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BESOV-MORREY SPACES ASSOCIATED WITH HERMITE OPERATORS AND APPLICATIONS TO FRACTIONAL HERMITE EQUATIONS

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Abstract. The purpose of this article is to establish the molecular decomposition of the homogeneous Besov-Morrey spaces associated with the Hermite operator $\mathbb{H} = -\Delta + |x|^2$ on the Euclidean space $\mathbb{R}^n$. Particularly, we obtain some estimates for the operator $\mathbb{H}$ on the Hermite-Besov-Morrey spaces and the regularity results to the fractional Hermite equations

$$(-\Delta + |x|^2)^s u = f,$$

and

$$(-\Delta + |x|^2 + I)^s u = f.$$

Our results generalize some results by Anh and Thinh [1].

1. Introduction

In this article, we study the Besov-Morrey spaces associated with the Hermite operator $\mathbb{H} = -\Delta + |x|^2$ on $\mathbb{R}^n$, $n \geq 1$. It is known that the classical theory of the Besov and Triebel-Lizorkin spaces plays a crucial role not only in the theory of function spaces, but also in the theory of partial differential equations and harmonic analysis, see e.g. [7, 9, 10, 11, 12, 14, 15], and the references therein.

Recently, the theory of the Besov and Triebel-Lizorkin spaces associated with the operators has been developed by many authors when one observed that the classical Besov and Triebel-Lizorkin spaces are not always the most suitable to investigate a number of operators, see [1, 2, 3, 4, 18, 10, 11, 19], and their references. For example, Petrusev and Xu [13] studied the characterization of the inhomogeneous Besov and Triebel-Lizorkin spaces in terms of Littlewood-Paley decomposition in the context of Hermite expansions that the frame elements have almost exponential localization. Note that these frame elements can be viewed as an analogue of the φ-transform of Frazier and Jawerth [7]. Another approach introduced by Anh and Thinh [1] is of defining the Besov and Triebel-Lizorkin spaces in terms of the heat kernels via square functions. Their approach adapted to the study of the theory of both homogeneous and inhomogeneous Besov and Triebel-Lizorkin spaces. This allows them to extend the range of indices $1 \leq p, q \leq \infty$ of the homogeneous Besov

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space $\mathrm{BM}^{\alpha,\mathbb{H}}_{p,q}$ (resp. Triebel-Lizorkin spaces $\mathrm{FM}^{\alpha,\mathbb{H}}_{p,q}$) to $0 < p, q \leq \infty$, compare to the results in [8].

One of the most interesting studies of the theory of Besov spaces is the Besov-Morrey spaces, introduced first by Kozono and Yamazaki [9] to investigate time-local solutions of the Navier-Stokes equations with the initial data in the spaces of this type. As a matter of fact, the Besov-Morrey spaces share several features of Besov and Morrey spaces. They represent the local oscillations and singularities of functions more precisely than the classical Besov spaces. Thus, they behav better in many aspects, particularly under the action of singular integrals and pseudo-differential operators. In addition, Mazzucato [11, 12] established the wavelet decompositions to characterize the homogeneous and inhomogeneous Besov-Morrey spaces. For more results on the Besov-Morrey spaces, we refer the reader to [9, 10, 11, 12, 14, 15, 17, 19] and the references therein.

Inspired by the above results, we would like to generalize the theory of the homogeneous Besov spaces associated with the Hermite operator $\mathrm{BM}^{\alpha,\mathbb{H}}_{p,q}$ to the one of the homogeneous Besov-Morrey spaces associated with the Hermite operator $\mathrm{BM}^{\alpha,\mathbb{H}}_{p,q,r}$ in this paper. To study $\mathrm{BM}^{\alpha,\mathbb{H}}_{p,q,r}$, we use the results in [1], specifically, the estimates on the heat kernels via the square functions. Beside, we also establish the molecular decompositions for $\mathrm{BM}^{\alpha,\mathbb{H}}_{p,q,r}$. As applications, we obtain the regularity of solutions to the fractional Hermite equations:

$$\mathbb{H}^s u = f,$$

and

$$(\mathbb{H} + I)^s u = f.$$

We organize this paper as follows: Section 2 contains some preliminary results and definitions of functional spaces. Section 3 is devoted to the study of the molecular decomposition for the Hermite-Besov-Morrey space. Finally, we investigate the regularity of solutions on Hermite-Besov-Morrey spaces to the fractional Hermite equations in Section 4.

Throughout this paper, we always use C and c to denote positive constants that are independent of the main parameters involved but whose values may differ from line to line. We write $A \lesssim B$ if there is a universal constant C such that $A \leq CB$; and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We use the following notation: $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}_+ = \{1, 2, 3, \dots\}$, $\mathbb{Z}^- = \{-1, -2, \dots\}$, $\mathbb{Z}^-_0 = \{0, -1, -2, \dots\}$ $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$, and $\mathrm{int}[a]$ is the integer part of a.

2. Preliminaries

2.1. **Dyadic cube.** The set of all dyadic cubes $\mathcal{D}$ in $\mathbb{R}^n$ is defined by

$$\mathcal{D} = \big\{ \prod_{j=1}^{n} [m_j 2^k, (m_j + 1)2^k) : m_1, m_2, \dots, m_n, k \in \mathbb{Z} \big\}.$$

For a dyadic cube $Q := \prod_{j=1}^{n} \Big[m_j 2^k, (m_j + 1)2^k\Big)$, for some $m_1, m_2, \dots, m_n, k \in \mathbb{Z}$ we denote by $\ell(Q)$ and x_Q the length and the center of the dyadic cube Q. In this case, $\ell(Q) = 2^k$ and $x_Q = \big((m_j + 1/2)2^k\big)_{j=1}^{n}$. Moreover, for every $\nu \in \mathbb{Z}$, we set

$$\mathcal{D}_\nu = \{Q \in \mathcal{D} : \ell(Q) = 2^\nu\}.$$

2.2. **Morrey space.** Let us first recall the definition of the Morrey spaces.

Definition 2.1. For every $0 < p \le r < \infty$, the Morrey space M_p^r is defined by

$$\mathrm{M}_p^r \equiv \big\{ f \in L^p_{\rm loc}(\mathbb{R}^n) : \|f\|_{\mathrm{M}_p^r} = \sup_{x_0\in\mathbb{R}^n} \sup_{R>0} R^{n(\frac{1}{r}-\frac{1}{p})} \|f\|_{L^p(B(x_0,R))} < \infty \big\}.$$

Next, we point out some known results about the Morrey norms.

Proposition 2.2. *Let $0 < p \le r < \infty$. Then*

$$\|f\|_{\mathrm{M}_p^r} \sim \sup_{Q\in\mathcal{D}} |Q|^{\frac{1}{r}-\frac{1}{p}} \|f\|_{L^p(Q)}, \tag{2.1}$$

$$\|f^\theta\|_{\mathrm{M}_p^r} = \|f\|^\theta_{\mathrm{M}^{r\theta}_{p\theta}}, \quad \forall \theta > 0, \tag{2.2}$$

$$\Big\| \Big(\int_a^b |F(\cdot,t)|^q \frac{dt}{t} \Big)^{1/q} \Big\|_{\mathrm{M}_p^r} \le \Big(\int_a^b \|F(\cdot,t)\|^q_{\mathrm{M}_p^r} \frac{dt}{t} \Big)^{1/q}, \quad \textit{for } 0 < q \le p. \tag{2.3}$$

Proof. Note that (2.1) and (2.2) follow from the definition of the Morrey spaces. While, (2.3) can be obtained by using Minkowski integral inequality, see also [8, (2.20)]. □

For $\theta > 0$, we denote by $\mathbb{M}_\theta$ the Hardy-Littlewood maximal function

$$\mathbb{M}_\theta f(x) = \sup_{x\in B} \Big(\frac{1}{|B|} \int_B |f(y)|^\theta dy \Big)^{1/\theta}, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x.

Then, we have a version of the Fefferman-Stein vector-valued maximal inequality for the Morrey spaces, see [16, Proposition 2.1].

Proposition 2.3. *Let $0 < q \le \infty$, $0 < p \le r < \infty$, and $0 < \theta < \min\{p, q\}$. Then*

$$\Big\| \Big(\sum_{k\in\mathbb{Z}} |\mathbb{M}_\theta f_k|^q \Big)^{1/q} \Big\|_{\mathrm{M}_p^r} \lesssim \Big\| \Big(\sum_{k\in\mathbb{Z}} |f_k|^q \Big)^{1/q} \Big\|_{\mathrm{M}_p^r}.$$

Remark 2.4. As a consequence of Proposition 2.3, the Hardy-Littlewood maximal operator $\mathbb{M}_\theta$ is bounded on M_p^r.

Next, we put

$$A_v = \Big(\sup_{J\in\mathcal{D},\ell(J)\geqslant 2^v} \Big(\frac{1}{|J|} \Big)^{1-p/r} \sum_{Q\in\mathcal{D}_v, Q\subset J} |Q|^{1-p/r} |s_Q|^p \Big)^{1/p}.$$

We borrow a result of Wang [19, p.779] involving the characterization of A_v in the Morrey norms.

Lemma 2.5. *Let $0 < p \le r < \infty$, and $\nu \in \mathbb{Z}$. Assume that the sequence $\{s_Q : Q \in \mathcal{D}_\nu\}$ satisfies*

$$\| \sum_{Q\in\mathcal{D}_\nu} |Q|^{-1/r} |s_Q| \chi_Q \|_{\mathrm{M}_p^r} < \infty.$$

Then

$$\| \sum_{Q\in\mathcal{D}_v} |Q|^{-1/r} |s_Q| \chi_Q \|_{\mathrm{M}_p^r} \sim A_\nu.$$

2.3. **Kernel estimates on Hermite operators.** For any $k \geq 0$ and for $t > 0$, we denote the kernel associated with $(t\sqrt{\mathbb{H}})^k e^{-t\sqrt{\mathbb{H}}}$ by $p_{t,k}(x,y)$. We recall here the results of [1, Lemma 2.1 and Propisition 2.2].

Proposition 2.6. *For $k \in \mathbb{N}$, there exist $C > 0$ and $\delta > 0$ so that*

(1) $|p_{t,k}(x,y)| \leq C \frac{t^k}{(t+|x-y|)^{n+k}}$, *for $x, y \in \mathbb{R}^n$.*
(2) *for any $|h| < t$, we have*

$$|p_{t,k}(x+h,y) - p_{t,k}(x,y)| \leq C\Big(\frac{|h|}{t}\Big)^{\delta} \frac{t^k}{(t+|x-y|)^{n+k}}, \quad \text{for } x, y \in \mathbb{R}^n.$$

Proposition 2.7. *For every $y \in \mathbb{R}^n$, we have $p_{t,k}(\cdot, y) \in \mathcal{S}$.*

2.4. **Calderón reproducing formulas.** In this part we recall Calderón's formula from [1], that is useful for studying the homogeneous Besov-Morrey spaces.

Proposition 2.8. *Let $m_1, m_2 \in \mathbb{N}^+$ and $f \in \mathcal{S}'$. Then*

$$f = -\frac{1}{2^{m-1}(m-1)!} \int_0^\infty (t\sqrt{\mathbb{H}})^{m_1} e^{-t\sqrt{\mathbb{H}}} (t\sqrt{\mathbb{H}})^{m_2} e^{-t\sqrt{\mathbb{H}}} f \frac{dt}{t} \quad \text{in } \mathcal{S}',$$

where $m = m_1 + m_2$, and $\mathcal{S}'$ is the dual space of the Schwartz functions $\mathcal{S}$ as usual.

3. Besov-Morrey Spaces associated with the Hermite operators

It is convenient for us to introduce first the homogeneous Besov-Morrey spaces corresponding to the Hermite operator $\mathbb{H}$.

Definition 3.1. Let $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$, $p \leq r \leq \infty$, and for every positive integer $m > n + \max\{\alpha, 0\} + \operatorname{int}[n(\frac{1}{\theta_0} - 1)] + 1$, with $\theta_0 = \min\{1, p, q\}$. Then, we define the homogeneous Hermite-Besov-Morrey space $\mathrm{BM}^{\alpha,\mathbb{H},m}_{p,q,r}$ as follows

$$\mathrm{BM}^{\alpha,\mathbb{H},m}_{p,q,r} := \Big\{ f \in \mathcal{S}' : \|f\|_{\mathrm{BM}^{\alpha,\mathbb{H},m}_{p,q,r}} = \Big(\int_0^\infty \Big(t^{-\alpha} \|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\|_{\mathrm{M}^r_p} \Big)^q \frac{dt}{t} \Big)^{1/q} < \infty \Big\}.$$

Remark 3.2. If $r = p$, then the space $\mathrm{BM}^{\alpha,\mathbb{H},m}_{p,q,r}$ is exactly the space $\mathrm{BM}^{\alpha,\mathbb{H},m}_{p,q}$ in [1].

We will show that $\mathrm{BM}^{\alpha,\mathbb{H},m}_{p,q,r}$ is independent of the choice of m when m is large enough. Precisely, we have the following result.

Theorem 3.3. *Let $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$, and $p \leq r \leq \infty$. Let m_1, m_2 be the positive integers such that*

$$m_1, m_2 > n + \max\{\alpha, 0\} + \operatorname{int}[n(\frac{1}{\theta_0} - 1)] + 1,$$

with $\theta_0 = \min\{1, p, q\}$. Then, the spaces $\mathrm{BM}^{\alpha,\mathbb{H},m_1}_{p,q,r}$ and $\mathrm{BM}^{\alpha,\mathbb{H},m_2}_{p,q,r}$ coincide with equivalent norms.

As a consequence of Theorem 3.3, we can define the Besov space $\mathrm{BM}^{\alpha,\mathbb{H}}_{p,q,r}$ as any space $\mathrm{BM}^{\alpha,\mathbb{H},m}_{p,q,r}$, for any positive integer $m > n + \max\{\alpha, 0\} + \operatorname{int}[n(\frac{1}{\theta_0} - 1)] + 1$.

We now recall the definition of the molecules associated with the Hermite operator in [1].

Definition 3.4. Let $0 < r \le \infty, \alpha \in \mathbb{R}$, and $N, M \in \mathbb{N}_+$. A function u is said to be an $(\mathbb{H}, M, N, \alpha, r)$ molecule if there exist a function b from the domain $(\sqrt{\mathbb{H}})^M$ and a dyadic cube $Q \in \mathcal{D}$ so that (i) $u = (\sqrt{\mathbb{H}})^M b$, and (ii)

$$|(\sqrt{\mathbb{H}})^k b(x)| \le \ell(Q)^{M-k} |Q|^{\alpha/n - 1/r} \Big(1 + \frac{|x - x_Q|}{\ell(Q)}\Big)^{-n-N}, \quad \text{for } k = 0, \dots, 2M.$$

Briefly, we denote $u = m_Q$, for every dyadic cube $Q \in \mathcal{D}$.

Next, we have some elementary estimates.

Lemma 3.5. *Let $N \in \mathbb{N}_+$ and $a > t > 0$. For any $x, z \in \mathbb{R}^n$,*

$$\int_{\mathbb{R}^n} \Big(1 + \frac{|x-y|}{t}\Big)^{-n-N} \Big(1 + \frac{|z-y|}{a}\Big)^{-n-N} dy \lesssim t^n \Big(1 + \frac{|x-z|}{t}\Big)^{-n-N}.$$

For a proof of the above lemma, we refer to [1, Lemma 3.6]. Next, we have a result of the molecular decomposition for $\mathrm{BM}^{\alpha,\mathbb{H},m}_{p,q,r}$.

Theorem 3.6. *Let $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, $p \le r \le \infty$, and $\theta_0 = \min\{1, p, q\}$.*

(i) For every $M, N \in \mathbb{N}_+$ and $m > n + \max\{\alpha, 0\} + \mathrm{int}[n(\frac{1}{\theta_0} - 1)] + 1$, if $f \in \mathrm{BM}^{\alpha,\mathbb{H},m}_{p,q,r}$, then there exist a sequence of $(\mathbb{H}, M, N, \alpha, r)$ molecules $\{m_Q\}_{Q \in \mathcal{D}_v, v \in \mathbb{Z}}$ and a sequence of coefficients $\{s_Q\}_{Q \in \mathcal{D}_v, v \in \mathbb{Z}}$ so that

$$f = \sum_{v \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_v} s_Q m_Q, \quad \text{in } \mathcal{S}'.$$

Moreover,

$$\Big(\sum_{v \in \mathbb{Z}} A_v^q\Big)^{1/q} \lesssim \|f\|_{\mathrm{BM}^{\alpha,\mathbb{H},m}_{p,q,r}}. \tag{3.1}$$

(ii) Conversely, if

$$f = \sum_{v \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_v} s_Q m_Q, \quad \text{in } \mathcal{S}',$$

where $\{m_Q\}_{Q \in \mathcal{D}_v, v \in \mathbb{Z}}$ is a sequence of $(\mathbb{H}, M, N, \alpha, r)$ molecules and $\{s_Q\}_{Q \in \mathcal{D}_v, v \in \mathbb{Z}}$ is a sequence of coefficients satisfying $\big(\sum_{v \in \mathbb{Z}} A_v^q\big)^{1/q} < \infty$, then $f \in \mathrm{BM}^{\alpha,\mathbb{H},m}_{p,q,r}$, and

$$\|f\|_{\mathrm{BM}^{\alpha,\mathbb{H},m}_{p,q,r}} \lesssim \Big(\sum_{v \in \mathbb{Z}} A_v^q\Big)^{1/q}, \tag{3.2}$$

provided that $N, M \in \mathbb{N}_+$ such that $\frac{n}{n+N} < \theta_0$, $M > \max\{\frac{n}{\theta_0} - \alpha, m\}$, with $m > \max\{\alpha, 0\} + N + n$.

Proof of part (i). For every $f \in \mathrm{BM}^{\alpha,\mathbb{H},m}_{p,q,r}$, it follows from Proposition 2.8 that

$$f = c_{m,M,N} \int_0^\infty (t\sqrt{\mathbb{H}})^{M+N} e^{-t\sqrt{\mathbb{H}}} (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f \frac{dt}{t}, \quad \text{in } \mathcal{S}',$$

with $c_{m,M,N} = -\frac{1}{2^{m+M+N-1}(m+M+N-1)!}$. Thus,

$$\begin{aligned} f &= c_{m,M,N} \sum_{v \in \mathbb{Z}} \int_{2^v}^{2^{v+1}} (t\sqrt{\mathbb{H}})^{M+N} e^{-t\sqrt{\mathbb{H}}} (t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f \frac{dt}{t} \\ &= c_{m,M,N} \sum_{v \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_v} \int_{2^v}^{2^{v+1}} (t\sqrt{\mathbb{H}})^{M+N} e^{-t\sqrt{\mathbb{H}}} [(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f . \chi_Q] \frac{dt}{t}. \end{aligned}$$

For any $v \in \mathbb{Z}$ and $Q \in \mathcal{D}_v$, we set

$$s_Q = 2^{-v(\alpha-n/r)} \sup_{(y,t)\in Q\times[2^v,2^{v+1})} |(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f(y)|, \tag{3.3}$$

and $m_Q = \mathbb{H}^{M/2} b_Q$, with

$$b_Q = \frac{1}{s_Q}\int_{2^v}^{2^{v+1}} t^M (t\sqrt{\mathbb{H}})^N e^{-t\sqrt{\mathbb{H}}}[(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f.\chi_Q]\frac{dt}{t}.$$

Obviously, we have

$$f = \sum_{v\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_v} s_Q m_Q, \quad \text{in } \mathcal{S}'.$$

Thus, it remains to show that m_Q is an $(\mathbb{H}, M, N, \alpha, r)$ molecule. Indeed, for $k = 0, \dots, 2M$, and for any $x \in \mathbb{R}^n$, from Proposition 2.6 we have

$$\begin{aligned} |\mathbb{H}^{k/2} b_Q(x)| &= \Big|\frac{1}{s_Q}\int_{2^v}^{2^{v+1}} t^{M-k}(t\sqrt{\mathbb{H}})^{N+k} e^{-t\sqrt{\mathbb{H}}}[(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f.\chi_Q]\frac{dt}{t}\Big| \\ &\le \frac{1}{s_Q}\int_{2^v}^{2^{v+1}} t^{M-k}\int_Q |p_{t,N+k}(x,y)|\big|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f(y)\big| dy\frac{dt}{t} \\ &\lesssim \frac{1}{s_Q}\sup_{(z,t)\in Q\times[2^v,2^{v+1})}\big|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f(z)\big| \\ &\quad\times \int_{2^v}^{2^{v+1}} t^{M-k}\int_Q \frac{t^N}{(t+|x-y|)^{n+N}} dy\frac{dt}{t}. \end{aligned} \tag{3.4}$$

On the other hand, it is not difficult to verify that

$$\int_Q \frac{t^N}{(t+|x-y|)^{n+N}} dy \le C(n,N)\Big(1+\frac{|x-x_Q|}{2^v}\Big)^{-n-N}, \quad \forall t\in[2^v,2^{v+1}). \tag{3.5}$$

Combination (3.3), (3.4) and (3.5) yields

$$|\mathbb{H}^{k/2} b_Q(x)| \lesssim 2^{v(\alpha+M-k-n/r)}\Big(1+\frac{|x-x_Q|}{2^v}\Big)^{-n-N}.$$

This implies that m_Q is an $(\mathbb{H}, M, N, \alpha, r)$ molecule.

Next, we prove (3.1). We observe that $w(x,t) \equiv \mathbb{H}^{m/2} e^{-t\sqrt{\mathbb{H}}} f(x)$ is a solution of the equation

$$-(\Delta_{x,t} + |x|^2)w = 0, \quad \text{with } \Delta_{x,t} w = w_{tt} + \Delta w.$$

So, w is a subharmonic function. Thanks to [5, Lemma 5.2], for every $\theta \in (0,\infty)$ we obtain

$$\sup_{(y,t)\in\widetilde{Q}} |\mathbb{H}^{m/2} e^{-t\sqrt{\mathbb{H}}} f(y)| \lesssim \Big(\frac{1}{|\widetilde{Q}|}\int_{\frac{3}{2}\widetilde{Q}} |\mathbb{H}^{m/2} e^{-t\sqrt{\mathbb{H}}} f(y)|^\theta dydt\Big)^{1/\theta},$$

where $\widetilde{Q} = Q \times [2^v, 2^{v+1})$ is a cube in $\mathbb{R}^{n+1}$.

Note that $|\widetilde{Q}| \sim 2^v|Q|$ and $t \sim 2^v$, for any $(y,t) \in \widetilde{Q}$. Hence, it follows from the last inequality that

$$\begin{aligned}\sup_{(y,t)\in\widetilde{Q}} |(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f(y)| &\lesssim \Big(\frac{1}{|Q|}\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \int_{\frac{3}{2}Q} |(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f(y)|^\theta dy \frac{dt}{t}\Big)^{1/\theta} \\ &\lesssim \Big(\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} [\mathbb{M}_\theta(|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f|)(x)]^\theta \frac{dt}{t}\Big)^{1/\theta},\end{aligned} \tag{3.6}$$

for any $x \in Q$. From (3.3) and (3.6), we obtain

$$|s_Q|\chi_Q(x) \lesssim 2^{-v(\alpha-n/r)}\Big(\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} [\mathbb{M}_\theta(|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f|)(x)]^\theta \frac{dt}{t}\Big)^{1/\theta}\chi_Q(x),$$

or

$$\sum_{Q\in\mathcal{D}_v} |Q|^{-1/r}|s_Q|\chi_Q(x) \lesssim 2^{-v\alpha}\Big(\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} [\mathbb{M}_\theta(|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f|)(x)]^\theta \frac{dt}{t}\Big)^{1/\theta}.$$

Thanks to Lemma 2.5, we have

$$A_v \lesssim 2^{-v\alpha}\Big\|\Big(\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} [\mathbb{M}_\theta(|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f|)]^\theta \frac{dt}{t}\Big)^{1/\theta}\Big\|_{\mathrm{M}_p^r}.$$

Next, Minkowski integral inequality (see (2.3)) yields

$$A_v \lesssim 2^{-v\alpha}\Big[\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \|\mathbb{M}_\theta\big(|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f|\big)\|_{\mathrm{M}_p^r}^\theta \frac{dt}{t}\Big]^{1/\theta}.$$

At the moment, for a fixed $\theta \in (0,\theta_0)$, then $\mathbb{M}_\theta$ is a bounded operator on M_p^r, likewise

$$\begin{aligned}A_v &\lesssim 2^{-v\alpha}\Big[\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\|_{\mathrm{M}_p^r}^\theta \frac{dt}{t}\Big]^{1/\theta} \\ &\lesssim \Big[\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \Big(t^{-\alpha}\|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\|_{\mathrm{M}_p^r}\Big)^\theta \frac{dt}{t}\Big]^{1/\theta} \\ &\lesssim \Big[\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \Big(t^{-\alpha}\|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\|_{\mathrm{M}_p^r}\Big)^q \frac{dt}{t}\Big]^{1/q},\end{aligned}$$

where the last inequality is obtained by using Hölder's inequality. Therefore,

$$\Big(\sum_{v\in\mathbb{Z}} A_v^q\Big)^{1/q} \lesssim \Big[\sum_{v\in\mathbb{Z}}\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \Big(t^{-\alpha}\|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\|_{\mathrm{M}_p^r}\Big)^q \frac{dt}{t}\Big]^{1/q}.$$

By noting that $\sum_{v\in\mathbb{Z}} \chi_{(\frac{3}{4}2^v,\frac{9}{8}2^{v+1})} \leq 2$, we obtain

$$\sum_{v\in\mathbb{Z}}\int_{\frac{3}{4}2^v}^{\frac{9}{8}2^{v+1}} \Big(t^{-\alpha}\|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\|_{\mathrm{M}_p^r}\Big)^q \frac{dt}{t} \leq 2\int_0^\infty \Big(t^{-\alpha}\|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\|_{\mathrm{M}_p^r}\Big)^q \frac{dt}{t},$$

which implies

$$\Big(\sum_{v\in\mathbb{Z}} A_v^q\Big)^{1/q} \lesssim \Big[\int_0^\infty \Big(t^{-\alpha}\|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} f\|_{\mathrm{M}_p^r}\Big)^q \frac{dt}{t}\Big]^{1/q} = \|f\|_{\mathrm{BM}_{p,q,r}^{\alpha,\mathbb{H},m}}.$$

This completes the proof of part (i). □

To prove (ii) of Theorem 3.6, we need the following auxiliary lemmas.

Lemma 3.7. *Let $N > 0$, and let $\eta, v \in \mathbb{Z}$ be such that $v \le \eta$. Let $\{f_Q\}_{Q\in\mathcal{D}_v}$ be a sequence of functions satisfying*

$$|f_Q(x)| \lesssim \left(1 + 2^{-\eta}|x - x_Q|\right)^{-n-N}.$$

Then, for any $\theta \in (\frac{n}{n+N}, \infty)$ and for a sequence of numbers $\{s_Q\}_{Q\in\mathcal{D}_v}$, we have

$$\sum_{Q\in\mathcal{D}_v} |s_Q||f_Q(x)| \lesssim 2^{\frac{(\eta-v)n}{\theta}} \mathbb{M}_\theta\Big(\sum_{Q\in\mathcal{D}_v} |s_Q|\chi_Q\Big)(x).$$

The proof of the above lemma can be found in [7, p.147]. Next, we recall [1, Lemma 3.6].

Lemma 3.8. *Under the assumptions as in (ii) of Theorem 3.6, we have*

$$|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q(x)| \lesssim |Q|^{\frac{\alpha}{n}-\frac{1}{r}}\Big(\frac{t}{2^v}\Big)^{m-N-n}\Big(1 + \frac{|x - x_Q|}{2^v}\Big)^{-n-N}, \quad \forall t < 2^v,$$
$$|(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q(x)| \lesssim |Q|^{\frac{\alpha}{n}-\frac{1}{r}}\Big(\frac{2^v}{t}\Big)^{M}\Big(1 + \frac{|x - x_Q|}{t}\Big)^{-n-N}, \quad \forall t \ge 2^v.$$

Proof of part (ii) of Theorem 3.6. We begin by writing

$$\begin{aligned}
\|f\|^q_{\mathrm{BM}^{\alpha,\mathbb{H},m}_{p,q,r}} &= \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \Big(t^{-\alpha}\|\sum_{v\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_v} s_Q(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q\|_{\mathrm{M}^r_p}\Big)^q \frac{dt}{t} \\
&\lesssim \sum_{k\in\mathbb{Z}} \Big(2^{-k\alpha}\|\sum_{v>k}\sum_{Q\in\mathcal{D}_v} |s_Q| \sup_{t\in[2^k,2^{k+1})} |(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q|\|_{\mathrm{M}^r_p}\Big)^q \\
&\quad + \sum_{k\in\mathbb{Z}} \Big(2^{-k\alpha}\|\sum_{v\le k}\sum_{Q\in\mathcal{D}_v} |s_Q| \sup_{t\in[2^k,2^{k+1})} |(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q|\|_{\mathrm{M}^r_p}\Big)^q \\
&:= I_1 + I_2.
\end{aligned}$$

Thus, the proof is complete if we can demonstrate that

$$I_1,\ I_2 \lesssim \sum_{v\in\mathbb{Z}} A_v^q. \tag{3.7}$$

We first prove (3.7) for I_1. Keep in mind that $v \ge k+1$ in this case. Since $\theta_0 > \frac{n}{n+N}$ and $M > \max\{\frac{n}{\theta_0} - \alpha, m\}$, we can choose a real number $\theta \in (\frac{n}{n+N}, \theta_0)$ such that $M > \frac{n}{\theta} - \alpha$. By noting that $2^v \ge 2^{k+1} > t$, Lemma 3.8 implies

$$\sup_{t\in[2^k,2^{k+1})} |(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q(x)| \lesssim |Q|^{\frac{\alpha}{n}-\frac{1}{r}} 2^{(k-v)(m-N-n)}\left(1 + 2^{-v}|x - x_Q|\right)^{-n-N}.$$

Thus,

$$\begin{aligned}
&\sum_{Q\in\mathcal{D}_v} |s_Q| \sup_{t\in[2^k,2^{k+1})} |(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q(x)| \\
&\lesssim \sum_{Q\in\mathcal{D}_v} |Q|^{\frac{\alpha}{n}-\frac{1}{r}} 2^{(k-v)(m-N-n)}|s_Q|\left(1 + 2^{-v}|x - x_Q|\right)^{-n-N} \\
&\lesssim 2^{v\alpha}2^{(k-v)(m-N-n)} \sum_{Q\in\mathcal{D}_v} |Q|^{-1/r}|s_Q|\left(1 + 2^{-v}|x - x_Q|\right)^{-n-N}.
\end{aligned} \tag{3.8}$$

Now, we apply Lemma 3.7 with $\eta = v$ and $f_Q(x) = (1 + 2^{-v}|x - x_Q|)^{-n-N}$ to obtain

$$\sum_{Q\in\mathcal{D}_v} |Q|^{-1/r}|s_Q| \left(1 + 2^{-v}|x - x_Q|\right)^{-n-N} \lesssim \mathbb{M}_\theta\Big(\sum_{Q\in\mathcal{D}_v} |Q|^{-1/r}|s_Q|\chi_Q\Big)(x), \quad (3.9)$$

for $\theta \in (\frac{n}{n+N}, \theta_0)$. Inserting (3.9) into (3.8) yields

$$\begin{aligned}&\sum_{Q\in\mathcal{D}_v} |s_Q| \sup_{t\in[2^k,2^{k+1})} |(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q(x)| \\ &\lesssim 2^{v\alpha}2^{(k-v)(m-N-n)}\mathbb{M}_\theta\Big(\sum_{Q\in\mathcal{D}_v} |Q|^{-1/r}|s_Q|\chi_Q\Big)(x).\end{aligned}$$

Then

$$\begin{aligned}I_1 &\lesssim \sum_{k\in\mathbb{Z}} \Big[2^{-k\alpha}\|\sum_{v>k} 2^{\alpha v}2^{(k-v)(m-N-n)}\mathbb{M}_\theta\Big(\sum_{Q\in\mathcal{D}_v} |Q|^{-1/r}|s_Q|\chi_Q\Big)\|_{\mathrm{M}^r_p}\Big]^q \\ &= \sum_{k\in\mathbb{Z}} \|\sum_{v>k} 2^{(k-v)(m-N-n-\alpha)}\mathbb{M}_\theta\Big(\sum_{Q\in\mathcal{D}_v} |Q|^{-1/r}|s_Q|\chi_Q\Big)\|^q_{\mathrm{M}^r_p} \\ &\lesssim \sum_{k\in\mathbb{Z}} \Big[\sum_{v>k} 2^{(k-v)(m-N-n-\alpha)}\|\mathbb{M}_\theta\Big(\sum_{Q\in\mathcal{D}_v} |Q|^{-1/r}|s_Q|\chi_Q\Big)\|_{\mathrm{M}^r_p}\Big]^q.\end{aligned} \quad (3.10)$$

Again the fact that $\mathbb{M}_\theta$ is bounded on M^r_p implies

$$\|\mathbb{M}_\theta\Big(\sum_{Q\in\mathcal{D}_v} |Q|^{-1/r}|s_Q|\chi_Q\Big)\|_{\mathrm{M}^r_p} \lesssim \|\sum_{Q\in\mathcal{D}_v} |Q|^{-1/r}|s_Q|\chi_Q\|_{\mathrm{M}^r_p} \sim A_v. \quad (3.11)$$

Combination (3.10) and (3.11) yields

$$I_1 \lesssim \sum_{k\in\mathbb{Z}} \Big[\sum_{v>k} 2^{(k-v)(m-N-n-\alpha)} A_v\Big]^q.$$

Applying Young's inequality yields

$$\begin{aligned}&\sum_{v>k} 2^{(k-v)(m-N-n-\alpha)} A_v \\ &\le \Big(\sum_{v>k} 2^{\frac{(k-v)(m-N-n-\alpha)q}{2(q-1)}}\Big)^{\frac{q-1}{q}} \Big(\sum_{v>k} 2^{\frac{(k-v)(m-N-n-\alpha)q}{2}} A_v^q\Big)^{1/q}.\end{aligned}$$

Since $m > N + n + \alpha$, $\sum_{v>k} 2^{\frac{(k-v)(m-N-n-\alpha)q}{2(q-1)}}$ is then bounded by a constant independent of k, v. Thus,

$$\begin{aligned}I_1 &\lesssim \sum_{k\in\mathbb{Z}}\sum_{v>k} 2^{\frac{(k-v)(m-N-n-\alpha)q}{2}} A_v^q \\ &= \sum_{v\in\mathbb{Z}} \left(\sum_{k<v} 2^{\frac{(k-v)(m-N-n-\alpha)q}{2}}\right) A_v^q \lesssim \sum_{v\in\mathbb{Z}} A_v^q.\end{aligned}$$

It remains to show that estimate (3.7) holds for I_2. Actually, the proof for I_2 is most likely to the one for I_1, with only one different point that we use Lemma 3.8 for $v \le k$, i.e.

$$\sup_{t\in[2^k,2^{k+1})} |(t\sqrt{\mathbb{H}})^m e^{-t\sqrt{\mathbb{H}}} m_Q(x)| \lesssim |Q|^{\frac{\alpha}{n}-\frac{1}{r}} 2^{(v-k)M}\Big(1 + \frac{|x - x_Q|}{2^v}\Big)^{-n-N}.$$

Proceed similarly to the proof (from (3.8) to (3.11)) above, we obtain

$$I_2 \lesssim \sum_{k\in\mathbb{Z}} \Big[\sum_{v\le k} 2^{(v-k)(M+\alpha)} A_v\Big]^q.$$

By noting that $M+\alpha>0$, apply Young's inequality yields the result. This completes the proof of Theorem 3.6. ☐

Proof of Theorem 3.3. Let $N = \mathrm{int}[n(\frac{1}{\theta_0}-1)]+1$, and $M > \max\{m_1, m_2, \frac{n}{\theta_0}-\alpha\}$. Because m_1 and m_2 play the same role, it then suffices to prove that $\mathrm{BM}^{\alpha,\mathbb{H},m_1}_{p,q} \hookrightarrow \mathrm{BM}^{\alpha,\mathbb{H},m_2}_{p,q}$.

In fact, for $f\in \mathrm{BM}^{\alpha,\mathbb{H},m_1}_{p,q,r}$, thanks to (i) of Theorem 3.6, there exist a sequence of $(\mathbb{H}, M, N, \alpha, r)$ molecules $\Big\{m_Q : Q\in\mathcal{D}_v, v\in\mathbb{Z}\Big\}$, and a sequence of coefficients $\Big\{s_Q : Q\in\mathcal{D}_v, v\in\mathbb{Z}\Big\}$ so that

$$f = \sum_{v\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_v} s_Q m_Q, \quad \text{in } \mathcal{S}',$$

and

$$\Big(\sum_{v\in\mathbb{Z}} A_v^q\Big)^{1/q} \lesssim \|f\|_{\mathrm{BM}^{\alpha,\mathbb{H},m_1}_{p,q,r}}.$$

In other words, $(\sum_{v\in\mathbb{Z}} A_v^q)^{1/q}$ is finite.

By (ii) of Theorem 3.6, we obtain $f\in \mathrm{BM}^{\alpha,\mathbb{H},m_2}_{p,q,r}$. Furthermore, f satisfies

$$\|f\|_{\mathrm{BM}^{\alpha,\mathbb{H},m_2}_{p,q,r}} \lesssim \Big(\sum_{v\in\mathbb{Z}} A_v^q\Big)^{1/q}.$$

Or, we obtain the result. ☐

4. Regularity on Besov-Morrey spaces for fractional Hermite equations

In this part, we study the regularity results of solutions of the two fractional Hermite equations:

$$\mathbb{H}^s u = f, \quad \text{and} \quad (I+\mathbb{H})^s = f, \quad \text{on } \mathbb{R}^n,$$

for any $s>0$, and for $f\in \mathrm{BM}^{\alpha,\mathbb{H}}_{p,q,r}$. To solve the indicated equations, it is necessary to investigate the operators $\mathbb{H}^{-s}$ and $(I+\mathbb{H})^{-s}$, named by the Riesz potential of Hermite operator and the Bessel potential of Hermite operator respectively.

In fact, by following [1, Proposition 2.5], we can define the operators $\mathbb{H}^{-s} : \mathcal{S}' \to \mathcal{S}'$ and $(I+\mathbb{H})^{-s} : \mathcal{S}'\to\mathcal{S}'$ by setting

$$\langle \mathbb{H}^{-s} f, \phi\rangle = \langle f, \mathbb{H}^{-s}\phi\rangle, \quad \text{and} \quad \langle (I+\mathbb{H})^{-s} f, \phi\rangle = \langle f, (I+\mathbb{H})^{-s}\phi\rangle,$$

for any $f\in\mathcal{S}'$, and for $\phi\in\mathcal{S}$. Note that $\langle\cdot,\cdot\rangle$ is the pair between a linear function in $\mathcal{S}'$ and a function in $\mathcal{S}$. Moreover, for any $\phi\in\mathcal{S}$ we have

$$\mathbb{H}^{-s}\phi = \frac{1}{\Gamma(s)}\int_0^\infty t^s e^{-t\mathbb{H}}\phi \frac{dt}{t} \in \mathcal{S},$$
$$(I+\mathbb{H})^{-s}\phi = \frac{1}{\Gamma(s)}\int_0^\infty t^s e^{-t} e^{-t\mathbb{H}}\phi \frac{dt}{t} \in \mathcal{S}.$$

Let $K_t(x,y)$ (resp. $K_{t,k}(x,y)$) be the kernel of $e^{-t\mathbb{H}}$ (resp. $(t\mathbb{H})^k e^{-t\mathbb{H}}$). Thanks to [6, Lemma 2.5], and [1, Lemma 2.4], we have the following results.

Lemma 4.1. *For $k\in\mathbb{N}$, there exists $c, C>0$ so that for all $y\in\mathbb{R}$*

$$|\partial_x^k K_t(x,y)| \le \begin{cases} Ct^{-\frac{k+1}{2}}\exp\big(-c\frac{|x-y|^2}{t}\big), & 0<t\le 1;\\ e^{-t}e^{-|x-y|^2}, & t>1. \end{cases} \tag{4.1}$$

$$K_{t,k}(x,y) \le \frac{C}{t^{n/2}}\exp\Big(-c\frac{|x-y|^2}{t}\Big), \tag{4.2}$$

Our regularity results are as follows.

Theorem 4.2. *Let $\alpha\in\mathbb{R}$, $0<q\le\infty$, $0<p\le r\le\infty$, and $f\in \mathrm{BM}^{\alpha,\mathbb{H}}_{p,q,r}$. Assume that u is a solution of equation $\mathbb{H}^s u = f$, Then, there exists a constant $C>0$ such that*

$$\|u\|_{\mathrm{BM}^{\alpha+2s,\mathbb{H}}_{p,q,r}} \le C\|f\|_{\mathrm{BM}^{\alpha,\mathbb{H}}_{p,q,r}}.$$

Theorem 4.3. *Let $\alpha\in\mathbb{R}$, $0<q\le\infty$, $0<p\le r\le\infty$, and $f\in \mathrm{BM}^{\alpha,\mathbb{H}}_{p,q,r}$. Assume that u is a solution of equation $(\mathbb{H}+I)^s u = f$. Then, there exists a constant $C>0$ such that*

$$\|u\|_{\mathrm{BM}^{\alpha+2s,\mathbb{H}}_{p,q,r}} \le C\|f\|_{\mathrm{BM}^{\alpha,\mathbb{H}}_{p,q,r}}.$$

Theorems 4.2 and 4.3 are just a consequence of the theorem below.

Theorem 4.4. *Let $\alpha\in\mathbb{R}$, $0<p\le r<\infty$, and $0<q\le\infty$. For any $s>0$, the operator $\mathbb{H}^{-s}$ (resp. $(I+\mathbb{H})^{-s}$) is bounded from $\mathrm{BM}^{\alpha,\mathbb{H}}_{p,q,r}$ to $\mathrm{BM}^{\alpha+2s,\mathbb{H}}_{p,q,r}$.*

Proof of Theorem 4.4. Let $\{m_Q : Q\in\mathcal{D}_v, v\in\mathbb{Z}\}$ be a sequence of $(\mathbb{H}, 4M, N, \alpha, r)$ molecules, with $M,N\in\mathbb{N}$, and $M>s+n/2+N/2$.

We first prove that $\mathbb{H}^{-s}(m_Q)$ is an $(H, 2M, N, \alpha+2s, r)$ molecule associated with the cube Q. Indeed, let $m_Q=\mathbb{H}^{2M}b_Q$ as in Definition 3.4, and put $y_Q=\mathbb{H}^{-s}\mathbb{H}^M b_Q$. Then

$$\mathbb{H}^{-s}m_Q = \mathbb{H}^M y_Q = (\sqrt{\mathbb{H}})^{2M} y_Q.$$

Thus, it suffices to show that

$$|(\sqrt{\mathbb{H}})^k y_Q(x)| \lesssim \ell(Q)^{2M-k}|Q|^{\frac{\alpha+2s}{n}-\frac{1}{r}}\Big(1+\frac{|x-x_Q|}{\ell(Q)}\Big)^{-n-N}, \tag{4.3}$$

for $k=0,\dots,4M$. In fact, we have

$$y_Q(x) = \mathbb{H}^{-s}\mathbb{H}^M b_Q = \frac{1}{\Gamma(s)}\int_0^\infty t^s e^{-t\mathbb{H}}\mathbb{H}^M b_Q(x)\frac{dt}{t}.$$

Therefore,

$$\begin{aligned} |(\sqrt{\mathbb{H}})^k y_Q(x)| &\le \frac{1}{\Gamma(s)}\int_0^{4^v} |t^s e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^{2M+k}b_Q(x)|\frac{dt}{t}\\ &\quad + \frac{1}{\Gamma(s)}\int_{4^v}^\infty |t^s e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^{2M+k}b_Q(x)|\frac{dt}{t} := I_1+I_2. \end{aligned}$$

First, we estimate I_1. Thanks to Lemma 4.1, we have

$$\begin{aligned} |e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^{2M+k}b_Q(x)| &= \int_{\mathbb{R}^n} |K_t(x,y)(\sqrt{\mathbb{H}})^{2M+k}b_Q(y)|dy\\ &\lesssim \int_{\mathbb{R}^n}\frac{1}{t^{n/2}}\exp\Big(-2c\frac{|x-y|^2}{t}\Big)|(\sqrt{\mathbb{H}})^{2M+k}b_Q(y)|dy. \end{aligned}$$

Taking Definition 3.4 into account, we obtain

$$|e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^{2M+k}b_Q(x)| \lesssim \int_{\mathbb{R}^n} \frac{1}{t^{n/2}} \exp\Big(-c\frac{|x-y|^2}{t}\Big)\Big(1+\frac{|x-y|}{\sqrt{t}}\Big)^{-n-N} \times |Q|^{\frac{\alpha}{n}-\frac{1}{r}}2^{v(2M-k)}\Big(1+\frac{|y-x_Q|}{2^v}\Big)^{-n-N}dy.$$

Next, we apply the inequality $(1+a+b)\le(1+a)(1+b)$, for all $a,b\ge 0$ and the fact $t<4^v$ to the right hand side of the above inequality to obtain

$$\begin{aligned}
&|e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^{2M+k}b_Q(x)| \\
&\lesssim |Q|^{\frac{\alpha}{n}-\frac{1}{r}}2^{v(2M-k)}\Big(1+\frac{|x-x_Q|}{2^v}\Big)^{-n-N}\int_{\mathbb{R}^n}\frac{1}{t^{n/2}}\exp\Big(-c\frac{|x-y|^2}{t}\Big)dy;
\end{aligned}$$

thus

$$|e^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^{2M+k}b_Q(x)| \lesssim |Q|^{\frac{\alpha}{n}-\frac{1}{r}}2^{v(2M-k)}\Big(1+\frac{|x-x_Q|}{2^v}\Big)^{-n-N}.$$

This implies

$$\begin{aligned}
I_1 &\lesssim |Q|^{\frac{\alpha}{n}-\frac{1}{r}}2^{v(2M-k)}\Big(1+\frac{|x-x_Q|}{2^v}\Big)^{-n-N}\int_0^{4^v} t^s\frac{dt}{t} \\
&\lesssim |Q|^{\frac{\alpha+2s}{n}-\frac{1}{r}}2^{v(2M-k)}\Big(1+\frac{|x-x_Q|}{2^v}\Big)^{-n-N}.
\end{aligned} \tag{4.4}$$

It remains to consider I_2. By (4.2), we have

$$\begin{aligned}
|\mathbb{H}^Me^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^kb_Q(x)| &= t^{-M}|(t\mathbb{H})^Me^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^kb_Q(x)| \\
&= t^{-M}\int_{\mathbb{R}^n}|K_{t,M}(x,y)(\sqrt{\mathbb{H}})^kb_Q(y)|dy \\
&\lesssim t^{-M}\int_{\mathbb{R}^n}\frac{1}{t^{n/2}}\exp\Big(-c\frac{|x-y|^2}{t}\Big)|(\sqrt{\mathbb{H}})^kb_Q(y)|dy.
\end{aligned}$$

In similar to the above proof, we also have

$$\begin{aligned}
&|\mathbb{H}^Me^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^kb_Q(x)| \\
&\lesssim t^{-M}\int_{\mathbb{R}^n}\frac{1}{t^{n/2}}\Big(1+\frac{|x-y|}{\sqrt{t}}\Big)^{-n-N}|Q|^{\frac{\alpha}{n}-\frac{1}{r}}2^{v(4M-k)}\Big(1+\frac{|y-x_Q|}{2^v}\Big)^{-n-N}dy.
\end{aligned}$$

By Lemma 3.5, and noting that $t\ge 4^v$, we obtain

$$\begin{aligned}
&|\mathbb{H}^Me^{-t\mathbb{H}}(\sqrt{\mathbb{H}})^kb_Q(x)| \\
&\lesssim t^{-M}|Q|^{\frac{\alpha}{n}-\frac{1}{r}}2^{v(4M-k)}\Big(1+\frac{|x-x_Q|}{\sqrt{t}}\Big)^{-n-N} \\
&\lesssim \big(\frac{t}{4^v}\big)^{(n+N)/2}t^{-M}|Q|^{\frac{\alpha}{n}-\frac{1}{r}}2^{v(4M-k)}\Big(1+\frac{|x-x_Q|}{2^v}\Big)^{-n-N}.
\end{aligned}$$

Thus

$$\begin{aligned}
I_2 &\lesssim |Q|^{\frac{\alpha}{n}-\frac{1}{r}}2^{v(4M-k-N-n)}\Big(1+\frac{|x-x_Q|}{2^v}\Big)^{-n-N}\int_{4^v}^\infty t^{s+\frac{n+N}{2}-M}\frac{dt}{t} \\
&\lesssim |Q|^{\frac{\alpha+2s}{n}-\frac{1}{r}}2^{v(2M-k)}\Big(1+\frac{|x-x_Q|}{2^v}\Big)^{-n-N}.
\end{aligned} \tag{4.5}$$

Hence, (4.3) follows from (4.4) and (4.5). Thus $\mathbb{H}^{-s}(m_Q)$ is an $(H,2M,N,\alpha+2s,r)$ molecule associated with the cube Q. By Theorem 3.6 and a suitable choice of M,N, we obtain the boundedness of $\mathbb{H}^{-s}$ from $\mathrm{BM}^{\alpha,\mathbb{H}}_{p,q,r}$ to $\mathrm{BM}^{\alpha+2s,\mathbb{H}}_{p,q,r}$.

Similarly, we can also establish the boundedness of the Bessel potential $(I+H)^{-s}$ from $\mathrm{BM}^{\alpha,\mathbb{H}}_{p,q,r}$ to $\mathrm{BM}^{\alpha+2s,\mathbb{H}}_{p,q,r}$. We leave the proof to the reader. $\square$

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