Electronic Journal of Differential Equations, Vol. 2018 (2018), No. 163, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE AND REGULARITY OF GLOBAL SOLUTIONS NONLINEAR HARTREE EQUATIONS WITH COULOMB POTENTIALS AND SUBLINEAR DAMPING

YANJUN ZHAO, BINHUA FENG

Communicated by Jesus Ildefonso Diaz

Abstract. In this article, we consider the nonlinear Hartree equation with a sublinear damping and a time-dependent Coulomb potential in $\mathbb{R}^3$. We first prove the existence of a global solution and then obtain the Σ^2-regularity.

1. Introduction

Because of their important applications in physics, nonlinear Schrödinger equations with damping have been extensively studied; see [3, 4, 12, 16, 22, 23]. In this article, we consider the nonlinear Hartree equation with a time-dependent Coulomb potential and a sublinear damping,

$$
\begin{gathered}
i\partial_t u + \Delta u = V(x)u + \frac{1}{|x-a(t)|}u + \lambda\Big(\frac{1}{|x|} * |u|^2\Big)u - ib\frac{u}{|u|^\alpha},\\
(t,x) \in [0,\infty)\times\mathbb{R}^3,\\
u(0) = u_0 \in \Sigma,
\end{gathered}
\tag{1.1}
$$

where $u(t,x)$ is a complex-valued function in $(t,x)\in[0,\infty)\times\mathbb{R}^3$, $\lambda\in\mathbb{R}$, $b>0$, $0<\alpha\leqslant 1$, $a\in W^{1,1}((0,\infty),\mathbb{R}^3)$, Σ denotes the energy space associated to the harmonic oscillator, i.e.,

$$
\Sigma := \{u\in H^1(\mathbb{R}^3) : xu\in L^2(\mathbb{R}^3)\},
$$

equipped with the norm

$$
\|u\|_\Sigma := \|u\|_{H^1} + \|xu\|_{L^2}.
$$

The external potential V is assumed to be harmonic,

$$
V(x) = \sum_{j=1}^{3}\omega_j^2 x_j^2, \quad \omega_j>0. \tag{1.2}
$$

Equation (1.1) has many interesting applications in the quantum theory of large systems of non-relativistic bosonic atoms and molecules. In particular, this equation arises in the study of mean-field limit of many-body quantum systems, see, e.g.,

2010 *Mathematics Subject Classification.* 35J60, 35Q55.
Key words and phrases. Nonlinear Hartree equation; Coulomb potential; sublinear damping.

Submitted August 18, 2017. Published September 11, 2018.

[1, 21, 25] and the references therein. An essential feature of equation (1.1) is that the convolution kernel $|x|^{-1}$ still retains the fine structure of micro two-body interactions of the quantum system. It is therefore of considerable interest to extend mathematical methods originally develop for nonlinear Schrödinger equations with local nonlinearities to the study of Hartree-type equation, see [18, 19, 20, 24, 13, 14].

In this article, we are interested in the existence and Σ^2-regularity of global solutions to (1.1). More precisely, we will prove that the solution $u(t)$ of (1.1) satisfies: $\|u(t)\|_\Sigma \leqslant C(\|u_0\|_\Sigma)$ for all $t>0$. Moreover, if $u_0 \in \Sigma^2$, then $u \in L^\infty((0,T);\Sigma^2)$ for any $T>0$, where

$$\Sigma^2 = \{u \in L^2 : x^j \nabla^k u \in L^2, \forall \text{ multi-indices } j \text{ and } k \text{ with } |j|+|k| \le 2\},$$

equipped with the norm

$$\|u\|_{\Sigma^2} := \sum_{|j|+|k|\le 2} \|x^j \nabla^k u\|_{L^2}^2.$$

To solve these problems, we mainly use the ideas from Carles et al. [7, 9]. Carles and Gallo [7] proved that the solution for the Schrödinger equation

$$i\partial_t u + \Delta u = -ib\frac{u}{|u|^\alpha}, \quad (t,x) \in [0,\infty) \times M, \tag{1.3}$$

becomes zero in finite time, where M is a compact manifold without boundary. Carles and Ozawa [9] extended this study to the equation

$$i\partial_t u + \Delta u = V(x)u + \lambda |u|^{2\sigma_1}u - ia|u|^{2\sigma_2}u - ib\frac{u}{|u|^\alpha}, \quad (t,x) \in [0,\infty) \times M, \tag{1.4}$$

where M is either a compact manifold without boundary, or the whole space in the presence of harmonic confinement $V(x)$, in space dimension one and two.

However, compared with the equations (1.3) and (1.4) in [7, 9], there exist some major difficulties in the analysis of the global existence and regularity of (1.1). For example, due to the appearance of a time-dependent Coulomb potential $\frac{1}{|x-a(t)|}$, it is difficult to obtain the Σ^2-regularity of (1.1) by differentiating equation (1.1) two times with respect to space variable. Therefore, we use the idea due to Kato [17] (see also [10]), based on the general idea for Schrödinger equation, that two space derivatives cost the same as one time derivative. However, due to the same reason, we cannot immediately calculate the time derivative of equation (1.1). To overcome this difficulty, we will use a change of variable $y = x - a(t)$ to avoid the time derivative of the time-dependent Coulomb potential. This leads to a more complicated equation (4.8). For this reason, we only prove the solution $u \in L^\infty_{\rm loc}((0,\infty);\Sigma^2)$. If the initial data $u_0 \neq 0$, and the corresponding solution $u \in L^\infty((0,\infty);\Sigma^2)$, then the solution of (1.1) becomes zero in finite time. Indeed, it follows from (2.1) that

$$\|u(t)\|_{L^2} \leqslant C\|u(t)\|_{L^{2-\alpha}}^{1-\frac{3\alpha}{8-\alpha}} \|u(t)\|_{H^2}^{\frac{3\alpha}{8-\alpha}} \leqslant C\|u\|_{L^\infty((0,\infty);\Sigma^2)}^{\frac{3\alpha}{8-\alpha}} \|u(t)\|_{L^{2-\alpha}}^{1-\frac{3\alpha}{8-\alpha}},$$

for $t \geqslant 0$. This and (1.5) imply that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^3} |u(t,x)|^2 dx &= -2b\int_{\mathbb{R}^3} |u(t,x)|^{2-\alpha} dx \\
&\leqslant -\frac{2b}{C\|u\|_{L^\infty((0,\infty);\Sigma^2)}^{3\alpha/4}} \Big(\int_{\mathbb{R}^3} |u(t,x)|^2 dx\Big)^{1-(\alpha/8)}.
\end{aligned}$$

This differential inequality can be solved explicitly:

$$\|u(t)\|_{L^2}^2 \leqslant \Big(\|u(0)\|_{L^2}^{\alpha/4} - \frac{b\alpha}{4C\|u\|^{3\alpha/4}_{L^\infty((0,\infty);\Sigma^2)}}t\Big)^{8/\alpha}.$$

This implies that $\|u(t)\|_{L^2}$ vanishes in finite time T, with

$$T \leqslant \frac{4C\|u_0\|_{L^2}^{\alpha/4}\|u\|^{3\alpha/4}_{L^\infty((0,\infty);\Sigma^2)}}{b\alpha}.$$

Therefore, if $u \in L^\infty((0,\infty);\Sigma^2)$, the solution of (1.1) becomes zero in finite time.

Before stating our main results, we give the notion of weak solution of (1.1).

Definition 1.1. Assume $0 < \alpha < 1$ and $a \in W^{1,1}((0,\infty),\mathbb{R}^3)$. A global weak solution to (1.1) is a function $u \in C([0,\infty);L^2(\mathbb{R}^3)) \cap L^\infty((0,\infty);\Sigma)$ solving (1.1) in $\mathcal{D}'((0,\infty)\times\mathbb{R}^3)$.

Definition 1.2. Assume $\alpha = 1$ and $a \in W^{1,1}((0,\infty),\mathbb{R}^3)$. A global weak solution to (1.1) is a function $u \in C([0,\infty);L^2(\mathbb{R}^3)) \cap L^\infty((0,\infty);\Sigma)$ solving

$$i\partial_t u + \Delta u = V(x)u + \frac{1}{|x-a(t)|}u + \lambda\Big(\frac{1}{|x|} * |u|^2\Big)u - ibF$$

in $\mathcal{D}'((0,\infty)\times\mathbb{R}^3)$, where F is such that

$$\|F\|_{L^\infty((0,\infty)\times\mathbb{R}^3)} \leqslant 1, \quad \text{and} \quad F = \frac{u}{|u|} \ \textit{if}\ u \neq 0.$$

Since $a \in W^{1,1}((0,\infty),\mathbb{R}^3) \hookrightarrow L^\infty((0,\infty),\mathbb{R}^3)$, it follows that $\frac{1}{|x-a(t)|} \in L^2_{\rm loc}(\mathbb{R}^3)$. This and $u \in C([0,\infty);L^2(\mathbb{R}^3)) \cap L^\infty((0,\infty);\Sigma)$ imply that $\frac{1}{|x-a(t)|}u \in L^1_{\rm loc}((0,\infty)\times\mathbb{R}^3)$. In addition, from Hardy's inequality, $(\frac{1}{|x|} * |u|^2) \in L^\infty((0,\infty);L^\infty(\mathbb{R}^3))$. This implies that $(\frac{1}{|x|} * |u|^2)u \in L^1_{\rm loc}((0,\infty)\times\mathbb{R}^3)$.

Our main results read as follows.

Theorem 1.3. *Let $0 < \alpha \leqslant 4/5$, $a \in W^{1,1}((0,\infty),\mathbb{R}^3)$ and $u_0 \in \Sigma$. Then the Cauchy problem* (1.1) *has a unique, global weak solution. In addition,*

$$\frac{d}{dt}\int_{\mathbb{R}^3} |u(t,x)|^2 dx = -2b\int_{\mathbb{R}^3} |u(t,x)|^{2-\alpha} dx, \tag{1.5}$$

$$\|u(t)\|_\Sigma \le C(\|u_0\|_\Sigma) \quad \forall t > 0. \tag{1.6}$$

Theorem 1.4. *Let $u_0 \in \Sigma^2$, $a \in W^{2,\infty}((0,\infty),\mathbb{R}^3)$, $b > 0$ and $0 < \alpha \leqslant \frac{1}{2}$. For every $0 < T < \infty$, the solution u of* (1.1) *belongs to $L^\infty((0,T),\Sigma^2)$.*

The H^s-regularity for nonlinear Schrödinger equations is well-known if the nonlinearity is sufficiently smooth, see [10]. The smooth condition on the nonlinearity can be improved (removed, if $s \le 2$) by estimating time derivatives of the equation instead of space derivatives, see [17, 10, 15]. Since the appearance of the time-dependent Coulomb potential $\frac{1}{|x-a(t)|}$ and the sublinear term $\frac{u}{|u|^\alpha}$, we will prove the Σ^2-regularity for (1.1) by estimating its time derivatives. However, compared to the classical Schrödinger equation, there are two major difficulties in proving this problem. Firstly, due to the presence of a sublinear damping term, the Strichartz's estimates cannot be applied to prove the Σ^2-regularity for (1.1) by the similar method as that in [10]. Secondly, for applying Kato's idea, the key point is obtain an L^2-estimate for the time derivative of solution u^δ for the regularizing equation

(3.1). For this aim, we will use a change of variable $y = x - a(t)$ to avoid the time derivative of the time-dependent Coulomb potential. This leads to a more complicated equation (4.8). We finally obtain the desired estimate which is independent of δ.

This article is organized as follows: in Section 2, we present some preliminaries and some estimates for Hartree nonlinearity. In section 3, we prove Theorem 1.3. In section 4, we prove Theorem 1.4.

Notation. Throughout this article, we use the following notation. $C > 0$ will stand for a constant that may be different from line to line when it does not cause any confusion. Since we exclusively deal with $\mathbb{R}^3$, we often abbreviate $L^q(\mathbb{R}^3)$, $\|\cdot\|_{L^q(\mathbb{R}^3)}$ and $H^s(\mathbb{R}^3)$ by L^q, $\|\cdot\|_{L^q}$ and H^s, respectively. We recall that a pair of exponents (q, r) is Schrödinger-admissible if $\frac{2}{q} = 3(\frac{1}{2} - \frac{1}{r})$ and $2 \leq r \leq 6$. Then, for any space-time slab $I \times \mathbb{R}^3$, we can define the Strichartz norm

$$\|u\|_{S(I)} = \sup_{(q,r)} \|u\|_{L_t^q L_x^r(I)},$$

where the supremum is taken over all admissible pairs of exponents (q, r). Furthermore, $\|f\|_{S_\Sigma(I)} := \|f\|_{S(I)} + \|\nabla f\|_{S(I)} + \|xf\|_{S(I)}$.

2. Preliminaries

In this section, we recall some useful results. Firstly, we recall the following Gagliardo-Nirenberg inequality (see [10]).

Lemma 2.1. *There exists a constant C such that*

$$\|f\|_{L^2} \leqslant C\|f\|_{L^{2-\alpha}}^{1-\theta} \|f\|_{H^2}^{\theta}, \quad \forall f \in L^{2-\alpha} \cap H^2, \tag{2.1}$$

where $\theta = 3\alpha/(8 - \alpha)$ with $\alpha \in (0, 1]$.

The following inequalities can be viewed as dual to the Gagliardo-Nirenberg inequalities.

Lemma 2.2. *There exist some constants C such that*

$$\|f\|_{L^{p'}} \leqslant C\|f\|_{L^2}^{1-\delta_1(p)} \|xf\|_{L^2}^{\delta_1(p)}, \quad \forall f \in \Sigma, \tag{2.2}$$

where $\delta_1(p) = 3\big(\frac{1}{2} - \frac{1}{p}\big)$ with $p \in [2, 6)$.

$$\|f\|_{L^{p'}} \leqslant C\|f\|_{L^2}^{1-\delta_2(p)} \|x^2 f\|_{L^2}^{\delta_2(p)}, \quad \forall f \in \Sigma^2, \tag{2.3}$$

where $\delta_2(p) = \frac{3}{2}\big(\frac{1}{2} - \frac{1}{p}\big)$ with $p \geqslant 2$.

Proof. The proof of (2.2) is similar to that of [8, Lemma 6.2]. We give the proof for the reader's convenience. Let $\lambda > 0$, and write

$$\int_{\mathbb{R}^3} |f(x)|^{p'} dx = \int_{|x| \leqslant \lambda} |f(x)|^{p'} dx + \int_{|x| > \lambda} |f(x)|^{p'} dx.$$

Estimate the first term by Hölder's inequality

$$\int_{|x| \leqslant \lambda} |f(x)|^{p'} dx \leqslant C\lambda^{3/r'} \Big(\int_{|x| \leqslant \lambda} |f(x)|^{p' r} dx \Big)^{1/r},$$

and choose $r = 2/p'$. Estimate the second term by the same Hölder's inequality, after inserting the factor x as follows,

$$\begin{aligned}\int_{|x|>\lambda} |f(x)|^{p'}dx &= \int_{|x|>\lambda} |x|^{-p'}|x|^{p'}|f(x)|^{p'}dx \\ &\leqslant \Big(\int_{|x|>\lambda} |x|^{-p'r'}dx\Big)^{1/r'}\Big(\int_{|x|>\lambda} |xf(x)|^2dx\Big)^{1/r} \\ &\leqslant C\lambda^{\frac{3}{r'}-p'}\|xf\|_{L^2}^{2/r}.\end{aligned}$$

In summary, we have the following estimate, for any $\lambda > 0$,

$$\|f\|_{L^{p'}} \leqslant C\lambda^{\frac{3}{r'p'}}\|f\|_{L^2} + C\lambda^{\frac{3}{r'p'}-1}\|xf\|_{L^2}. \tag{2.4}$$

Notice that $\frac{3}{r'p'} = \delta(p)$, taking $\lambda = \frac{\|xf\|_{L^2}}{\|f\|_{L^2}}$, this yields (2.2). By a similar argument, we can obtain (2.3). $\square$

The following lemma is vital for proving the uniqueness for (1.1); it was first proved in [5].

Lemma 2.3. *Let $\sigma \geqslant -1$. For all $z_1, z_2 \in \mathbb{C}$,*

$$\mathrm{Re}((|z_1|^\sigma z_1 - |z_2|^\sigma z_2)(\overline{z_1 - z_2})) \geqslant 0.$$

3. Proof of Theorem 1.3

By a similar idea as that in [7, 9], we modify (1.1) by regularizing the singular potential and sublinear nonlinearity:

$$\begin{aligned}i\partial_t u^\delta + \Delta u^\delta &= V(x)u^\delta + \frac{u^\delta}{(|x-a(t)|^2+\delta)^{1/2}} \\ &\quad + \lambda\left(\frac{1}{|x|} * |u^\delta|^2\right)u^\delta - ib\frac{u^\delta}{(|u^\delta|^2+\delta)^{\alpha/2}}, \\ u^\delta(0) &= u_0.\end{aligned} \tag{3.1}$$

By the energy estimates, we can infer the following global existence for (3.1).

Lemma 3.1. *Let $\delta > 0$ and $a \in W^{1,1}((0,\infty),\mathbb{R}^3)$. For every $u_0 \in \Sigma$, there exists a unique global solution $u^\delta \in C([0,\infty);\Sigma)$ to (3.1). In addition, for every $t > 0$, we have*

$$\int_{\mathbb{R}^3} |u^\delta(t,x)|^2dx + 2b\int_0^t\int_{\mathbb{R}^3} \frac{|u^\delta(s,x)|^2}{(|u^\delta(s,x)|^2+\delta)^{\alpha/2}}dxds = \|u_0\|_{L^2}^2, \tag{3.2}$$

$$\|u^\delta(t)\|_\Sigma \leq C(\|u_0\|_\Sigma). \tag{3.3}$$

Proof. Since the external potential $V(x)$ is quadratic, local in time Strichartz's estimates hold for the Hamiltonian $-\Delta + V(x)$. The local existence can be proved by applying a fixed-point argument in space of the type $S_\Sigma(0,T)$, which can be found for instance in [10] in the case of $V = 0$ (see [6] in the presence of a potential, see [2, 11] for a time-dependent Coulomb potential $\frac{1}{|x-a(t)|}$). So we omit it.

On the other hand, multiplying (3.1) by $\overline{u^\delta}$, integrating on $\mathbb{R}^3$ and taking the imaginary part, we can obtain (3.2).

To show (3.3), we first assume that $u(t)$ is sufficiently regular and decaying so that all of the following formal manipulations can be carried out. Once the final result is established, a standard density argument allows to conclude that it also holds for $u \in C([0,T],\Sigma)$.

We deduce from (3.1) that

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla u^\delta(t)\|_{L^2}^2 \\
&= -2\operatorname{Re}\int_{\mathbb{R}^3} \Delta u^\delta(t,x)\partial_t\overline{u^\delta}(t,x)dx \\
&= -2\operatorname{Re}\int_{\mathbb{R}^3} \Big(V(x)u^\delta(t,x) + \frac{u^\delta(t,x)}{(|x-a(t)|^2+\delta)^{1/2}}\Big)\partial_t\overline{u^\delta}(t,x)dx \\
&\quad - 2\operatorname{Re}\int_{\mathbb{R}^3} \Big(\lambda\Big(\frac{1}{|x|} * |u^\delta(t)|^2\Big)u^\delta(t,x) - ibf_\delta(u^\delta)(t,x)\Big)\partial_t\overline{u^\delta}(t,x)dx \\
&= -\frac{d}{dt}\int_{\mathbb{R}^3} V(x)|u^\delta(t,x)|^2dx - \frac{d}{dt}\int_{\mathbb{R}^3}\frac{|u^\delta(t,x)|^2}{(|x-a(t)|^2+\delta)^{1/2}}dx \\
&\quad - \frac{\lambda}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|} * |u^\delta(t)|^2\Big)|u^\delta(t,x)|^2dx \\
&\quad - 2b\operatorname{Im}\int_{\mathbb{R}^3} f_\delta(u^\delta)(t,x)\partial_t\overline{u^\delta}(t,x)dx \\
&\quad + \int_{\mathbb{R}^3}\partial_t\Big(\frac{1}{(|x-a(t)|^2+\delta)^{1/2}}\Big)|u^\delta(t,x)|^2dx,
\end{aligned}
\tag{3.4}
$$

where

$$
f_\delta(v) = \frac{v}{(|v|^2+\delta)^{\alpha/2}}.
$$

For the last term in (3.4), we deduce from Hardy's inequality that

$$
\begin{aligned}
\int_{\mathbb{R}^3}\partial_t\Big(\frac{1}{(|x-a(t)|^2+\delta)^{1/2}}\Big)|u^\delta(t,x)|^2dx &\leqslant |\frac{da}{dt}(t)|\int_{\mathbb{R}^3}\frac{|u^\delta(t,x)|^2}{|x-a(t)|^2}dx \\
&\leqslant 4|\frac{da}{dt}(t)|\int_{\mathbb{R}^3}|\nabla u^\delta(t,x)|^2dx.
\end{aligned}
\tag{3.5}
$$

In addition, using (3.1) again, we have

$$
\begin{aligned}
&-2b\operatorname{Im}\int_{\mathbb{R}^3} f_\delta(u^\delta)(t,x)\partial_t\overline{u^\delta}(t,x)dx \\
&= 2b\operatorname{Re}\int_{\mathbb{R}^3} f_\delta(u^\delta)(t,x)\Delta\overline{u^\delta}(t,x)dx - 2b\int_{\mathbb{R}^3}\frac{V(x)|u^\delta(t,x)|^2}{(|u^\delta(t,x)|^2+\delta)^{\alpha/2}}dx \\
&\quad - 2b\int_{\mathbb{R}^3}\frac{1}{(|x-a(t)|^2+\delta)^{1/2}}\frac{|u^\delta(t,x)|^2}{(|u^\delta(t,x)|^2+\delta)^{\alpha/2}}dx \\
&\quad - 2b\lambda\int_{\mathbb{R}^3}\big(\frac{1}{|x|} * |u^\delta(t)|^2\big)\frac{|u^\delta(t,x)|^2}{(|u^\delta(t,x)|^2+\delta)^{\alpha/2}}dx,
\end{aligned}
\tag{3.6}
$$

where

$$
\begin{aligned}
&2b\operatorname{Re}\int_{\mathbb{R}^3} f_\delta(u^\delta)(t,x)\Delta\overline{u^\delta}(t,x)dx \\
&= -2b\operatorname{Re}\int_{\mathbb{R}^3}\frac{|\nabla u^\delta(t,x)|^2}{(|u^\delta(t,x)|^2+\delta)^{\alpha/2}}dx + 2b\alpha\int_{\mathbb{R}^3}\frac{|\operatorname{Re}(\overline{u^\delta}(t,x)\nabla u^\delta(t,x))|^2}{(|u^\delta(t,x)|^2+\delta)^{\alpha/2+1}}dx \\
&= -2b\int_{\mathbb{R}^3}\frac{\delta|\nabla u^\delta(t,x)|^2 + (1-\alpha)|\operatorname{Re}(\overline{u^\delta}\nabla u^\delta)(t,x)|^2 + |\operatorname{Im}(\overline{u^\delta}\nabla u^\delta)(t,x)|^2}{(|u^\delta(t,x)|^2+\delta)^{\alpha/2+1}}dx.
\end{aligned}
\tag{3.7}
$$

Now, we define

$$E^\delta(t)=\int_{\mathbb{R}^3}|\nabla u^\delta(t,x)|^2dx+\int_{\mathbb{R}^3}V(x)|u^\delta(t,x)|^2dx.$$

Collecting (3.4)-(3.7), we derive

$$\begin{aligned}
&\frac{d}{dt}E^\delta(t)\\
&=-\frac{d}{dt}\Big(\int_{\mathbb{R}^3}\frac{|u^\delta(t,x)|^2}{(|x-a(t)|^2+\delta)^{1/2}}dx+\frac{\lambda}{2}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|}*|u^\delta(t)|^2\Big)|u^\delta(t,x)|^2dx\Big)\\
&\quad+\int_{\mathbb{R}^3}\partial_t\Big(\frac{1}{(|x-a(t)|^2+\delta)^{1/2}}\Big)|u^\delta(t,x)|^2dx-2b\int_{\mathbb{R}^3}\frac{V(x)|u^\delta(t,x)|^2}{(|u^\delta(t,x)|^2+\delta)^{\alpha/2}}dx\\
&\quad-2b\int_{\mathbb{R}^3}\frac{\delta|\nabla u^\delta(t,x)|^2+(1-\alpha)|\operatorname{Re}(\overline{u^\delta}\nabla u^\delta)(t,x)|^2+|\operatorname{Im}(\overline{u^\delta}\nabla u^\delta)(t,x)|^2}{(|u^\delta(t,x)|^2+\delta)^{\alpha/2+1}}dx\\
&\quad-2b\int_{\mathbb{R}^3}\frac{1}{(|x-a(t)|^2+\delta)^{1/2}}\frac{|u^\delta(t,x)|^2}{(|u^\delta(t,x)|^2+\delta)^{\alpha/2}}dx \qquad (3.8)\\
&\quad-2b\lambda\int_{\mathbb{R}^3}\Big(\frac{1}{|x|}*|u^\delta(t)|^2\Big)\frac{|u^\delta(t,x)|^2}{(|u^\delta(t,x)|^2+\delta)^{\alpha/2}}dx\\
&\leqslant-\frac{d}{dt}\Big(\int_{\mathbb{R}^3}\frac{|u^\delta(t,x)|^2}{(|x-a(t)|^2+\delta)^{1/2}}dx+\frac{\lambda}{2}\int_{\mathbb{R}^3}\big(\frac{1}{|x|}*|u^\delta(t)|^2\big)|u^\delta(t,x)|^2dx\Big)\\
&\quad+4|\frac{da}{dt}(t)|\int_{\mathbb{R}^3}|\nabla u^\delta(t,x)|^2dx+C\|\nabla u^\delta(t)\|_{L^2}\int_{\mathbb{R}^3}\frac{|u^\delta(t,x)|^2}{(|u^\delta(t,x)|^2+\delta)^{\alpha/2}}dx.
\end{aligned}$$

Integrating this inequality on $(0,t)$ with respect to time t, it follows that

$$\begin{aligned}
&E^\delta(t)\\
&\leqslant E^\delta(0)-\Big(\int_{\mathbb{R}^3}\frac{|u^\delta(t,x)|^2}{(|x-a(t)|^2+\delta)^{1/2}}dx+\frac{\lambda}{2}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|}*|u^\delta(t)|^2\Big)|u^\delta(t,x)|^2dx\Big)\\
&\quad+\Big(\int_{\mathbb{R}^3}\frac{|u_0(x)|^2}{(|x-a(0)|^2+\delta)^{1/2}}dx+\frac{\lambda}{2}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|}*|u_0|^2\Big)|u_0(x)|^2dx\Big)\\
&\quad+4\int_0^t|\frac{da}{ds}(s)|\|\nabla u^\delta(s)\|_{L^2}^2ds+C\int_0^t\|\nabla u^\delta(s)\|_{L^2}g(s)ds \qquad (3.9)\\
&\leqslant C(\|u_0\|_\Sigma)+\frac{|\lambda|}{2}\int_{\mathbb{R}^3}\Big(\frac{1}{|x|}*|u^\delta(t)|^2\Big)|u^\delta(t,x)|^2dx\\
&\quad+4\int_0^t|\frac{da}{ds}(s)|\|\nabla u^\delta(s)\|_{L^2}^2ds+C\int_0^t g(s)\|\nabla u^\delta(s)\|_{L^2}^2ds+C\int_0^t g(s)\,ds,
\end{aligned}$$

where

$$g(t)=\int_{\mathbb{R}^3}\frac{|u^\delta(t,x)|^2}{(|u^\delta(t,x)|^2+\delta)^{\alpha/2}}dx.$$

Using Hardy's and Young's inequalities, for all $t>0$, we have

$$\begin{aligned}
\int_{\mathbb{R}^3}\Big(\frac{1}{|x|}*|u^\delta(t)|^2\Big)|u^\delta(t,x)|^2dx&\leqslant\|\frac{1}{|x|}*|u^\delta(t)|^2\|_{L^\infty}\int_{\mathbb{R}^3}|u^\delta(t,x)|^2dx\\
&\leqslant\varepsilon\|\nabla u^\delta(t)\|_{L^2}^2+C_\varepsilon\|u_0\|_{L^2}^2.
\end{aligned}\qquad (3.10)$$

This and (3.9) imply

$$\|\nabla u^\delta(t)\|_{L^2}^2 \leqslant C(\|u_0\|_\Sigma) + 4\int_0^t |\frac{da}{ds}(s)|\|\nabla u^\delta(s)\|_{L^2}^2 ds + C\int_0^t g(s)\|\nabla u^\delta(s)\|_{L^2}^2 ds. \tag{3.11}$$

In view of (3.2) and $a \in W^{1,1}((0,\infty),\mathbb{R}^3)$, from Gronwall's inequality we deduce that

$$\|\nabla u^\delta(t)\|_{L^2}^2 \leqslant C(\|u_0\|_\Sigma) \quad \forall t > 0,$$

which, together with (3.9) imply

$$E^\delta(t) \leqslant C(\|u_0\|_\Sigma) \quad \forall t > 0. \tag{3.12}$$

This yields

$$\|u^\delta(t)\|_\Sigma \leqslant C(\|u_0\|_\Sigma) \quad \forall t > 0, \tag{3.13}$$

which implies that the solution of (3.1) exists globally. □

Proof of Theorem 1.3. Firstly, it follows from (3.2) and (3.3) that $(u^\delta)_{0<\delta\leqslant 1}$ is uniformly bounded in $L^\infty([0,\infty);\Sigma) \cap L^{2-\alpha}([0,\infty);L^{2-\alpha})$. Therefore, there exist $u \in L^\infty([0,\infty);\Sigma)$ and a subsequence (u^{δ_n}) such that

$$u^{\delta_n} \rightharpoonup u \quad \text{in } w*\ L^\infty([0,\infty);\Sigma),\ \text{as } n\to\infty; \tag{3.14}$$

this and Lemma 3.1, imply

$$\|u\|_{L^\infty([0,\infty);H^1)} \leqslant C(\|u_0\|_\Sigma).$$

In addition, we infer from Hardy's and Hölder's inequalities that

$$\begin{aligned}
&\Big\|\frac{1}{|x|} * |u|^2 u - \frac{1}{|x|} * |v|^2 v\Big\|_{L^2}\\
&\leqslant \Big\|\frac{1}{|x|} * |u|^2\Big\|_{L^\infty}\|u-v\|_{L^2} + \Big\|\frac{1}{|x|} * |u|^2 - \frac{1}{|x|} * |v|^2\Big\|_{L^\infty}\|v\|_{L^2}\\
&\leqslant C\|u\|_{L^2}\|\nabla u\|_{L^2}\|u-v\|_{L^2}\\
&\quad + C\sup_{x\in\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}\frac{(|u(y)|+|v(y)|)^2}{|x-y|^2}dy\Big)^{1/2}\|v\|_{L^2}\|u-v\|_{L^2}\\
&\leqslant C\|u\|_{H^1}^2\|u-v\|_{L^2} + C(\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2})\|v\|_{L^2}\|u-v\|_{L^2}\\
&\leqslant C(\|u\|_{H^1}^2 + \|v\|_{H^1}^2)\|u-v\|_{L^2}.
\end{aligned} \tag{3.15}$$

Thus, from (3.14) and the compact embedding $\Sigma \hookrightarrow L^2$ it follows that

$$\Big(\frac{1}{|x|} * |u^{\delta_n}(t)|^2\Big)u^{\delta_n}(t) \to \Big(\frac{1}{|x|} * |u(t)|^2\Big)u(t) \tag{3.16}$$

in L^2 for a.e. $t\in[0,T]$, for every $0<T<\infty$. Moreover, we have

$$\frac{1}{(|x-a(t)|^2+\delta)^{1/2}} \to \frac{1}{|x-a(t)|} \tag{3.17}$$

in $L^\infty([0,\infty);L^q+L^\infty)$, $q\in(1,3)$ as $\delta\to 0$.

Since $\frac{u^{\delta_n}}{(|u^{\delta_n}|^2+\delta_n)^{\alpha/2}}$ is uniformly bounded in $L^\infty([0,\infty);L^{\frac{2}{1-\alpha}})$, there exists $F \in L^\infty([0,\infty);L^{\frac{2}{1-\alpha}})$ such that

$$\frac{u^{\delta_n}}{(|u^{\delta_n}|^2+\delta_n)^{\alpha/2}} \rightharpoonup F \quad \text{in } w*L^\infty([0,\infty);L^{\frac{2}{1-\alpha}}), \tag{3.18}$$

and $\|F\|_{L^\infty([0,\infty);L^{\frac{2}{1-\alpha}})} \leqslant \|u_0\|_{L^2}^{1-\alpha}$.

On the other hand, it follows from (3.1) that for every $\omega \in C_c^\infty(\mathbb{R}^3)$ and $\eta \in C_c^\infty([0,\infty))$,

$$\begin{aligned}\int_0^T &[\langle iu^{\delta_n}, \omega\rangle_{\Sigma^*,\Sigma}\eta'(t) + \langle -\Delta u^{\delta_n} + V(x)u^{\delta_n} + \frac{u^{\delta_n}}{(|x-a(t)|^2+\delta_n)^{1/2}} \\ &+ \lambda\Big(\frac{1}{|x|} * |u^{\delta_n}|^2\Big)u^{\delta_n} - ib\frac{u^{\delta_n}}{(|u^{\delta_n}|^2+\delta_n)^{\alpha/2}}, \omega\rangle_{\Sigma^*,\Sigma}\eta(t)]dt = 0.\end{aligned} \tag{3.19}$$

Applying (3.14)-(3.18), and the dominated convergence theorem, we deduce easily that

$$\begin{aligned}\int_0^T &[\langle iu, \omega\rangle_{\Sigma^*,\Sigma}\eta'(t) + \langle -\Delta u + V(x)u + \frac{u}{|x-a(t)|} \\ &+ \lambda\left(\frac{1}{|x|} * |u|^2\right)u - ibF, \omega\rangle_{\Sigma^*,\Sigma}\eta(t)]dt = 0.\end{aligned} \tag{3.20}$$

This implies that u satisfies

$$iu_t + \Delta u = V(x)u + \frac{1}{|x-a(t)|}u + \lambda\big(\frac{1}{|x|} * |u|^2\big)u - ibF \tag{3.21}$$

for a.e. $t \in [0,\infty)$. We next show that $F = u/|u|^\alpha$. Fix $t' \in [0,\infty)$ and $\delta > 0$. It follows from (3.2) that for any $t \in [0,\infty)$,

$$\begin{aligned}\frac{d}{dt}\|u^\delta(t) - u^\delta(t')\|_{L^2}^2 &\leqslant \frac{d}{dt}(-2\operatorname{Re}\langle u^\delta(t), u^\delta(t')\rangle_{L^2}) \\ &= 2\operatorname{Re}\langle -i\Delta u^\delta + iV(x)u^\delta + i\frac{u^\delta}{(|x-a(t)|^2+\delta)^{1/2}} \\ &\quad + i\lambda\big(\frac{1}{|x|} * |u^\delta|^2\big)u^\delta + b\frac{u^\delta}{(|u^\delta|^2+\delta)^{\alpha/2}}, u^\delta\rangle_{L^2}.\end{aligned}$$

Integrating this inequality with respect to time, applying Hardy's inequality and (2.2), we obtain

$$\begin{aligned}\|u^\delta(t) - u^\delta(t')\|_{L^2}^2 \leqslant& C|t-t'|(\|\Delta u^\delta\|_{L^\infty([0,\infty);H^{-1})}\|u^\delta\|_{L^\infty([0,\infty);H^1)} \\ &+ \|xu^\delta\|_{L^\infty([0,\infty);L^2)} + \|\frac{|u^\delta|^2}{|x-a(t)|}\|_{L^\infty([0,\infty);L^1)} \\ &+ \|\Big(\frac{1}{|x|} * |u^\delta|^2\Big)|u^\delta|^2\|_{L^\infty([0,\infty);L^1)} + \|u^\delta\|_{L^\infty([0,\infty);L^{2-\alpha})}^{2-\alpha}) \\ \leqslant& C|t-t'|(\|\Delta u^\delta\|_{L^\infty([0,\infty);H^{-1})}\|u^\delta\|_{L^\infty([0,\infty);H^1)} \\ &+ \|xu^\delta\|_{L^\infty([0,\infty);L^2)} + \|\nabla u^\delta\|_{L^\infty([0,\infty);L^2)}\|u^\delta\|_{L^\infty([0,\infty);L^2)} \\ &+ \|\nabla u^\delta\|_{L^\infty([0,\infty);L^2)}\|u^\delta\|_{L^\infty([0,\infty);L^2)}^3 \\ &+ \|u^\delta\|_{L^\infty([0,\infty);L^2)}^{\frac{4-5\alpha}{2}}\|xu^\delta\|_{L^\infty([0,\infty);L^2)}^{\frac{3\alpha}{2}}),\end{aligned}$$

which, together with (3.3) imply

$$\|u^\delta(t) - u^\delta(t')\|_{L^2} \leqslant C|t-t'|^{1/2}.$$

Therefore, for any $T > 0$, $(u^\delta)_{0<\delta\leqslant 1}$ is a bounded sequence in $C([0,T];\Sigma)$ and is uniformly equicontinuous from $[0,T]$ to L^2. In addition, since the embedding $\Sigma \hookrightarrow L^2$ is compact, the set $\{u^\delta(t)|\delta \in (0,1]\}$ is relatively compact in L^2. Applying

Arzelà-Ascoli Theorem, it follows that (u^{δ_n}) is relatively compact in $C([0,T];L^2)$. Thus, we deduce from (3.14) that

$$u^{\delta_n} \to u \quad \text{in } C([0,T];L^2).$$

This yields that $u \in C([0,T];L^2)$ as well as $u(0) = u^\delta(0) = u_0$. Note that this holds for any $T > 0$, therefore, $u \in C([0,\infty);L^2)$. Finally, up to subsequence, $u^{\delta_n}(t,x) \to u(t,x)$ in $(t,x) \in [0,\infty) \times \mathbb{R}^3$. Hence, for a.e. $(t,x) \in [0,\infty) \times \mathbb{R}^3$ such that $u(t,x) \neq 0$, it follows

$$\frac{u^{\delta_n}}{(|u^{\delta_n}|^2 + \delta_n)^{\alpha/2}}(t,x) \to \frac{u}{|u|^\alpha}(t,x), \quad \text{as } n \to \infty.$$

This and (3.18) imply that $F(t,x) = \frac{u}{|u|^\alpha}(t,x)$. This completes the proof of existence.

Next we show uniqueness. Let u and v be two solutions to (1.1) with the same initial data u_0. We set $w = u - v$ and it satisfies

$$\begin{gathered} i\partial_t w + \Delta w = V(x)w + \frac{1}{|x-a(t)|}w + \lambda\Big(\frac{1}{|x|} * |u|^2 u - \frac{1}{|x|} * |v|^2 v\Big) \\ - ib\Big(\frac{u}{|u|^\alpha} - \frac{v}{|v|^\alpha}\Big), \\ w(0,x) = 0. \end{gathered} \tag{3.22}$$

Multiplying this equation by $\overline{w}$, integrating over $\mathbb{R}^3$ and taking the imaginary part, by (3.15) and Lemma 4.2 we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |w(t,x)|^2 dx &= \operatorname{Im}\int_{\mathbb{R}^3} \Big(\frac{1}{|x|} * |u|^2 u - \frac{1}{|x|} * |v|^2 v\Big)\overline{w} dx \\ &\quad - b \operatorname{Re}\int_{\mathbb{R}^3} \Big(\frac{u}{|u|^\alpha} - \frac{v}{|v|^\alpha}\Big)\overline{w} dx \\ &\leqslant \|\frac{1}{|x|} * |u|^2 u - \frac{1}{|x|} * |v|^2 v\|_{L^2} \|w\|_{L^2} \\ &\leqslant C\|w(t)\|_{L^2}^2. \end{aligned}$$

This and Gronwall's inequality impliy $w(t) = 0$ for all $t \in [0,\infty)$. Therefore, there exists a unique global weak solution of (1.1). $\square$

4. Proof of Theorem 1.4

Firstly, we have the following estimates.

Lemma 4.1. *Assume $0 < \alpha \leqslant 1/2$. There exists $C > 0$ such that the solution u^δ of (3.1) satisfies*

$$\|u^\delta(t)\|_{H^2} + \|x^2 u^\delta(t)\|_{L^2} \leqslant C\|\partial_t u^\delta(t)\|_{L^2} + C, \quad \text{for all } t > 0, \tag{4.1}$$

where C depends only on u_0.

Proof. Since u^δ is the solution of (3.1), it follows that for all $t > 0$

$$
\begin{aligned}
&\|u^\delta(t)\|_{H^2} + \|x^2 u^\delta(t)\|_{L^2} \\
&\leqslant C\|\Delta u^\delta(t)\|_{L^2} + C\|u^\delta(t)\|_{L^2} + C\|V u^\delta(t)\|_{L^2} \\
&\leqslant C\|\partial_t u^\delta(t)\|_{L^2} + C\|u^\delta(t)\|_{L^2} + C\|\frac{u^\delta(t)}{|x-a(t)|}\|_{L^2} \\
&\quad + C\|\Big(\frac{1}{|x|} * |u^\delta(t)|^2\Big)u^\delta(t)\|_{L^2} + C\||u^\delta(t)|^{1-\alpha}\|_{L^2}.
\end{aligned} \tag{4.2}
$$

Applying Hardy's and Young's inequalities, we obtain that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$
\|\frac{u^\delta(t)}{|x-a(t)|}\|_{L^2} \leqslant 2\|\nabla u^\delta(t)\|_{L^2} \leqslant \varepsilon\|u^\delta(t)\|_{H^2} + C_\varepsilon\|u^\delta(t)\|_{L^2}, \tag{4.3}
$$

and

$$
\|\frac{1}{|x|} * |u^\delta|^2 u^\delta(t)\|_{L^2} \leqslant C\|\nabla u^\delta(t)\|_{L^2}\|u^\delta(t)\|_{L^2}^2 \leqslant \varepsilon\|u^\delta(t)\|_{H^2} + C_\varepsilon\|u^\delta(t)\|_{L^2}. \tag{4.4}
$$

When $\alpha \leqslant 1/2$, we deduce from (2.3) and Young's inequality that

$$
\begin{aligned}
\||u^\delta(t)|^{1-\alpha}\|_{L^2} &= \|u^\delta(t)\|_{L^{2-2\alpha}}^{1-\alpha} \\
&\leqslant C\|x^2 u^\delta(t)\|_{L^2}^{3\alpha/4}\|u^\delta(t)\|_{L^2}^{\frac{4-7\alpha}{4}} \\
&\leqslant \varepsilon\|x^2 u^\delta(t)\|_{L^2} + C_\varepsilon\|u^\delta(t)\|_{L^2}.
\end{aligned} \tag{4.5}
$$

Taking $\varepsilon = \frac{1}{6}$ in (4.3)-(4.5), (4.1) follows from (4.2)-(4.5). $\square$

Lemma 4.2. *Let $u_0 \in \Sigma^2$, $a \in W^{2,\infty}((0,\infty),\mathbb{R}^3)$, $b > 0$ and $0 < \alpha \leqslant \frac{1}{2}$. Then, for every $T < \infty$, there exists $C > 0$ such that the solution u^δ of (3.1) satisfies*

$$
\|\partial_t u^\delta(t)\|_{L^2} \leqslant C(\|a\|_{W^{2,\infty}((0,\infty),\mathbb{R}^3)}, T, \|u_0\|_{\Sigma^2}), \quad \text{for all } t \in [0,T].
$$

Proof. We make the change of variables $y = x - a(t)$ and set $u^\delta(t,x) = v^\delta(t,y)$. Then,

$$
\partial_t v^\delta(t,y) = \partial_t u^\delta(t,x) + \frac{da}{dt}(t)\cdot\nabla u^\delta(t,x), \tag{4.6}
$$

and $\nabla u^\delta(t,x) = \nabla v^\delta(t,y)$. Therefore, v^δ satisfies the equation

$$
\begin{gathered}
i\partial_t v^\delta + \Delta v^\delta = V(y+a(t))v^\delta + \frac{v^\delta}{(|y|^2+\delta)^{1/2}} + \lambda\frac{1}{|x|} * |v^\delta|^2 v^\delta \\
+ ib\frac{v^\delta}{(|v^\delta|^2+\delta)^{\alpha/2}} + i\frac{da}{dt}(t)\cdot\nabla v^\delta, \\
v^\delta(0,y) = u_0(y+a(0)).
\end{gathered} \tag{4.7}
$$

Now, we set $w^\delta(t,y) = \partial_t v^\delta(t,y)$ and since

$$
\partial_t V(y+a(t)) = \frac{da}{dt}(t)\cdot\nabla V(y+a(t)),
$$

it follows that w satisfies

$$
\begin{aligned}
i\partial_t w^\delta + \Delta w^\delta &= V(y+a(t))w^\delta + (\frac{da}{dt}(t)\cdot \nabla V(y+a(t)))v^\delta \\
&\quad + \frac{w^\delta}{(|y|^2+\delta)^{1/2}} + \lambda \partial_t\Big(\frac{1}{|x|} * |v^\delta|^2 v^\delta\Big) + ib\partial_t\Big(\frac{v^\delta}{(|v^\delta|^2+\delta)^{\alpha/2}}\Big) \\
&\quad + i\frac{d^2a}{dt^2}(t)\cdot\nabla v^\delta + i\frac{da}{dt}(t)\cdot \nabla w^\delta, \\
w^\delta(0,y) &= i\Delta v^\delta(0) - iV(y+a(0))v^\delta(0) - i\frac{v^\delta(0)}{(|y|^2+\delta)^{1/2}} \\
&\quad - i\lambda \frac{1}{|x|} * |v^\delta(0)|^2 v^\delta(0) + b\frac{v^\delta(0)}{(|v^\delta(0)|^2+\delta)^{\alpha/2}} + \frac{da}{dt}(t)\cdot \nabla v^\delta(0).
\end{aligned}
\tag{4.8}
$$

Multiplying by $\overline{w^\delta}$, integrating on $\mathbb{R}^3$ and taking the imaginary part we have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |w^\delta(t,y)|^2 dy &= \operatorname{Im}\int_{\mathbb{R}^3} \Big(\frac{da}{dt}(t)\cdot\nabla V(y+a(t))\Big)v^\delta(t,y)\overline{w^\delta}(t,y)dy \\
&\quad + \lambda \operatorname{Im}\int_{\mathbb{R}^3} \partial_t\Big(\frac{1}{|x|} * |v^\delta|^2 v^\delta\Big)(t,y)\overline{w^\delta}(t,y)dy \\
&\quad + b\operatorname{Re}\int_{\mathbb{R}^3} \partial_t\Big(\frac{v^\delta}{(|v^\delta|^2+\delta)^{\alpha/2}}\Big)(t,y)\overline{w^\delta}(t,y)dy \\
&\quad + \operatorname{Re}\int_{\mathbb{R}^3} \frac{d^2a}{dt^2}(t)\cdot\nabla v^\delta(t,y)\overline{w^\delta}(t,y)dy \\
&\quad + \operatorname{Re}\int_{\mathbb{R}^3} \frac{da}{dt}(t)\cdot\nabla w^\delta(t,y)\overline{w^\delta}(t,y)dy.
\end{aligned}
\tag{4.9}
$$

We deduce from the Hölder and Hardy inequalities that

$$
\begin{aligned}
&\operatorname{Im}\int_{\mathbb{R}^3} \partial_t\Big(\frac{1}{|x|} * |v^\delta|^2 v^\delta\Big)(t,y)\overline{w^\delta}(t,y)dy \\
&= \operatorname{Im}\int_{\mathbb{R}^3} \left(\frac{1}{|x|} * 2\operatorname{Re}(\overline{v^\delta}w^\delta)\right)(t,y)v^\delta(t,y)\overline{w^\delta}(t,y)dy \\
&\leqslant C\|\frac{1}{|x|} * \operatorname{Re}(\overline{v^\delta}(t)w^\delta(t))\|_{L^\infty}\|v^\delta(t)\|_{L^2}\|w^\delta(t)\|_{L^2} \\
&\leqslant C\|\frac{1}{|x|^2} * |v^\delta(t)|^2\|_{L^\infty}^{1/2}\|v^\delta(t)\|_{L^2}\|w^\delta(t)\|_{L^2}^2 \\
&\leqslant C\|\nabla v^\delta(t)\|_{L^2}\|v^\delta(t)\|_{L^2}\|w^\delta(t)\|_{L^2}^2.
\end{aligned}
\tag{4.10}
$$

After some computations, we have

$$
\begin{aligned}
&-b\operatorname{Re}\int_{\mathbb{R}^3} \partial_t\Big(\frac{v^\delta}{(|v^\delta|^2+\delta)^{\alpha/2}}\Big)(t,y)\overline{w^\delta}(t,y)dy \\
&= -b\operatorname{Re}\int_{\mathbb{R}^3} \Big(\frac{w^\delta}{(|v^\delta|^2+\delta)^{\alpha/2}} - \alpha v^\delta\frac{Rev^\delta\overline{w^\delta}}{(|v^\delta|^2+\delta)^{\alpha/2+1}}\Big)(t,y)\overline{w^\delta}(t,y)dy \\
&= -b\int_{\mathbb{R}^3} \frac{(1-\alpha)|Rev^\delta\overline{w^\delta}|^2(t,y) + |Imv^\delta\overline{w^\delta}|^2(t,y) + \delta|w^\delta|^2(t,y)}{(|v^\delta(t,y)|^2+\delta)^{\alpha/2+1}}dy,
\end{aligned}
\tag{4.11}
$$

where we use the decomposing

$$
|v^\delta|^2|w^\delta|^2 = |\operatorname{Re} v^\delta\overline{w^\delta}|^2 + |\operatorname{Im} v^\delta\overline{w^\delta}|^2.
$$

In addition,

$$\operatorname{Re}\int_{\mathbb{R}^3}\frac{da}{dt}(t)\cdot\nabla w^\delta(t,y)\overline{w^\delta}(t,y)dy=\frac{1}{2}\int_{\mathbb{R}^3}\frac{da}{dt}(t)\cdot\nabla|w^\delta(t,y)|^2dy=0. \tag{4.12}$$

Collecting (4.9)-(4.12), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|w^\delta(t,y)|^2dy\\
&\leqslant \operatorname{Im}\int_{\mathbb{R}^3}\Big(\frac{da}{dt}(t)\cdot\nabla V(y+a(t))\Big)v^\delta(t,y)\overline{w^\delta}(t,y)dy\\
&\quad + C\|\nabla v^\delta(t)\|_{L^2}\|v^\delta(t)\|_{L^2}\|w^\delta(t)\|_{L^2}^2\\
&\quad + \operatorname{Re}\int_{\mathbb{R}^3}\frac{d^2a}{dt^2}(t)\cdot\nabla v^\delta(t,y)\overline{w^\delta}(t,y)dy\\
&\leqslant |\frac{da}{dt}|\|xu^\delta(t)\|_{L^2}\|w^\delta(t)\|_{L^2}+C\|\nabla v^\delta(t)\|_{L^2}\|v^\delta(t)\|_{L^2}\|w^\delta(t)\|_{L^2}^2\\
&\quad + |\frac{d^2a}{dt^2}|\|\nabla v^\delta(t)\|_{L^2}\|w^\delta(t)\|_{L^2}.
\end{aligned} \tag{4.13}$$

Integrating in the time variable on $(0,t)$, we deduce from Lemma 3.1 that

$$\begin{aligned}
\|w^\delta(t)\|_{L^2}^2 &\leqslant C\|u_0\|_{\Sigma^2}^2 + C\int_0^t|\frac{da}{ds}(s)|\|xu^\delta(s)\|_{L^2}\|w^\delta(s)\|_{L^2}ds\\
&\quad + C\int_0^t\|\nabla v^\delta(s)\|_{L^2}\|v^\delta(s)\|_{L^2}\|w^\delta(s)\|_{L^2}^2ds\\
&\quad + \int_0^t|\frac{d^2a}{ds^2}|\|\nabla v^\delta(s)\|_{L^2}\|w^\delta(s)\|_{L^2}ds\\
&\leqslant C\|u_0\|_{\Sigma^2}^2 + C\int_0^t\Big(|\frac{da}{ds}(s)|+|\frac{d^2a}{ds^2}(s)|\Big)\|w^\delta(s)\|_{L^2}ds\\
&\quad + C\int_0^t\|w^\delta(s)\|_{L^2}^2ds.
\end{aligned} \tag{4.14}$$

Thus, it follows from Gronwall's inequality that

$$\|w^\delta(t)\|_{L^2}\leqslant C(T,\|u_0\|_{\Sigma^2})\quad \forall t\in[0,T], \tag{4.15}$$

for every $T<\infty$.

Finally, it follows from (4.6) and Theorem 1.3 that

$$\|\partial_t u^\delta(t)\|_{L^2}\leqslant\|\partial_t v^\delta(t)\|_{L^2}+C\|\nabla u^\delta(t)\|_{L^2}\leqslant C(\|a\|_{W^{2,\infty}(0,\infty)},T,\|u_0\|_{\Sigma^2}).$$

□

Proof of Theorem 1.4. Combining Lemmas 4.1 and 4.2, for every $0<T<\infty$, there exists $C>0$ such that the solution u^δ of (3.1) satisfies for all $t\in[0,T]$

$$\|u^\delta(t)\|_{H^2}+\|x^2u^\delta(t)\|_{L^2}+\|\partial_t u^\delta(t)\|_{L^2}\leqslant C, \tag{4.16}$$

where C is independent of δ. This implies that there exist $u\in L^\infty((0,T);\Sigma^2)$ and a subsequence (u^{δ_n}) such that, for a.e. $t\in[0,T]$

$$u^{\delta_n}(t)\rightharpoonup u(t),\quad \text{in } \Sigma^2 \text{ as } n\to\infty. \tag{4.17}$$

Thus, we can pass to the limit in the distributions sense in (3.1) as $\delta \to 0$. Thus, u is the solution of (1.1) in the sense of distributions and satisfies $u \in L^\infty((0,T);\Sigma^2)$, $\partial_t u \in L^\infty((0,T);L^2)$ and

$$\|u(t)\|_{\Sigma^2} + \|\partial_t u(t)\|_{L^2} \leqslant C, \ \forall t \in [0,T].$$

This completes the proof. $\square$

Acknowledgments. This work is supported by the NSFC Grants (Nos. 11601435 and 11401478), by Gansu Provincial Natural Science Foundation (1606RJZA010), and by NWNU-LKQN-14-6.

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Yanjun Zhao
College of Humanities and Sciences, Northeast Normal University, Changchun 130117, China
E-mail address: 940508086@qq.com

Binhua Feng
Department of Mathematics, Northwest Normal University, Lanzhou 730070, China
E-mail address: binhuaf@163.com