec{v}h;T) + \widetilde{E}_h(\vec{v}h;0)) \end{aligned} \tag{4.9}$$

for each $\varepsilon_1 > 0$.

Concerning the term $\int_0^T \|(-\partial_h^2)^{-1/2}\vec{v}h'(t)\|^2_{\mathbb{R}^N,h} dt$, we take the inner product of (1.6)-2 with $(-\partial_h^2)^{-1}\vec{v}h(t)$ in $(\mathbb{R}^N, \|\cdot\|_{\mathbb{R}^N,h})$ to obtain

$$\int_0^T \langle \vec{v}h''(t) - \partial_h^2\vec{v}h(t) + \alpha\vec{u}h(t), (-\partial_h^2)^{-1}\vec{v}h(t)\rangle_{\mathbb{R}^N,h} dt = 0.$$

This gives

$$\begin{aligned} &\int_0^T \langle (-\partial_h^2)^{-1/2}\vec{v}h''(t), (-\partial_h^2)^{-1/2}\vec{v}h(t)\rangle_{\mathbb{R}^N,h} dt \\ &+ \int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt + \alpha \int_0^T \langle \vec{u}h(t), (-\partial_h^2)^{-1}\vec{v}h(t)\rangle_{\mathbb{R}^N,h} dt = 0. \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} &\int_0^T \|(-\partial_h^2)^{-1/2}\vec{v}h'(t)\|^2_{\mathbb{R}^N,h} dt \\ &= [Y_h(t)]_0^T + \int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt + \alpha \int_0^T \langle \vec{u}h(t), (-\partial_h^2)^{-1}\vec{v}h(t)\rangle_{\mathbb{R}^N,h} dt, \end{aligned} \tag{4.10}$$

with $Y_h(t) = \langle(-\partial_h^2)^{-1/2}\vec{v}h'(t), (-\partial_h^2)^{-1/2}\vec{v}h(t)\rangle_{\mathbb{R}^N,h}$. However, for this term we have

$$\begin{aligned}
|[Y_h(t)]_0^T| &\le |\langle(-\partial_h^2)^{-1/2}\vec{v}h'(T), (-\partial_h^2)^{-1/2}\vec{v}h(T)\rangle_{\mathbb{R}^N,h}| \\
&\quad + |\langle(-\partial_h^2)^{-1/2}\vec{v}h'(0), (-\partial_h^2)^{-1/2}\vec{v}h(0)\rangle_{\mathbb{R}^N,h}| \\
&\le \frac{1}{2\sqrt{\alpha_0}}[\|(-\partial_h^2)^{-1/2}\vec{v}h'(T)\|^2_{\mathbb{R}^N,h} + \|(-\partial_h^2)^{-1/2}\vec{v}h'(0)\|^2_{\mathbb{R}^N,h}] \\
&\quad + \frac{\sqrt{\alpha_0}}{2}\big[\|(-\partial_h^2)^{-1/2}\vec{v}h(T)\|^2_{\mathbb{R}^N,h} + \|(-\partial_h^2)^{-1/2}\vec{v}h(0)\|^2_{\mathbb{R}^N,h}\big].
\end{aligned} \tag{4.11}$$

Moreover, according to Theorem 4.1, we have

$$\|(-\partial_h^2)^{-1/2}\vec{v}h(T)\|^2_{\mathbb{R}^N,h} + \|(-\partial_h^2)^{-1/2}\vec{v}h(0)\|^2_{\mathbb{R}^N,h} \le \frac{1}{\alpha_0}(\|\vec{v}h(T)\|^2_{\mathbb{R}^N,h} + \|\vec{v}h(0)\|^2_{\mathbb{R}^N,h}).$$

Inserting this last inequality into (4.11), we obtain

$$|[Y_h(t)]_0^T| \le \frac{1}{\sqrt{\alpha_0}}(\widetilde{E}_h(\vec{v}h;T) + \widetilde{E}_h(\vec{v}h;0)). \tag{4.12}$$

On the other hand,

$$\begin{aligned}
&|\alpha\int_0^T \langle\vec{u}h(t), (-\partial_h^2)^{-1}\vec{v}h(t)\rangle_{\mathbb{R}^N,h} dt| \\
&\le \frac{|\alpha|}{2}\int_0^T \|\vec{u}h(t)\|^2_{\mathbb{R}^N,h} dt + \frac{|\alpha|}{2}\int_0^T \|(-\partial_h^2)^{-1}\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt\,.
\end{aligned}$$

In view of inequality (4.1), we can write

$$\begin{aligned}
&|\alpha\int_0^T \langle\vec{u}h(t), (-\partial_h^2)^{-1}\vec{v}h(t)\rangle_{\mathbb{R}^N,h} dt| \\
&\le \frac{|\alpha|}{2}\int_0^T \|\vec{u}h(t)\|^2_{\mathbb{R}^N,h} dt + \frac{|\alpha|}{2\alpha_0^2}\int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt.
\end{aligned} \tag{4.13}$$

Using (4.10), (4.12), (4.13) and (4.9), we obtain

$$\begin{aligned}
&\int_0^T \|(-\partial_h^2)^{-1/2}\vec{v}h'(t)\|^2_{\mathbb{R}^N,h} dt \\
&\le \frac{C_2}{\varepsilon_1|\alpha|}(E_h(\vec{u}h;T) + E_h(\vec{u}h;0)) + C_1\int_0^T \|\vec{u}h(t)\|^2_{\mathbb{R}^N,h} dt \\
&\quad + \Big(\frac{1}{\sqrt{\alpha_0}} + \frac{C_2\varepsilon_1}{|\alpha|}\Big)(\widetilde{E}_h(\vec{v}h;T) + \widetilde{E}_h(\vec{v}h;0)),
\end{aligned} \tag{4.14}$$

with

$$C_1 = 1 + \frac{\alpha_0(1+\alpha_0^2)}{2\alpha_0^2}, \quad C_2 = \frac{2\alpha_0^2 + \alpha_0}{2\alpha_0^2}.$$

Next, we estimate $\widetilde{E}_h(\vec{v}h;T) + \widetilde{E}_h(\vec{v}h;0)$. For this purpose, we take the inner product of (1.6)-2 with $(-\partial_h^2)^{-1}\vec{v}h'(t)$ in space $(\mathbb{R}^N, \|\cdot\|_{\mathbb{R}^N,h})$ to obtain

$$\frac{d}{dt}\widetilde{E}_h(\vec{v}h;t) = -\alpha\langle(-\partial_h^2)^{-1/2}\vec{u}h(t), (-\partial_h^2)^{-1/2}\vec{v}h'(t)\rangle_{\mathbb{R}^N,h}.$$

It follows that

$$\widetilde{E}_h(\vec{v}h;T) + \widetilde{E}_h(\vec{v}h;0)$$

$$= 2\widetilde{E}_h(\vec{v}h;0) - \alpha\int_0^T \langle(-\partial_h^2)^{-1/2}\vec{u}h(t),(-\partial_h^2)^{-1/2}\vec{v}h'(t)\rangle_{\mathbb{R}^N,h}dt.$$

We estimate now the second member of the right-hand side of this equation in the following way

$$\begin{aligned}
&\big|\alpha\int_0^T \langle(-\partial_h^2)^{-1/2}\vec{u}h(t),(-\partial_h^2)^{-1/2}\vec{v}h'(t)\rangle_{\mathbb{R}^N,h}dt\big| \\
&\le \frac{|\alpha|}{2}\int_0^T \|(-\partial_h^2)^{-1/2}\vec{u}h(t)\|^2_{\mathbb{R}^N,h}dt + \frac{|\alpha|}{2}\int_0^T \|(-\partial_h^2)^{-1/2}\vec{v}h'(t)\|^2_{\mathbb{R}^N,h}dt \\
&\le \frac{|\alpha|}{2\alpha_0}\int_0^T \|\vec{u}h(t)\|^2_{\mathbb{R}^N,h}dt + \frac{|\alpha|}{2}\int_0^T \|(-\partial_h^2)^{-1/2}\vec{v}h'(t)\|^2_{\mathbb{R}^N,h}dt,
\end{aligned} \tag{4.15}$$

where in the last step we have used inequality (4.1). Moreover, by (4.14) and having in mind equation (4.15) we can write

$$\begin{aligned}
&\big[1 - \frac{|\alpha|}{2\sqrt{\alpha_0}} - \frac{\varepsilon_1 C_2}{2}\big]\Big(\widetilde{E}_h(\vec{v}h;T) + \widetilde{E}_h(\vec{v}h;0)\Big) \\
&\le 2\widetilde{E}_h(\vec{v}h;0) + \frac{(\alpha_0 C_1+1)|\alpha|}{2\alpha_0}\int_0^T \|\vec{u}h(t)\|^2_{\mathbb{R}^N,h}dt + \frac{C_2}{2\varepsilon_1}(E_h(\vec{u}h;T) + E_h(\vec{u}h;0)).
\end{aligned}$$

Taking $\varepsilon_1 = \frac{1}{C_2}$ in the above inequality, we have

$$\begin{aligned}
&(1 - \frac{|\alpha|}{\sqrt{\alpha_0}})(\widetilde{E}_h(\vec{v}h;T) + \widetilde{E}_h(\vec{v}h;0)) \\
&\le \frac{(\alpha_0 C_1+1)|\alpha|}{\alpha_0}\int_0^T \|\vec{u}h(t)\|^2_{\mathbb{R}^N,h}dt + 4\widetilde{E}_h(\vec{v}h;0) \quad + C_2^2(E_h(\vec{u}h;T) + E_h(\vec{u}h;0)).
\end{aligned}$$

when $|\alpha| < \sqrt{\alpha_0}$, this implies,

$$\begin{aligned}
&\widetilde{E}_h(\vec{v}h;T) + \widetilde{E}_h(\vec{v}h;0) \\
&\le \frac{C_3|\alpha|}{\sqrt{\alpha_0}-|\alpha|}\int_0^T \|\vec{u}h(t)\|^2_{\mathbb{R}^N,h}dt + \frac{4\sqrt{\alpha_0}}{\sqrt{\alpha_0}-|\alpha|}\widetilde{E}_h(\vec{v}h;0) \\
&\quad + \frac{C_4}{\sqrt{\alpha_0}-|\alpha|}(E_h(\vec{u}h;T) + E_h(\vec{u}h;0)),
\end{aligned} \tag{4.16}$$

with $C_3 = (\alpha_0 C_1 + 1)/\sqrt{\alpha_0}$ and $C_4 = \sqrt{\alpha_0}C_2^2$.

Step 2. Improvement of estimates (4.9) and (4.14). Taking $\varepsilon_1 = 1$ in equation (4.9) yields

$$\begin{aligned}
\int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h}dt &\le \int_0^T \|\vec{u}h(t)\|^2_{\mathbb{R}^N,h}dt + \frac{1}{|\alpha|}(E_h(\vec{u}h;T) + E_h(\vec{u}h;0)) \\
&\quad + \frac{1}{|\alpha|}(\widetilde{E}_h(\vec{v}h;T) + \widetilde{E}_h(\vec{v}h;0)).
\end{aligned}$$

Inserting (4.16) in this last inequality, we obtain

$$
\begin{aligned}
&\int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h}dt \\
&\le \frac{C_7}{|\alpha|(\sqrt{\alpha_0}-|\alpha|)}(E_h(\vec{u}h;T)+E_h(\vec{u}h;0)) \\
&\quad + \frac{C_5}{|\alpha|(\sqrt{\alpha_0}-|\alpha|)}\widetilde{E}_h(\vec{v}h;0) + \frac{C_6}{\sqrt{\alpha_0}-|\alpha|}\int_0^T \|\vec{u}h(t)\|^2_{\mathbb{R}^N,h}dt,
\end{aligned}
\tag{4.17}
$$

with $C_5 = 4\sqrt{\alpha_0}$, $C_6 = \sqrt{\alpha_0}+C_3$ and $C_7 = \sqrt{\alpha_0}+C_4$. On the other hand, equation (4.14), with $\varepsilon_1 = 1$, implies

$$
\begin{aligned}
&\int_0^T \|(-\partial_h^2)^{-1/2}\vec{v}h'(t)\|^2_{\mathbb{R}^N,h}dt \\
&\le \Big(\frac{1}{\sqrt{\alpha_0}}+\frac{C_2}{|\alpha|}\Big)(\widetilde{E}_h(\vec{v}h;T)+\widetilde{E}_h(\vec{v}h;0)) \\
&\quad + \frac{C_2}{|\alpha|}(E_h(\vec{u}h;T)+E_h(\vec{u}h;0)) + C_1\int_0^T \|\vec{u}h(t)\|^2_{\mathbb{R}^N,h}dt
\end{aligned}
$$

and in view of (4.16), we can write

$$
\begin{aligned}
&\int_0^T \|(-\partial_h^2)^{-1/2}\vec{v}h'(t)\|^2_{\mathbb{R}^N,h}dt \\
&\le \frac{C_{10}}{|\alpha|(\sqrt{\alpha_0}-|\alpha|)}(E_h(\vec{u}h;T)+E_h(\vec{u}h;0)) \\
&\quad + \frac{C_8}{|\alpha|(\sqrt{\alpha_0}-|\alpha|)}\widetilde{E}_h(\vec{v}h;0) + \frac{C_9}{\sqrt{\alpha_0}-|\alpha|}\int_0^T \|\vec{u}h(t)\|^2_{\mathbb{R}^N,h}dt,
\end{aligned}
\tag{4.18}
$$

with $C_8 = 4\sqrt{\alpha_0}(1+C_2)$, $C_9 = C_1 + (\sqrt{\alpha_0}+C_2)C_3$ and $C_{10} = (\alpha_0 + C_2\sqrt{\alpha_0})C_2^2 + C_2\sqrt{\alpha_0}$.

Step 3. Estimate for $E_h(\vec{u}h;T)+E_h(\vec{u}h;0)$ and improvement of (4.16), (4.17) and (4.18). Using the characteristics of system (1.6), we obtain

$$
\frac{d}{dt}E_h(\vec{u}h;t) = -\alpha\langle \vec{v}h(t), \vec{u}h'(t)\rangle_{\mathbb{R}^N,h}. \tag{4.19}
$$

This gives

$$
E_h(\vec{u}h;T) - E_h(\vec{u}h;0) = -\alpha\int_0^T \langle \vec{v}h(t), \vec{u}h'(t)\rangle_{\mathbb{R}^N,h}dt.
$$

It follows that

$$
\begin{aligned}
&E_h(\vec{u}h;T)+E_h(\vec{u}h;0) \\
&\le 2E_h(\vec{u}h;0) + \frac{|\alpha|}{2\varepsilon_2}\int_0^T \|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h}dt + \frac{|\alpha|\varepsilon_2}{2}\int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h}dt
\end{aligned}
$$

for each $\varepsilon_2 > 0$, and in view of (4.17) we can write

$$
\begin{aligned}
&\big[1 - \frac{\varepsilon_2 C_7}{2(\sqrt{\alpha_0}-|\alpha|)}\big](E_h(\vec{u}h;T)+E_h(\vec{u}h;0)) \\
&\le 2E_h(\vec{u}h;0) + \frac{|\alpha|}{2\varepsilon_2}\int_0^T \|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h}dt + \frac{C_5\varepsilon_2}{2(\sqrt{\alpha_0}-|\alpha|)}\widetilde{E}_h(\vec{v}h;0)
\end{aligned}
$$

$$+ \frac{|\alpha|\varepsilon_2 C_6}{2(\sqrt{\alpha_0} - |\alpha|)} \int_0^T \|\vec{u}h(t)\|^2_{\mathbb{R}^N,h} dt.$$

Next, taking $\varepsilon_2 = \big(\sqrt{\alpha_0} - |\alpha|\big)/C_7$ in the above inequality, we have

$$\begin{aligned} E_h(\vec{u}h;T) + E_h(\vec{u}h;0) &\le C_{11}(E_h(\vec{u}h;0) + \widetilde{E}_h(\vec{v}h;0)) \\ &\quad + \frac{C_{12}|\alpha|}{\sqrt{\alpha_0} - |\alpha|} \int_0^T \big(\|\vec{u}h(t)\|^2_{\mathbb{R}^N,h} + \|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h}\big) dt, \end{aligned} \tag{4.20}$$

with $C_{11} = \max(C_7, \sqrt{\alpha_0}\frac{C_6}{C_7})$ and $C_{12} = \max(4, \frac{C_5}{C_7})$. Inserting this last inequality in equations (4.16)-(4.18), we obtain

$$\begin{aligned} \int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt &\le \frac{C_{13}}{|\alpha|(\sqrt{\alpha_0} - |\alpha|)}(E_h(\vec{u}h;0) + \widetilde{E}_h(\vec{v}h;0)) \\ &\quad + \frac{C_{14}}{(\sqrt{\alpha_0} - |\alpha|)^2} \int_0^T (\|\vec{u}h(t)\|^2_{\mathbb{R}^N,h} + \|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h}) dt, \end{aligned} \tag{4.21}$$

$$\begin{aligned} &\int_0^T \|(-\partial_h^2)^{-1/2}\vec{v}h'(t)\|^2_{\mathbb{R}^N,h} dt \\ &\le \frac{C_{15}}{|\alpha|(\sqrt{\alpha_0} - |\alpha|)}(E_h(\vec{u}h;0) + \widetilde{E}_h(\vec{v}h;0)) \\ &\quad + \frac{C_{16}}{(\sqrt{\alpha_0} - |\alpha|)^2} \int_0^T \big(\|\vec{u}h(t)\|^2_{\mathbb{R}^N,h} + \|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h}\big) dt, \end{aligned} \tag{4.22}$$

$$\begin{aligned} &\widetilde{E}_h(\vec{v}h;T) + \widetilde{E}_h(\vec{v}h;0) \\ &\le \frac{C_{17}}{\sqrt{\alpha_0} - |\alpha|}(E_h(\vec{u}h;0) + \widetilde{E}_h(\vec{v}h;0)) \\ &\quad + \frac{C_{18}|\alpha|}{(\sqrt{\alpha_0} - |\alpha|)^2} \int_0^T \big(\|\vec{u}h(t)\|^2_{\mathbb{R}^N,h} + \|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h}\big) dt, \end{aligned} \tag{4.23}$$

with the notation

$$\begin{gathered} C_{13} = \max(C_7C_{11}, C_5), \quad C_{14} = \max(C_7C_{12}, \sqrt{\alpha_0}C_6), \\ C_{15} = \max(C_8, C_{10}C_{11}), \quad C_{16} = \max(C_{10}C_{12}, \sqrt{\alpha_0}C_9), \\ C_{17} = \max(C_4C_{11}, 4\sqrt{\alpha_0}), \quad C_{18} = \max(C_4C_{12}, \sqrt{\alpha_0}C_3). \end{gathered}$$

Step 4. Proof of estimate (4.31). From (4.19), we deduce

$$E_h(\vec{u}h;t) = E_h(\vec{u}h;0) - \alpha \int_0^t \langle \vec{v}h(s), \vec{u}h'(s)\rangle_{\mathbb{R}^N,h} ds.$$

Tt follows that

$$E_h(\vec{u}h;t) \ge E_h(\vec{u}h;0) - \frac{|\alpha|}{2\varepsilon_3} \int_0^T \|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h} dt - \frac{|\alpha|\varepsilon_3}{2} \int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt \tag{4.24}$$

for all $\varepsilon_3 > 0$. Integrating the latter inequality between 0 and T, we obtain

$$\begin{aligned} \int_0^T E_h(\vec{u}h;t)\, dt &\ge TE_h(\vec{u}h;0) - \frac{|\alpha|T}{2\varepsilon_3} \int_0^T \|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h} dt \\ &\quad - \frac{|\alpha|\varepsilon_3 T}{2} \int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt, \end{aligned}$$

and having in mind equation (4.21), we can improve the last estimate as follows

$$\begin{aligned}\int_0^T E_h(\vec{u}h;t)\,dt \ge{}& T\Big[1-\frac{\varepsilon_3 C_{13}}{2(\sqrt{\alpha_0}-|\alpha|)}\Big]E_h(\vec{u}h;0)-\frac{\varepsilon_3 C_{13}T}{2(\sqrt{\alpha_0}-|\alpha|)}\\ &\times \widetilde{E}_h(\vec{v}h;0)-\frac{|\alpha|C_{14}\varepsilon_3 T}{(\sqrt{\alpha_0}-|\alpha|)^2}\int_0^T \|\vec{u}h(t)\|^2_{\mathbb{R}^N,h}dt\\ &-\frac{|\alpha|T}{2}\Big[\frac{1}{\varepsilon_3}+\frac{C_{14}\varepsilon_3}{(\sqrt{\alpha_0}-|\alpha|)^2}\Big]\int_0^T \|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h}dt.\end{aligned}$$

Moreover, in view of Theorem 4.1, we deduce

$$\begin{aligned}\int_0^T E_h(\vec{u}h;t)\,dt \ge{}& T\Big[1-\frac{\varepsilon_3 C_{13}}{2(\sqrt{\alpha_0}-|\alpha|)}\Big]E_h(\vec{u}h;0)-\frac{\varepsilon_3 C_{13}T}{2(\sqrt{\alpha_0}-|\alpha|)}\\ &\times \widetilde{E}_h(\vec{v}h;0)-\frac{|\alpha|C_{14}\varepsilon_3 T}{\alpha_0(\sqrt{\alpha_0}-|\alpha|)^2}\|(-\partial_h^2)^{1/2}\vec{u}h(t)\|^2_{\mathbb{R}^N,h}\\ &-\frac{|\alpha|T}{2}\Big[\frac{1}{\varepsilon_3}+\frac{C_{14}\varepsilon_3}{(\sqrt{\alpha_0}-|\alpha|)^2}\Big]\int_0^T \|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h}dt.\end{aligned}$$

Choosing $\varepsilon_3=(\sqrt{\alpha_0}-|\alpha|)/C_{13}$ in the above inequality, yields

$$\int_0^T E_h(\vec{u}h;t)\,dt \ge \frac{T}{2}(E_h(\vec{u}h;0)-\widetilde{E}_h(\vec{v}h;0))-\frac{|\alpha|C_{19}T}{\sqrt{\alpha_0}-|\alpha|}\int_0^T E_h(\vec{u}h;t)\,dt,$$

where $C_{19}=\max(\frac{C_{14}}{\alpha_0 C_{13}},\frac{C_{13}^2+C_{14}}{2C_{13}})$. In other words

$$\Big[1+\frac{|\alpha|C_{19}T}{\sqrt{\alpha_0}-|\alpha|}\Big]\int_0^T E_h(\vec{u}h;t)\,dt \ge \frac{T}{2}(E_h(\vec{u}h;0)-\widetilde{E}_h(\vec{v}h;0)).$$

Since $|\alpha|\le\sqrt{\alpha_0}/2$, it follows that

$$\int_0^T E_h(\vec{u}h;t)\,dt \ge \frac{C_1'T}{2(1+|\alpha|T)}(E_h(\vec{u}h;0)-\widetilde{E}_h(\vec{v}h;0)),$$

with $C_1'=\max(\sqrt{\alpha_0},C_{19})/\sqrt{\alpha_0}$. This completes the proof. □

The next lemma indicates that the natural and weakened total energies of system (1.6) are conserved in time.

Lemma 4.4 (Conservation of energies). *For all solutions* $(\vec{u}h,\vec{v}h)$ *of system* (1.6), *we have*

$$E_{T,h}(t)=E_{T,h}(0),\quad \forall t\in[0,T], \tag{4.25}$$
$$\widetilde{E}_{T,h}(t)=\widetilde{E}_{T,h}(0),\quad \forall t\in[0,T]. \tag{4.26}$$

Proof. (1) Multiplying the first equation in (1.6) by $\vec{u}h'$, we obtain

$$\langle \vec{u}h''(t)-\partial_h^2\vec{u}h(t)+\alpha\vec{v}h(t),\vec{u}h'(t)\rangle_{\mathbb{R}^N,h}=0.$$

It follows that

$$\begin{aligned}&\langle \vec{u}h''(t),\vec{u}h'(t)\rangle_{\mathbb{R}^N,h}+\langle(-\partial_h^2)^{1/2}\vec{u}h(t),(-\partial_h^2)^{1/2}\vec{u}h'(t)\rangle_{\mathbb{R}^N,h}\\ &+\alpha\langle\vec{v}h(t),\vec{u}h'(t)\rangle_{\mathbb{R}^N,h}=0.\end{aligned}$$

Therefore,

$$\frac{d}{dt}E_h(\vec{u}h;t)+\alpha\langle\vec{v}h(t),\vec{u}h'(t)\rangle_{\mathbb{R}^N,h}=0. \tag{4.27}$$

Analogously, multiplying (1.6)-2 by $\vec{v}h'$ leads to

$$\frac{d}{dt}E_h(\vec{v}h;t) + \alpha\langle \vec{u}h(t), \vec{v}h'(t)\rangle_{\mathbb{R}^N,h} = 0. \tag{4.28}$$

Adding (4.27) and (4.28), we can write

$$\frac{d}{dt}E_{T,h}(t) = 0,$$

which is equivalent to (4.25).

(2) Analogously to (4.25) multiplying equations (1.6)-1 and (1.6)-2, respectively, by $(-\partial_h^2)^{-1}\vec{u}h'$ and $(-\partial_h^2)^{-1}\vec{v}h'$ and taking the sum of the resulting two identities we obtain

$$\begin{aligned} &\frac{d}{dt}\widetilde{E}_h(\vec{u}h;t) + \frac{d}{dt}\widetilde{E}_h(\vec{v}h;t) + \alpha\langle \vec{v}h(t), (-\partial_h^2)^{-1}\vec{u}h'(t)\rangle_{\mathbb{R}^N,h} \\ &+ \alpha\langle \vec{u}h(t), (-\partial_h^2)^{-1}\vec{v}h'(t)\rangle_{\mathbb{R}^N,h} = 0, \end{aligned}$$

and using the symmetry of the matrix $(-\partial_h^2)^{-1}$ we obtain

$$\frac{d}{dt}\widetilde{E}_{T,h}(t) = 0.$$

□

From Lemma 4.4, we deduce the following result.

Lemma 4.5. *For all* $0 \le |\alpha| \le \frac{\alpha_0}{3}$,

$$\int_0^T (E_h(\vec{u}h;t) + \widetilde{E}_h(\vec{v}h;t))dt \ge \frac{C_2'T}{2}(\widetilde{E}_h(\vec{u}h;0) + \widetilde{E}_h(\vec{v}h;0)), \tag{4.29}$$

where $C_2' = \min(1, \alpha_0)$.

Proof. We recall that

$$E_h(\vec{u}h;t) = \frac{1}{2}\|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h} + \frac{1}{2}\|(-\partial_h^2)^{1/2}\vec{u}h(t)\|^2_{\mathbb{R}^N,h},$$

and according to Theorem 4.1, we can write

$$E_h(\vec{u}h;t) \ge \frac{\alpha_0}{2}\big\|(-\partial_h^2)^{-1/2}\vec{u}h'(t)\big\|^2_{\mathbb{R}^N,h} + \frac{\alpha_0}{2}\|\vec{u}h(t)\|^2_{\mathbb{R}^N,h} = \alpha_0\widetilde{E}_h(\vec{u}h;t).$$

It follows that

$$\int_0^T (E_h(\vec{u}h;t) + \widetilde{E}_h(\vec{v}h;t))dt \ge C_2'\int_0^T (\widetilde{E}_h(\vec{u}h;t) + \widetilde{E}_h(\vec{v}h;t))dt, \tag{4.30}$$

with $C_2' = \min(1, \alpha_0)$. On the other hand,

$$|\widetilde{E}_{T,h}(t) - (\widetilde{E}_h(\vec{u}h;t) + \widetilde{E}_h(\vec{v}h;t))| = \big|\alpha\langle(-\partial_h^2)^{-1}\vec{u}h(t), \vec{v}h(t)\rangle_{\mathbb{R}^N,h}\big|,$$

and thanks to Theorem 4.1, one has

$$|\widetilde{E}_{T,h}(t) - (\widetilde{E}_h(\vec{u}h;t) + \widetilde{E}_h(\vec{v}h;t))| \le \frac{|\alpha|}{\alpha_0}(\widetilde{E}_h(\vec{u}h;t) + \widetilde{E}_h(\vec{v}h;t)). \tag{4.31}$$

Hence,

$$\widetilde{E}_h(\vec{u}h;t) + \widetilde{E}_h(\vec{v}h;t) \ge \frac{\alpha_0}{\alpha_0 + |\alpha|}\widetilde{E}_{T,h}(t).$$

Integrating this last inequality over $t \in [0,T]$ and using the fact that the energy $\widetilde{E}_{T,h}(t)$ is conservative, we deduce that

$$\int_0^T (\widetilde{E}_h(\vec{u}h;t) + \widetilde{E}_h(\vec{v}h;t))dt \geq \frac{\alpha_0 T}{\alpha_0 + |\alpha|}\widetilde{E}_{T,h}(0). \tag{4.32}$$

Moreover, thanks to inequality (4.31), we have

$$\widetilde{E}_{T,h}(0) \geq \frac{\alpha_0 - |\alpha|}{\alpha_0}(\widetilde{E}_h(\vec{u}h;0) + \widetilde{E}_h(\vec{v}h;0)),$$

and inserting this last equation into (4.32) yields

$$\int_0^T (\widetilde{E}_h(\vec{u}h;t) + \widetilde{E}_h(\vec{v}h;t))dt \geq \frac{\alpha_0 - |\alpha|}{\alpha_0 + |\alpha|}T(\widetilde{E}_h(\vec{u}h;0) + \widetilde{E}_h(\vec{v}h;0)). \tag{4.33}$$

However, since

$$\frac{\alpha_0 - |\alpha|}{\alpha_0 + |\alpha|} \geq \frac{1}{2}$$

for all $|\alpha| \leq \frac{\alpha_0}{3}$, we deduce from (4.33) that

$$\int_0^T (\widetilde{E}_h(\vec{u}h;t) + \widetilde{E}_h(\vec{v}h;t))dt \geq \frac{T}{2}(\widetilde{E}_h(\vec{u}h;0) + \widetilde{E}_h(\vec{v}h;0)).$$

Inserting this inequality into (4.30), the desired estimate (4.29) is obtained. □

We complete this subsection with the following lemma.

Lemma 4.6. *For all $0 \leq |\alpha| \leq \min(\alpha_0, \sqrt{\alpha_0})$, we have*

$$\begin{aligned}\int_0^T \widetilde{E}_h(\vec{v}h;t)\,dt &\leq \frac{C_3'}{|\alpha|(\sqrt{\alpha_0} - |\alpha|)}(E_h(\vec{u}h;0) + \widetilde{E}_h(\vec{v}h;0)) \\ &\quad + \frac{C_4'}{(\sqrt{\alpha_0} - |\alpha|)^2}\int_0^T E_h(\vec{u}h;t)\,dt,\end{aligned} \tag{4.34}$$

$$\begin{aligned}\int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h}dt &\leq \frac{C_{13}}{|\alpha|(\sqrt{\alpha_0} - |\alpha|)}(E_h(\vec{u}h;0) + \widetilde{E}_h(\vec{v}h;0)) \\ &\quad + \frac{C_5'}{(\sqrt{\alpha_0} - |\alpha|)^2}\int_0^T E_h(\vec{u}h;t)\,dt,\end{aligned} \tag{4.35}$$

$$\begin{aligned}\widetilde{E}_h(\vec{v}h;T) + \widetilde{E}_h(\vec{v}h;0) &\leq \frac{C_{17}}{\sqrt{\alpha_0} - |\alpha|}(E_h(\vec{u}h;0) + \widetilde{E}_h(\vec{v}h;0)) \\ &\quad + \frac{C_6'|\alpha|}{(\sqrt{\alpha_0} - |\alpha|)^2}\int_0^T E_h(\vec{u}h;t)\,dt,\end{aligned} \tag{4.36}$$

where the constants C_3'-C_6' will be explicitly given.

Proof. First, we recall estimates (4.21) and (4.22) from the proof of Lemma 4.3:

$$\begin{aligned}\int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h}dt &\leq \frac{C_{13}}{|\alpha|(\sqrt{\alpha_0} - |\alpha|)}(E_h(\vec{u}h;0) + \widetilde{E}_h(\vec{v}h;0)) \\ &\quad + \frac{C_{14}}{(\sqrt{\alpha_0} - |\alpha|)^2}\int_0^T \big(\|\vec{u}h(t)\|^2_{\mathbb{R}^N,h} + \|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h}\big)dt,\end{aligned} \tag{4.37}$$

$$\int_0^T \|(-\partial_h^2)^{-1/2}\vec{v}h'(t)\|^2_{\mathbb{R}^N,h}dt \leq \frac{C_{15}}{|\alpha|(\sqrt{\alpha_0} - |\alpha|)}(E_h(\vec{u}h;0) + \widetilde{E}_h(\vec{v}h;0))$$

$$+\frac{C_{16}}{(\sqrt{\alpha_0}-|\alpha|)^2}\int_0^T(\|\vec{u}h(t)\|^2_{\mathbb{R}^N,h}+\|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h})dt.$$

Taking the sum of these two inequalities, we obtain

$$\begin{aligned}\int_0^T\widetilde{E}_h(\vec{v}h;t)\,dt &\le \frac{C_3'}{|\alpha|(\sqrt{\alpha_0}-|\alpha|)}(E_h(\vec{u}h;0)+\widetilde{E}_h(\vec{v}h;0))\\ &\quad+\frac{C_{20}}{(\sqrt{\alpha_0}-|\alpha|)^2}\int_0^T(\|\vec{u}h(t)\|^2_{\mathbb{R}^N,h}+\|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h})dt.\end{aligned}\tag{4.38}$$

where $C_3'=(C_{13}+C_{15})/2$ and $C_{20}=(C_{14}+C_{16})/2$. And thanks to Theorem 4.1, we improve (4.38) as follows

$$\begin{aligned}\int_0^T\widetilde{E}_h(\vec{v}h;t)\,dt &\le \frac{C_3'}{|\alpha|(\sqrt{\alpha_0}-|\alpha|)}(E_h(\vec{u}h;0)+\widetilde{E}_h(\vec{v}h;0))\\ &\quad+\frac{C_4'}{(\sqrt{\alpha_0}-|\alpha|)^2}\int_0^T E_h(\vec{u}h;t)\,dt,\end{aligned}$$

with $C_4'=2\max(\frac{1}{\alpha_0},1)C_{20}$, which proves the inequality (4.34). The other estimates (4.35) and (4.36), are obtained easily from equations (4.23), (4.37) and the relation

$$\int_0^T(\|\vec{u}h(t)\|^2_{\mathbb{R}^N,h}+\|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h})dt\le\max(\frac{1}{\alpha_0},1)\int_0^T E_h(\vec{u}h;t)dt,$$

with the constants $C_5'=2\max(\frac{1}{\alpha_0},1)C_{14}$ and $C_6'=2\max(\frac{1}{\alpha_0},1)C_{12}$. □

4.3. **Uniform observability for the non homogeneous wave equation.** This section deals with the uniform observability, in a filtered space of initial conditions, of the finite-difference space semi-discretization of a $1-d$ non homogeneous single wave equation.

Let us consider the first equation in system (1.6)

$$\begin{gathered}u_j''+(-\partial_h^2\vec{u}h)_j=-\alpha v_j\quad\text{for } j=1,\dots,N,\ t\in(0,T)\\ u_0(t)=0,\ u_{N+1}(t)=0\quad\text{for } 0<t<T\\ u_j(0)=u_j^0,\ u_j'(0)=u_j^1\quad\text{for } j=1,\dots,N,\end{gathered}\tag{4.39}$$

where the initial datums $\vec{u}h^0$, $\vec{u}h^1$ are considered in the class $\mathcal{G}_h$ defined by (2.2).

We expand the solution $\vec{u}h$ on the basis $\vec{\varphi}k$ as

$$\vec{u}h(t)=\sum_{k=1}^N\widehat{u}_k(t)\vec{\varphi}k,\tag{4.40}$$

with

$$\begin{aligned}\widehat{u}_k(t)&=\widehat{u}_k^0\cos(t\sqrt{\lambda_k(h)})+\frac{\widehat{u}_k^1}{\sqrt{\lambda_k(h)}}\sin(t\sqrt{\lambda_k(h)})\\ &\quad+\frac{\alpha}{\sqrt{\lambda_k(h)}}\int_0^t\sin((t-s)\sqrt{\lambda_k(h)})\widehat{v}_k(s)\,ds,\end{aligned}$$

where $\widehat{u}_k^0$, $\widehat{u}_k^1$ and $\widehat{v}_k$ are the Fourier coefficients. More precisely, we have

$$\vec{u}h^0=\sum_{\lambda_kh^2<\gamma}\widehat{u}_k^0\vec{\varphi}k,\quad \vec{u}h^1=\sum_{\lambda_kh^2<\gamma}\widehat{u}_k^1\vec{\varphi}k,\quad \vec{v}h(t)=\sum_{k=1}^N\widehat{v}_k(t)\vec{\varphi}k.$$

However, we set

$$\vec{x}h(t) = \sum_{\lambda_k h^2 < \gamma} [\widehat{u}_k^0 \cos(t\sqrt{\lambda_k(h)}) + \frac{\widehat{u}_k^1}{\sqrt{\lambda_k(h)}} \sin(t\sqrt{\lambda_k(h)})],$$

$$\vec{y}h(t) = \sum_{k=1}^{N} \frac{\alpha}{\sqrt{\lambda_k(h)}} \int_0^t \sin((t-s)\sqrt{\lambda_k(h)})\widehat{v}_k(s)\,ds.$$

In this way, equation (4.40) becomes

$$\vec{u}h(t) = \vec{x}h(t) + \vec{y}h(t). \tag{4.41}$$

From [5], we have the following result.

Theorem 4.7. *For $h > 0$ sufficiently small and for all $T > 0$, it holds*

$$TE_h(\vec{x}h;0) \le \frac{C_7'(\gamma)}{4-\gamma} E_h(\vec{x}h;0) + \frac{2L}{4-\gamma} \int_0^T |\frac{x_N(t)}{h}|^2 dt, \tag{4.42}$$

with

$$C_7'(\gamma) = 4\sqrt{L^2 + \frac{3\gamma}{16\alpha_0} - \frac{\gamma h^2}{16}}.$$

Concerning the other term in decomposition (4.41), namely $\vec{y}h$, we have the following lemma.

Lemma 4.8. *For $h > 0$ and for all $T > 0$, it holds*

$$L\int_0^T |\frac{y_N(t)}{h}|^2 dt \le C_8'(T)|\alpha|^2 \int_0^T \|\vec{v}h(t)\|_{\mathbb{R}^N,h}^2 dt, \tag{4.43}$$

with $C_8'(T) = 2 + L(2T^2 + 4T + 1)$.

Proof. Proceeding as in the proof of [5, Lemma 2.1], we obtain the discrete identity

$$\frac{L}{2}\int_0^T |\frac{y_N(t)}{h}|^2 dt = A + [X_h(t)]_0^T - B, \tag{4.44}$$

with

$$A = \frac{h}{2}\sum_{j=0}^{N} \int_0^T \Big[|\frac{y_{j+1}(t) - y_j(t)}{h}|^2 + y_j'(t)y_{j+1}'(t)\Big] dt,$$

$$X_h(t) = h\sum_{j=1}^{N} j(\frac{y_{j+1}(t) - y_{j-1}(t)}{2})y_j'(t),$$

$$B = \alpha h \sum_{j=1}^{N} \int_0^T j(\frac{y_{j+1}(t) - y_{j-1}(t)}{2})v_j(t)\,dt.$$

We now estimate separately A, X_h and B.

Estimate for A. We have

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^T \|(-\partial_h^2)^{1/2}\vec{y}h(t)\|^2_{\mathbb{R}^N,h}dt + \frac{1}{2}\int_0^T \|\vec{y}h'(t)\|^2_{\mathbb{R}^N,h}dt \\
&\quad - \frac{h}{2}\sum_{j=0}^{N}\int_0^T (y_j'y_{j+1}' - |y_j'|^2)dt \\
&= \frac{1}{2}\int_0^T \|(-\partial_h^2)^{1/2}\vec{y}h(t)\|^2_{\mathbb{R}^N,h}dt + \frac{1}{2}\int_0^T \|\vec{y}h'(t)\|^2_{\mathbb{R}^N,h}dt \\
&\quad - \frac{h}{2}\sum_{j=0}^{N}\int_0^T |y_{j+1}' - y_j'|^2 dt \\
&\le \frac{1}{2}\int_0^T \|(-\partial_h^2)^{1/2}\vec{y}h(t)\|^2_{\mathbb{R}^N,h}dt + \frac{1}{2}\int_0^T \|\vec{y}h'(t)\|^2_{\mathbb{R}^N,h}dt \\
&= \int_0^T E_h(\vec{y}h;t)\,dt.
\end{aligned}
\tag{4.45}
$$

Estimate for X_h. We remark that

$$
\begin{aligned}
X_h(t) &= h\sum_{j=1}^{N} j(\frac{y_{j+1}-y_j}{2})y_j' + h\sum_{j=1}^{N} j(\frac{y_j - y_{j-1}}{2})y_j' \\
&= h\sum_{j=0}^{N}(jh)(\frac{y_{j+1}-y_j}{2h})y_j' + h\sum_{j=0}^{N}((j+1)h)(\frac{y_{j+1}-y_j}{2h})y_{j+1}'.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|X_h(t)| &\le \frac{L}{2}h\sum_{j=0}^{N}|\frac{y_{j+1}-y_j}{h}||y_j'| + \frac{L}{2}h\sum_{j=0}^{N}|\frac{y_{j+1}-y_j}{h}||y_{j+1}'| \\
&\le \frac{L}{4}h\sum_{j=0}^{N}|\frac{y_{j+1}-y_j}{h}|^2 + \frac{L}{4}h\sum_{j=0}^{N}|y_j'|^2 \\
&\quad + \frac{L}{4}h\sum_{j=0}^{N}|\frac{y_{j+1}-y_j}{h}|^2 + \frac{L}{4}h\sum_{j=0}^{N}|y_{j+1}'|^2 \\
&= \frac{L}{2}\|(-\partial_h^2)^{1/2}\vec{y}h(t)\|^2_{\mathbb{R}^N,h} + \frac{L}{2}\|\vec{y}h'(t)\|^2_{\mathbb{R}^N,h}.
\end{aligned}
\tag{4.46}
$$

Estimate for B. We have

$$\begin{aligned}
B &= \alpha h \sum_{j=1}^{N} \int_0^T j\Big(\frac{y_{j+1}-y_j}{2}\Big) v_j\,dt + \alpha h \sum_{j=1}^{N} \int_0^T j\Big(\frac{y_j-y_{j-1}}{2}\Big) v_j\,dt \\
&= \alpha h \sum_{j=1}^{N} \int_0^T j\Big(\frac{y_{j+1}-y_j}{2}\Big) v_j\,dt + \alpha h \sum_{j=0}^{N} \int_0^T j\Big(\frac{y_{j+1}-y_j}{2}\Big) v_{j+1}\,dt \\
&\le \frac{L}{4} h \sum_{j=0}^{N} \int_0^T \big|\frac{y_{j+1}-y_j}{h}\big|^2 dt + \frac{L|\alpha|^2}{4} h \sum_{j=0}^{N} \int_0^T |v_j|^2 dt \\
&\quad + \frac{L}{4} h \sum_{j=0}^{N} \int_0^T \big|\frac{y_{j+1}-y_j}{h}\big|^2 dt + \frac{L|\alpha|^2}{4} h \sum_{j=0}^{N} \int_0^T |v_{j+1}|^2 dt \\
&= \frac{L}{2} \int_0^T \|(-\partial_h^2)^{1/2} \vec{y}h(t)\|^2_{\mathbb{R}^N,h} dt + \frac{L|\alpha|^2}{2} \int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt.
\end{aligned} \tag{4.47}$$

Next, thanks to (4.44) and to (4.45)-(4.47), we obtain

$$\begin{aligned}
\frac{L}{2} \int_0^T \big|\frac{y_N(t)}{h}\big|^2 dt &\le (1+L) \int_0^T E_h(\vec{y}h;t)\,dt + L(E_h(\vec{y}h;T) + E_h(\vec{y}h;0)) \\
&\quad + \frac{L|\alpha|^2}{2} \int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt.
\end{aligned} \tag{4.48}$$

Moreover, we claim that

$$E_h(\vec{y}h;t) \le T|\alpha|^2 \int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt. \tag{4.49}$$

Indeed, we have

$$\begin{aligned}
\frac{h}{2} \sum_{j=0}^{N} \big|\frac{y_{j+1}-y_j}{h}\big|^2 &= \frac{h}{2} \sum_{j=0}^{N} \Big|\sum_{k=1}^{N} \frac{\widehat{A}_k}{h} (\varphi_{k,j+1} - \varphi_{k,j})\Big|^2 \\
&= \frac{h}{2} \sum_{j=0}^{N} \sum_{k=1}^{N} \widehat{A}_k^2 \big|\frac{\varphi_{k,j+1} - \varphi_{k,j}}{h}\big|^2 \\
&\quad + \frac{h}{2} \sum_{j=0}^{N} \sum_{\substack{k,k'=1 \\ k\neq k'}}^{N} \frac{\widehat{A}_k \widehat{A}_{k'}}{h} (\varphi_{k,j+1} - \varphi_{k,j})(\varphi_{k',j+1} - \varphi_{k',j}),
\end{aligned}$$

where

$$\widehat{A}_k = \widehat{A}_k(t) = \frac{\alpha}{\sqrt{\lambda_k(h)}} \int_0^t \sin((t-s)\sqrt{\lambda_k(h)}) \widehat{v}_k(s)\,ds.$$

Thanks to identities (3.8) and (4.2), we obtain

$$\begin{aligned}\frac{h}{2}\sum_{j=0}^{N}|\frac{y_{j+1}-y_j}{h}|^2 &= \frac{h}{2}\sum_{k=1}^{N}\lambda_k(h)|\widehat{A}_k|^2\sum_{j=1}^{N}|\varphi_{k,j}|^2\\ &= \frac{h|\alpha|^2}{2}\sum_{k=1}^{N}\Big|\int_0^t \sin((t-s)\sqrt{\lambda_k(h)})\widehat{v}_k(s)\,ds\Big|^2\sum_{j=1}^{N}|\varphi_{k,j}|^2\\ &\le \frac{T|\alpha|^2}{2}\int_0^T\sum_{k=1}^{N}|\widehat{v}_k(t)|^2dt\; h\sum_{j=1}^{N}|\varphi_{k,j}|^2\\ &= \frac{T|\alpha|^2}{2}\int_0^T\|\vec{v}h(t)\|^2_{\mathbb{R}^N,h}dt.\end{aligned} \tag{4.50}$$

On the other hand,

$$\frac{1}{2}\|\vec{y}h'(t)\|^2_{\mathbb{R}^N,h} = \frac{h}{2}\sum_{k=1}^{N}\lambda_k(h)|\widehat{A}'_k(t)|^2\sum_{j=1}^{N}|\varphi_{k,j}|^2 \le \frac{T|\alpha|^2}{2}\int_0^T\|\vec{v}h(t)\|^2_{\mathbb{R}^N,h}dt. \tag{4.51}$$

From (4.50)-(4.51) we deduce (4.49). Next using (4.48) together with (4.49), we obtain the desired estimate (4.43). □

The following result provides a uniform observability inequality for the non homogeneous discrete wave equation (4.39).

Lemma 4.9. *For all $h>0$, it holds*

$$\begin{aligned}\frac{4L}{4-\gamma}\int_0^T|\frac{u_N(t)}{h}|^2dt \ge (1-\varepsilon_4)\int_0^T E_h(\vec{u}h;t)\,dt - \frac{C'_7(\gamma)}{4-\gamma}E_h(\vec{u}h;0)\\ -\frac{C'_9(1)|\alpha|^2}{\varepsilon_4(4-\gamma)}\int_0^T\|\vec{v}h(t)\|^2_{\mathbb{R}^N,h}dt\end{aligned} \tag{4.52}$$

for all $\varepsilon_4\in(0,1)$, with $C'_9(1)=4C'_8(1)+2$.

Proof. First, thanks to (4.19) and Cauchy-Schwartz inequality, we have

$$\begin{aligned}E_h(\vec{u}h;t) &\le E_h(\vec{u}h;0)+\frac{\varepsilon_4}{2T}\int_0^T\|\vec{u}h'(t)\|^2_{\mathbb{R}^N,h}dt+\frac{|\alpha|^2T}{2\varepsilon_4}\int_0^T\|\vec{v}h(t)\|^2_{\mathbb{R}^N,h}dt\\ &\le E_h(\vec{u}h;0)+\frac{\varepsilon_4}{T}\int_0^T E_h(\vec{u}h;t)\,dt+\frac{|\alpha|^2T}{2\varepsilon_4}\int_0^T\|\vec{v}h(t)\|^2_{\mathbb{R}^N,h}dt\end{aligned} \tag{4.53}$$

for all $\varepsilon_4>0$. Integrating equation (4.53) over $t\in(0,T)$ provides

$$(1-\varepsilon_4)\int_0^T E_h(\vec{u}h;t)\,dt \le TE_h(\vec{u}h;0)+\frac{|\alpha|^2T^2}{2\varepsilon_4}\int_0^T\|\vec{v}h(t)\|^2_{\mathbb{R}^N,h}dt \tag{4.54}$$

for all $\varepsilon_4<1$. Remarking that $E_h(\vec{u}h;0)=E_h(\vec{x}h;0)$ and using inequality (4.42) together with (4.43), we estimate the energy $E_h(\vec{u}h;0)$ as

$$\begin{aligned}&TE_h(\vec{u}h;0)\\ &\le \frac{C'_7(\gamma)}{4-\gamma}E_h(\vec{x}h;0)+\frac{2L}{4-\gamma}\int_0^T|\frac{x_N(t)}{h}|^2dt\end{aligned}$$

$$\begin{aligned}
&\le \frac{C_7'(\gamma)}{4-\gamma}E_h(\vec{x}h;0) + \frac{4L}{4-\gamma}\int_0^T |\frac{u_N(t)}{h}|^2 dt + \frac{4L}{4-\gamma}\int_0^T |\frac{y_N(t)}{h}|^2 dt \\
&\le \frac{C_7'(\gamma)}{4-\gamma}E_h(\vec{x}h;0) + \frac{4L}{4-\gamma}\int_0^T |\frac{u_N(t)}{h}|^2 dt + \frac{4C_8'(T)|\alpha|^2}{4-\gamma}\int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt.
\end{aligned}$$

Inserting this last inequality into (4.54), we obtain

$$\begin{aligned}
\frac{4L}{4-\gamma}\int_0^T \big|\frac{u_N(t)}{h}\big|^2 dt \ge (1-\varepsilon_4)\int_0^T E_h(\vec{u}h;t)\,dt - \frac{C_7'(\gamma)}{4-\gamma}E_h(\vec{u}h;0) \\
- \frac{(4\varepsilon_4 C_8'(T)+2T^2)|\alpha|^2}{\varepsilon_4(4-\gamma)}\int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt.
\end{aligned}$$

Moreover, using that $\varepsilon_4 < 1$, we write

$$\begin{aligned}
\frac{4L}{4-\gamma}\int_0^T |\frac{u_N(t)}{h}|^2 dt \ge (1-\varepsilon_4)\int_0^T E_h(\vec{u}h;t)dt - \frac{C_7'(\gamma)}{4-\gamma}E_h(\vec{u}h;0) \\
- \frac{C_9'(T)|\alpha|^2}{\varepsilon_4(4-\gamma)}\int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt,
\end{aligned} \tag{4.55}$$

with $C_9'(T) = 4C_8'(T)+2T^2$. However, following the ideas of [2], the constant $C_9'(T)$ can be chosen independent of time T, indeed from (4.55) we deduce the following more general inequality

$$\begin{aligned}
\frac{4L}{4-\gamma}\int_{T_1}^{T_2} |\frac{u_N(t)}{h}|^2 dt \ge (1-\varepsilon_4)\int_{T_1}^{T_2} E_h(\vec{u}h;t)\,dt - \frac{C_7'(\gamma)}{4-\gamma}E_h(\vec{u}h;0) \\
- \frac{C_9'(T_2-T_1)|\alpha|^2}{\varepsilon_4(4-\gamma)}\int_{T_1}^{T_2} \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt,
\end{aligned} \tag{4.56}$$

for all $T_2 > T_1$. Let $k_0 := \mathbb{E}(T)$ be the integer part of T. If $k_0 \ge 1$, we write

$$\frac{4L}{4-\gamma}\int_0^T |\frac{u_N(t)}{h}|^2 dt = \frac{4L}{4-\gamma}\sum_{k=0}^{k_0-1}\int_k^{k+1} |\frac{u_N(t)}{h}|^2 dt + \frac{4L}{4-\gamma}\int_{k_0}^T |\frac{u_N(t)}{h}|^2 dt, \tag{4.57}$$

and in view of (4.56), this yields

$$\begin{aligned}
\frac{4L}{4-\gamma}\int_0^T |\frac{u_N(t)}{h}|^2 dt \ge (1-\varepsilon_4)\int_0^T E_h(\vec{u}h;t)\,dt - \frac{C_7'(\gamma)}{4-\gamma}E_h(\vec{u}h;0) \\
- \frac{C_9'(1)|\alpha|^2}{\varepsilon_4(4-\gamma)}\sum_{k=0}^{k_0-1}\int_k^{k+1} \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt \\
- \frac{C_9'(T-k_0)|\alpha|^2}{\varepsilon_4(4-\gamma)}\int_{k_0}^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt.
\end{aligned} \tag{4.58}$$

We remark that the function $T \mapsto -C_9(T)$ is decreasing in $(0,\infty)$ and that $T-k_0 < 1$. Hence, equation (4.58) becomes

$$\begin{aligned}
\frac{4L}{4-\gamma}\int_0^T |\frac{u_N(t)}{h}|^2 dt \ge (1-\varepsilon_4)\int_0^T E_h(\vec{u}h;t)\,dt - \frac{C_7'(\gamma)}{4-\gamma}E_h(\vec{u}h;0) \\
- \frac{C_9'(1)|\alpha|^2}{\varepsilon_4(4-\gamma)}\int_0^T \|\vec{v}h(t)\|^2_{\mathbb{R}^N,h} dt.
\end{aligned} \tag{4.59}$$

On the other hand, if $k_0 < 1$, it follows from (4.55) and $-C_9'(T) \geq -C_9'(1)$, that (4.59) is still true. This completes the proof. $\square$

4.4. **Proof of Theorem 2.2.** Now, the desired result on the uniform observability of system (1.6) can be derived in a straightforward manner. Indeed, by estimates (4.35), (4.36) and (4.52), one has

$$\begin{aligned}
&\frac{4L}{4-\gamma}\int_0^T |\frac{u_N(t)}{h}|^2 dt \\
&\geq \Big[1-\varepsilon_4+\frac{C_7'(\gamma)C_6'|\alpha|}{(\sqrt{\alpha_0}-|\alpha|)(4-\gamma)}-\frac{C_9'(1)C_5'|\alpha|^2}{\varepsilon_4(\sqrt{\alpha_0}-|\alpha|)^2(4-\gamma)}\Big] \\
&\quad\times\int_0^T E_h(\vec{u}h;t)\,dt-\Big[\frac{C_7'(\gamma)C_{17}}{(\sqrt{\alpha_0}-|\alpha|)(4-\gamma)}+\frac{C_9'(1)C_{13}|\alpha|}{\varepsilon_4(\sqrt{\alpha_0}-|\alpha|)(4-\gamma)}\Big] \\
&\quad\times(E_h(\vec{u}h;0)+\widetilde{E}_h(\vec{v}h;0)).
\end{aligned}$$

We take now $\varepsilon_4 = |\alpha|$ in the above inequality to obtain

$$\frac{8L}{4-\gamma}\int_0^T |\frac{u_N(t)}{h}|^2 dt \geq \int_0^T E_h(\vec{u}h;t)\,dt-\frac{C_{22}}{4-\gamma}(E_h(\vec{u}h;0)+\widetilde{E}_h(\vec{v}h;0)) \tag{4.60}$$

for sufficiently small $|\alpha|$ and where $C_{22} = (4/\sqrt{\alpha_0})(C_7'(\gamma)C_{17}+C_9'(1)C_{13})$. We remark that the second term of the right hand side of equation (4.60) has the wrong sign, to overcome this problem we introduce a small number ε_5 as follows

$$\begin{aligned}
&\frac{8L}{4-\gamma}\int_0^T |\frac{u_N(t)}{h}|^2 dt \\
&\geq (1-\varepsilon_5)\int_0^T E_h(\vec{u}h;t)\,dt+\varepsilon_5\int_0^T \Big(E_h(\vec{u}h;t) \\
&\quad+\widetilde{E}_h(\vec{v}h;t)\Big)\,dt-\varepsilon_5\int_0^T E_h(\vec{v}h;t)\,dt-\frac{C_{22}}{4-\gamma}\Big(E_h(\vec{u}h;0)+\widetilde{E}_h(\vec{v}h;0)\Big).
\end{aligned}$$

Then, from (4.29) and (4.34), we deduce

$$\begin{aligned}
&\frac{8L}{4-\gamma}\int_0^T |\frac{u_N(t)}{h}|^2 dt \\
&\geq (1-\varepsilon_5 C_{23})\int_0^T E_h(\vec{u}h;t)\,dt+\Big[\frac{\varepsilon_5 C_2'T}{2}-C_{24}\frac{\varepsilon_5+|\alpha|}{|\alpha|(4-\gamma)}\Big]\widetilde{E}_h(\vec{v}h;0) \\
&\quad-C_{24}\frac{\varepsilon_5+|\alpha|}{|\alpha|(4-\gamma)}E_h(\vec{u}h;0)+\frac{\varepsilon_5 C_2'T}{2}\widetilde{E}_h(\vec{u}h;0),
\end{aligned}$$

where $C_{23} = 1+\frac{4C_4'}{\alpha_0}$ and $C_{24} = \max(\frac{8C_3'}{\sqrt{\alpha_0}}, \sqrt{\alpha_0}C_{22})$. Since the term $\frac{\varepsilon_5 C_2'T}{2}\widetilde{E}_h(\vec{u}h;0)$ is positive,

$$\begin{aligned}
&\frac{8L}{4-\gamma}\int_0^T |\frac{u_N(t)}{h}|^2 dt \\
&\geq (1-\varepsilon_5 C_{23})\int_0^T E_h(\vec{u}h;t)\,dt+\Big[\frac{\varepsilon_5 C_2'T}{2}-C_{24}\frac{\varepsilon_5+|\alpha|}{|\alpha|(4-\gamma)}\Big]\widetilde{E}_h(\vec{v}h;0) \\
&\quad-C_{24}\frac{\varepsilon_5+|\alpha|}{|\alpha|(4-\gamma)}E_h(\vec{u}h;0)\,.
\end{aligned}$$

Using (4.6), we can write

$$\frac{8L}{4-\gamma}\int_0^T |\frac{u_N(t)}{h}|^2 dt \geq a(\alpha,T,\gamma)\widetilde{E}_h(\vec{v}h;0) + b(\alpha,T,\gamma)E_h(\vec{u}h;0). \tag{4.61}$$

where

$$a(\alpha,T,\gamma) = \frac{\varepsilon_5 C_2' T}{2} - C_{24}\frac{\varepsilon_5+|\alpha|}{|\alpha|(4-\gamma)} - \frac{C_1'(1-\varepsilon_5 C_{23})T}{2(1+|\alpha|T)},$$

$$b(\alpha,T,\gamma) = \frac{C_1'(1-\varepsilon_5 C_{23})T}{2(1+|\alpha|T)} - C_{24}\frac{\varepsilon_5+|\alpha|}{|\alpha|(4-\gamma)}.$$

Lemma 4.10. *For T large enough, and $|\alpha|$ sufficiently small, we have*

$$a(\alpha,T,\gamma) > 0, \quad \text{and} \quad b(\alpha,T,\gamma) > 0.$$

Proof. Indeed, we have

$$a(\alpha,T,\gamma) = \frac{Q_{\alpha,\gamma}(T)}{1+|\alpha|T},$$

where the polynomial Q_α is given by

$$Q_{\alpha,\gamma}(T) = a_2|\alpha|T^2 + \Big(a_2 - a_1 - \frac{C_{24}(\varepsilon_5+|\alpha|)}{4-\gamma}\Big)T - \frac{C_{24}(\varepsilon_5+|\alpha|)}{|\alpha|(4-\gamma)},$$

with

$$a_1 = \frac{C_1'(1-\varepsilon_5 C_{23})}{2} \quad \text{and} \quad a_2 = \frac{\varepsilon_5 C_2'}{2}.$$

For $\varepsilon_5 \to 0$, this polynomial $Q_{\alpha,\gamma}(T)$ has two real roots $T_1(\alpha,\gamma) \geq 0$ and $T_2(\alpha,\gamma) \leq 0$. Therefore

$$a(\alpha,T,\gamma) = \frac{a_2|\alpha|(T-T_1(\alpha,\gamma))(T-T_2(\alpha,\gamma))}{1+|\alpha|T},$$

which is positive for $T > T_1(\alpha,\gamma)$. We now turn to the term $b(\alpha,T,\gamma)$. We set

$$T_3(\alpha,\gamma) = \frac{C_{24}(\varepsilon_5+|\alpha|)}{|\alpha|\big((4-\gamma)a_1 - C_{24}(\varepsilon_5+|\alpha|)\big)}.$$

In this way, we have

$$b(\alpha,T,\gamma) = \frac{a_1(T-T_3(\alpha,\gamma))}{(1+|\alpha|T)(1+|\alpha|T_3(\alpha,\gamma))}.$$

Then remarking that $T_3(\alpha) > 0$, for $\varepsilon_5 \to 0$ and $|\alpha| \to 0$, we obtain $b(\alpha,T) > 0$ if $T > T_3(\alpha)$. This completes the proof. □

In view of (4.61) and by Lemma 4.10, we have the desired uniform inequality (2.3), and this completes the proof of Theorem 2.2.

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