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EXISTENCE AND GLOBAL ATTRACTIVITY OF POSITIVE PERIODIC SOLUTIONS FOR A PREDATOR-PREY MODEL WITH CROWLEY-MARTIN FUNCTIONAL RESPONSE

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Abstract. In this article, we consider a predator-prey system with mutual interference and Crowley-Martin functional response. We obtain positive solutions for the system by using the comparison principle. The existence of periodic solutions is established by applying coincidence degree theory. In addition, we obtain that the system has only one positive periodic solution which is a global attractor by constructing a proper Lyapunov function.

1. Introduction

Predator-prey model is one of the dominant themes in both ecology and mathematical ecology because of its universal existence and importance with many concerned biological studies [1]. In recent years, classical predator-prey models have been extensively studied; see [2, 3, 4, 5] and the references cited therein. Hassell [6] discussed the following model with the mutual interference between predator and prey,

$$\begin{aligned} \dot{x} &= xg(x) - \psi(x)y^m, \\ \dot{y} &= y(-d + k\psi(x)y^{m-1} - q(y)), \end{aligned} \tag{1.1}$$

where the mutual interference constant $m \in (0, 1]$ is a real number. During his research of the capturing behavior between hosts and parasites, he found that the hosts or parasites had the tendency to leave from each other when they met, which interfered with the hosts capturing effects. It is obvious that the mutual interference will be stronger while the size of the parasite becomes larger.

Mathematicians and ecologists have explored the dynamical behavior of predator-prey models with the Holling type II [7, 8] and Holling type III functional responses. Lv [9, 10] investigated the existence and globally attractivity of the positive periodic solutions of the predator-prey model with mutual interference and Holling type III,

$$\begin{aligned} \dot{x}(t) &= x(t)(r_1(t) - b_1(t)x(t)) - \frac{c_1(t)x^2(t)}{k^2 + x^2(t)}y^m, \\ \dot{y}(t) &= y(t)(-r_2(t) - b_2(t)y(t)) + \frac{c_2(t)x^2(t)}{k^2 + x^2(t)}y^m. \end{aligned} \tag{1.2}$$

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Later, the permanence and existence of a unique globally attractive positive almost periodic solution of model (1.2) were considered by Zhang et al. [11]. There are many other types of functional response such as Beddington-DeAngelis which has been discussed [12, 13, 14, 15, 16, 17]. For example, Lin and Chen [16] studied an almost periodic Volterra model with mutual interference and Beddington-DeAngelis functional response as follows

$$\begin{aligned} \dot{x}(t) &= x(t)(r_1(t) - b_1(t)x(t)) - \frac{k_1(t)x(t)}{a(t) + d(t)x(t) + c(t)y(t)} y^m, \\ \dot{y}(t) &= y(t)(-r_2(t) - b_2(t)y(t)) + \frac{k_2(t)x(t)}{a(t) + d(t)x(t) + c(t)y(t)} y^m. \end{aligned} \tag{1.3}$$

Guo and Chen [17] investigated a special case of system (1.3) and proved the existence and global attractivity of positive periodic solutions for the model. System (1.3) with Beddington-DeAngelis functional response adopts that handing and interference are exclusive activities.

Crowley-Martin [18] assumed that predator's predation will decrease due to high predator density (interference among the predator individuals) even when prey density is high (presence of handing or searching of prey by predator individual) [19]. There are very few literature available on predator-prey model with Crowley-Martin functional response [20, 21, 22]. The Crowley-Martin functional response is predator dependent. The per capita fedding rate for predator y in this formulation is

$$\eta(x, y) = \frac{bx}{(1 + a_1 x)(1 + b_1 y)},$$

where b, a_1, b_1 are positive parameters that are used for effects of capture rate, handling time and magnitude of interference among predators, respectively, on the fedding rate. If we consider $a_1 x \gg 1 + b_1 y$ along with absence of mutual interference among predators at high prey density (i.e. when $a_1 b_1 xy$ becomes too small) then the food supply (prey population) will be superabundant i.e. the increase in prey density (x) will not increase the feeding rate per predator $\eta(x, y)$. In this case, the predation rate per unit of predator $\eta(x, y)$ becomes constant and $\eta(x, y) = \frac{b}{a_1}$. As $y \to 0$, the limiting value of $\eta(x, y)$ becomes a function of x only and when $y \to \infty$, $\eta(x, y) \to 0$. One can easily observe that $\eta(x, y)$ varies inversely with respect to y. Crowley-Martin response function represents classical response function for $a_1 = 0$, $b_1 = 0$ while Michaelis-Menten (Holling type II) functional response for $a_1 > 0$, $b_1 = 0$ [1, 23].

Tripathi et al. [24] investigated the globally stability of the predator-prey model with Crowley-Martin response function with time delay of the form

$$\begin{aligned} \frac{dX}{dT} &= X(A - BX - \frac{CY}{A_1 + B_1 X + C_1 Y + B_1 C_1 XY}), \\ \frac{dY}{dT} &= Y(-D - EY + \frac{FX}{A_1 + B_1 X + C_1 Y + B_1 C_1 XY}). \end{aligned} \tag{1.4}$$

Egami and Hirano [25] considered the almost periodic solution and global attractivity on the basis of system (1.4).

Motivated by the above results, in this article we consider a predator-prey model with Crowley-Martin functional response

$$
\begin{aligned}
\dot{x}(t) &= x(t)[r_1(t) - a_1(t)x(t) - a_2(t)x(t-\tau)] \\
&\quad - \frac{C(t)x(t)y^m(t)}{A_1(t) + B_1(t)x(t) + C_1(t)y(t) + B_1(t)C_1(t)x(t)y(t)}, \\
\dot{y}(t) &= y(t)[-r_2(t) - b_1(t)y(t) - b_2(t)y(t-\tau)] \\
&\quad + \frac{D(t)x(t)y^m(t)}{A_1(t) + B_1(t)x(t) + C_1(t)y(t) + B_1(t)C_1(t)x(t)y(t)},
\end{aligned}
\tag{1.5}
$$

where $0 < m \leq 1$, $x(t)$ and $y(t)$ represent prey and predator densities at time t, respectively, r_1 is intrinsic growth rate of the prey in the absence of the predator and r_2 is the death rate of the predator, a_1 and b_1 are decay rates of the prey and the predator in competition among their own populations, a_2 and b_2 are decay rates of the prey and the predator effected by harmful environmental for a period of past time. $r_i(t)$, $a_i(t)$, $b_i(t)$ $(i = 1, 2)$, $A_1(t)$, $B_1(t)$, $C_1(t)$, $C(t)$, $D(t)$ are positive ω-periodic function, $t \in [0, \infty)$, delay $\tau > 0$, $n \geq 2$.

This article is organized as follow. In section 2, by using the comparison principle in ordinary differential equation and some analytical techniques, we study the permanence of positive solutions of delayed predator-prey model (1.5). In section 3, we prove the existence of positive periodic solutions to systems (1.5) by applying the coincidence degree theory. Section 4 is devoted to the global attractivity by constructing a suitable Lyapunov function.

2. Permanence

For the sake of convenience and simplicity, we introduce some notation as follows:

$$
f^L = \min_{t\in[0,\omega]} f(t), \quad f^M = \max_{t\in[0,\omega]} f(t), \quad |f|_0 = \max_{t\in[0,\omega]} \{|f(t)|\},
$$
$$
\hat{f} = \frac{1}{\omega}\int_0^\omega |f(t)|dt, \quad \bar{f} = \frac{1}{\omega}\int_0^\omega f(t)dt,
$$

where f is a continuous ω-periodic function. To obtain the permanence of positive solutions of system (1.5), we state some lemmas.

Lemma 2.1 ([26]). *If $a > 0$, $b > 0$, $\dot{z} \geq (\leq) z(b - az)$ and $z(0) > 0$, then, for any small constant $\varepsilon > 0$, there exists a positive constant T, such that*

$$
z(t) \geq \frac{b}{a} - \varepsilon, \quad (\leq \frac{b}{a} + \varepsilon), \quad \text{for } t \geq T. \tag{2.1}
$$

Lemma 2.2 ([26]). *If $a > 0$, $b > 0$, $\dot{z} \geq (\leq) z^m(b - az^{1-m})$ and $z(0) > 0$, then, for any small constant $\varepsilon > 0$, there exists a positive constant T, such that*

$$
z(t) \geq (\frac{b}{a})^{\frac{1}{1-m}} - \varepsilon, \quad (\leq (\frac{b}{a})^{\frac{1}{1-m}} + \varepsilon), \quad \text{for } t \geq T. \tag{2.2}
$$

Theorem 2.3. *System (1.5) is permanent; which means that for any positive solution $(x(t), y(t))^T$ of (1.5), there exist positive constants K_i, $i = 1, 2, 3, 4$, and $T > 0$ such that*

$$
K_3 \leq x(t) \leq K_1, \quad K_4 \leq y(t) \leq K_2, \quad \text{for } t \geq T.
$$

Proof. Assume $(x(t), y(t))^T$ is an arbitrary positive solution of system (1.5), then the first equation in (1.5) yields

$$\dot{x}(t) \le x(t)(r_1^M - a_1^L x(t)).$$

From Lemma 2.1, for any small constant $\varepsilon_0 > 0$, there exists a positive constant T_1, such that

$$x(t) \le (\frac{r_1^M}{a_1^L}) + \varepsilon_0 := K_1, \quad \text{for } t \ge T_1. \tag{2.3}$$

Similarly, from the second equation in (1.5), we obtain

$$\dot{y}(t) \le y^m(t)\Big(\frac{D^M K_1}{A_1^L} - r_2^L y^{1-m}(t)\Big).$$

By Lemma 2.2, for the ε_0 chosen above, there exists a positive constant T_2, such that

$$y(t) \le \Big(\frac{D^M K_1}{A_1^L r_2^L}\Big)^{\frac{1}{1-m}} + \varepsilon_0 := K_2, \quad \text{for } t \ge T_2. \tag{2.4}$$

From (2.3) and (2.4), for any small enough positive constant ϵ, there exists a positive number T_3 such that $x(t) \le K_1 + \epsilon$ and $y(t) \le K_2 + \epsilon$ for all $t \ge T_3$. From the first equation in (1.5), one has

$$\frac{\dot{x}(t)}{x(t)} \ge r_1^L - \frac{C^M (K_2 + \epsilon)^m}{A_1^L} - (a_1 + a_2)^M (K_1 + \epsilon).$$

Denoting

$$\delta(\epsilon) = r_1^L - \frac{C^M (K_2 + \epsilon)^m}{A_1^L} - (a_1 + a_2)^M (K_1 + \epsilon),$$

and integrating above inequality from $t - \tau$ to t, we obtain $x(t-\tau) \le e^{-\delta(\epsilon)\tau} x(t)$. From the first equation in (1.5), we obtain that

$$\dot{x}(t) \ge x(t)(r_1^L - \frac{C^M (K_2 + \epsilon)^m}{A_1^L} - (a_1^M + a_2^M e^{-\delta(\epsilon)\tau}) x(t)). \tag{2.5}$$

According to Lemma 2.1, for any positive constant

$$\varepsilon_1 \ll (r_1^L - \frac{C^M (K_2 + \epsilon)^m}{A_1^L})/(a_1^M + a_2^M e^{-\delta(\epsilon)\tau}),$$

when $\epsilon \to 0$, there exists a positive constant T_4, such that

$$x(t) \ge (r_1^L - \frac{C^M K_2^m}{A_1^L})/(a_1^M + a_2^M e^{-\delta\tau}) - \varepsilon_1 := K_3, \quad \text{for } t \ge T_4. \tag{2.6}$$

From the second equation in (1.5), we obtain

$$\begin{aligned} \dot{y}(t) \ge y^m(t)\Big(&\frac{D^L K_3}{A_1^M + B_1^M (K_1 + \epsilon) + C_1^M (K_2 + \epsilon) + C_1^M (K_2 + \epsilon) B_1^M (K_1 + \epsilon)} \\ &- (b_1^M + b_2^M)(K_2 + \epsilon)^{2-m} - r_2^M y^{1-m}(t)\Big). \end{aligned}$$

Then by Lemma 2.2, letting $\epsilon \to 0$ and for any positive constant ε_2, we have

$$\varepsilon_2 \ll (\frac{D^L K_3}{A_1^M + B_1^M K_1 + C_1^M K_2 + C_1^M K_2 B_1^M K_1} - (b_1^M + b_2^M) K_2^{2-m})/r_2^M,$$

and there exists a positive constant T_5, such that

$$\begin{aligned} y(t) \geq \Big[\Big(&\frac{D^L K_3}{A_1^M + B_1^M K_1 + C_1^M K_2 + C_1^M K_2 B_1^M K_1} \\ &- (b_1^M + b_2^M) K_2^{2-m}\Big)/r_2^M\Big]^{\frac{1}{1-m}} - \varepsilon_2 := K_4, \quad \text{for } t \geq T_5. \end{aligned} \tag{2.7}$$

Let $T = \max\{T_1, T_2, T_3, T_4, T_5\}$, and chose $\varepsilon_0 \ll \min\{\varepsilon_1, \varepsilon_2\}$, then we have

$$K_3 \leq x(t) \leq K_1, \quad K_4 \leq y(t) \leq K_2.$$

□

3. Existence of positive periodic solutions

To understand the sufficient conditions for guaranteeing the existence of positive periodic solutions, we will introduce the coincidence degree briefly as follows.

Definition 3.1 ([27]). Let X and Y be two Banach spaces, $L : \operatorname{Dom} L \subset X \to Y$ be a linear map. If the following conditions are satisfied

(a) $\operatorname{Im} L$ is a closed subspace of Y;
(b) $\dim \ker L = co\dim \operatorname{Im} L < \infty$, then we call the operator L is a Fredholm operator of index zero.

If L is a Fredholm operator with index zero and there exists continuous projections

$$P : X \to X \quad \text{and} \quad Q : Y \to Y$$

such that $X = \ker L \oplus \ker P$, $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$, $\operatorname{Im} P = \ker L$ and $\operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q)$, then $L|_{\operatorname{Dom} L \cap \ker P} : (I - P)X \to \operatorname{Im} L$ has an inverse function, and we set it as K_p, then $Kp : \operatorname{Im} L \to \operatorname{Dom} L \cap \ker P$.

Definition 3.2 ([27]). Let $N : X \to Y$ be a continuous map and $\Omega \times [0,1] \subset X$ is an open set. If $QN(\overline{\Omega} \times [0,1])$ is bounded and $K_p(I - Q)N(\overline{\Omega} \times [0,1]) \subset X$ is relatively compact, then we say that $N(\overline{\Omega} \times [0,1])$ is L-compact.

Lemma 3.3 ([27]). *Let both X and Y be Banach spaces, $L : \operatorname{Dom} L \subset X \to Y$ be a Fredholm operator with index zero, $\Omega \subset X$ be an open bounded set, and $N : \overline{\Omega} \to Y$ be L-compact on $\overline{\Omega}$. If all the following conditions hold:*

(1) *$Lx \neq \lambda N(x)$, for $x \in \partial\Omega \cap \operatorname{Dom} L, \lambda \in [0,1]$;*
(2) *$Nx \notin \operatorname{Im} L$, $x \in \partial\Omega \cap \ker L$;*
(3) *$\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$, where $J : \operatorname{Im} Q \to \ker L$ is an isomorphism, then the equation $Lx = N(x)$ has at least one solution on $\overline{\Omega} \cap \operatorname{Dom} L$.*

Suppose $(x(t), y(t))^T$ is an arbitrary positive solution of (1.5), let $u(t) = \ln x(t)$ and $v(t) = \ln y(t)$, then system (1.5) can be changed into

$$\begin{aligned} \dot{u}(t) &= r_1(t) - a_1(t)e^{u(t)} - a_2(t)e^{u(t-\tau)} \\ &\quad - \frac{C(t)e^{mv(t)}}{A_1(t) + B_1(t)e^{u(t)} + C_1(t)e^{v(t)} + B_1(t)e^{u(t)}C_1(t)e^{v(t)}}, \\ \dot{v}(t) &= -r_2(t) - b_1(t)e^{v(t)} - b_2(t)e^{v(t-\tau)} \\ &\quad + \frac{D(t)e^{u(t)}e^{(m-1)v(t)}}{A_1(t) + B_1(t)e^{u(t)} + C_1(t)e^{v(t)} + B_1(t)e^{u(t)}C_1(t)e^{v(t)}}. \end{aligned} \tag{3.1}$$

Denoting the right terms of first equation and second equation in (3.1) by $F_1(t, u(t), v(t))$ and $F_2(t, u(t), v(t))$ respectively and considering system

$$\begin{aligned}\dot{u}(t) &= \lambda F_1(t, u(t), v(t)),\\ \dot{v}(t) &= \lambda F_2(t, u(t), v(t)),\end{aligned} \tag{3.2}$$

where $\lambda \in (0, 1]$.

Lemma 3.4. *Suppose $(u(t), v(t))^T$ is a ω-periodic solution of (3.2), then there exists a positive number R_1, such that $|u(t)|+|v(t)| \le R_1$, where R_1 will be calculated as in the proof.*

Proof. Since $(u(t), v(t))^T$ is periodic, the following discussion will be restricted to $t \in [0, \omega]$. Integrating the first equation of (3.2) from 0 to ω and in view of $\int_0^\omega \dot{u}(t)dt = 0$, we obtain

$$\begin{aligned}\int_0^\omega r_1(t)dt = \int_0^\omega &(a_1(t)e^{u(t)} + a_2(t)e^{u(t-\tau)}\\ &+ \frac{C(t)e^{mv(t)}}{A_1(t) + B_1(t)e^{u(t)} + C_1(t)e^{v(t)} + B_1(t)e^{u(t)}C_1(t)e^{v(t)}})dt.\end{aligned} \tag{3.3}$$

Thus,

$$\int_0^\omega |\dot{u}(t)|dt = \lambda \int_0^\omega |F_1(t)|dt \le \int_0^\omega 2r_1(t)dt = 2\bar{r}_1\omega. \tag{3.4}$$

Note that $z = (u, v)^T \in X$, there exist $\underline{\xi}$, $\underline{\eta}$, $\overline{\xi}$, $\overline{\eta}$, such that

$$\begin{aligned}u(\underline{\xi}) &= \min_{t\in[0,\omega]} u(t), \quad u(\overline{\xi}) = \max_{t\in[0,\omega]} u(t),\\ v(\underline{\eta}) &= \min_{t\in[0,\omega]} v(t), \quad v(\overline{\eta}) = \max_{t\in[0,\omega]} v(t).\end{aligned} \tag{3.5}$$

So $\dot{u}(\underline{\xi}) = \dot{u}(\overline{\xi}) = \dot{v}(\underline{\eta}) = \dot{v}(\overline{\eta}) = 0$. From (3.3) and (3.5), we obtain

$$\begin{aligned}\int_0^\omega r_1(t)dt &\ge \int_0^\omega (a_1(t)e^{u(t)} + a_2(t)e^{u(t-\tau)})dt\\ &\ge \int_0^\omega (a_1(t) + a_2(t))e^{u(\underline{\xi})}dt = \omega(\bar{a}_1 + \bar{a}_2)e^{u(\underline{\xi})}.\end{aligned}$$

Hence we have

$$e^{u(\underline{\xi})} \le \frac{1}{\omega(\bar{a}_1 + \bar{a}_2)} \int_0^\omega r_1(t)dt = \frac{\bar{r}_1}{\bar{a}_1 + \bar{a}_2}, \tag{3.6}$$

i.e.,

$$u(\underline{\xi}) \le \ln \frac{\bar{r}_1}{\bar{a}_1 + \bar{a}_2}. \tag{3.7}$$

From (3.4) and (3.7), we have

$$u(t) \le u(\underline{\xi}) + \int_0^\omega |\dot{u}(t)|dt \le \ln \frac{\bar{r}_1}{\bar{a}_1 + \bar{a}_2} + 2\bar{r}_1\omega := U_1. \tag{3.8}$$

Letting $t = \overline{\eta}$ in the second equation of (3.2), one has

$$\begin{aligned}r_2(\overline{\eta}) &\le \frac{D(\overline{\eta})e^{u(\overline{\eta})}e^{(m-1)v(\overline{\eta})}}{A_1(\overline{\eta}) + B_1(\overline{\eta})e^{u(\overline{\eta})} + C_1(\overline{\eta})e^{v(\overline{\eta})} + B_1(\overline{\eta})e^{u(\overline{\eta})}C_1(\overline{\eta})e^{v(\overline{\eta})}}\\ &\le \frac{D(\overline{\eta})e^{(m-1)v(\overline{\eta})}}{B_1(\overline{\eta})}.\end{aligned}$$

So, we obtain

$$v(\overline{\eta}) \leq \frac{1}{m-1} \ln[\frac{r_2 B_1}{D}]^L := H_1. \tag{3.9}$$

Meanwhile, we know that

$$\begin{aligned} b_1(\overline{\eta})e^{v(\overline{\eta})} &\leq \frac{D(\overline{\eta})e^{u(\overline{\eta})}e^{(m-1)v(\overline{\eta})}}{A_1(\overline{\eta}) + B_1(\overline{\eta})e^{u(\overline{\eta})} + C_1(\overline{\eta})e^{v(\overline{\eta})} + B_1(\overline{\eta})e^{u(\overline{\eta})}C_1(\overline{\eta})e^{v(\overline{\eta})}} \\ &\leq \frac{D(\overline{\eta})e^{(m-1)v(\overline{\eta})}}{B_1(\overline{\eta})}, \end{aligned}$$

and

$$v(\overline{\eta}) \leq \frac{1}{m-2} \ln\big[\frac{b_1 B_1}{D}\big]^L := H_2. \tag{3.10}$$

Combining inequalities (3.9) with (3.10), we obtain

$$v(t) \leq \max\{H_1, H_2\} := H_3. \tag{3.11}$$

On the other hand, in view of (3.2), one has

$$\begin{aligned} r_1(\underline{\xi}) &= a_1(\underline{\xi})e^{u(\underline{\xi})} + a_2(\underline{\xi})e^{u(\underline{\xi}-\tau)} \\ &\quad + \frac{C(\underline{\xi})e^{mv(\underline{\xi})}}{A_1(\underline{\xi}) + B_1(\underline{\xi})e^{u(\underline{\xi})} + C_1(\underline{\xi})e^{v(\underline{\xi})} + B_1(\underline{\xi})e^{u(\underline{\xi})}C_1(\underline{\xi})e^{v(\underline{\xi})}}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} r_2(\underline{\eta}) &= -b_1(\underline{\eta})e^{v(\underline{\eta})} - b_2(\underline{\eta})e^{v(\underline{\eta}-\tau)} \\ &\quad + \frac{D(\underline{\eta})e^{u(\underline{\eta})}e^{(m-1)v(\underline{\eta})}}{A_1(\underline{\eta}) + B_1(\underline{\eta})e^{u(\underline{\eta})} + C_1(\underline{\eta})e^{v(\underline{\eta})} + B_1(\underline{\eta})e^{u((\underline{\eta})}C_1(\underline{\eta})e^{v(\underline{\eta})}}. \end{aligned} \tag{3.13}$$

By (3.12), we have

$$r_1(\underline{\xi}) \leq (a_1(\underline{\xi}) + a_2(\underline{\xi}))e^{u(\overline{\xi})} + \frac{C(\underline{\xi})e^{mv(\underline{\xi})}}{A_1(\underline{\xi})}.$$

So

$$e^{u(\overline{\xi})} \geq [\frac{A_1(\underline{\xi})r_1(\underline{\xi}) - C(\underline{\xi})e^{mH_3}}{(a_1(\underline{\xi}) + a_2(\underline{\xi}))A_1(\underline{\xi})}] := S_0,$$

i.e. $u(\overline{\xi}) \geq \ln S_0 := S_1$. Then

$$u(t) \geq u(\overline{\xi}) - \int_0^\omega |\dot{u}(t)|dt \geq S_1 - 2\bar{r}_1\omega := U_2. \tag{3.14}$$

$\square$

Next we estimate the lower bound of $v(t)$. If $v(\underline{\eta}) \geq 0$, then the lower bound of $v(t)$ is 0.

If $v(\underline{\eta}) < 0$, then $e^{(1-m)v(\underline{\eta})} \geq e^{(2-m)v(\underline{\eta})}$ and $e^{nv(\underline{\eta})} < 1$. Combined with (3.13), we obtain

$$\begin{aligned} &(r_2(\underline{\eta}) + b_1(\underline{\eta}) + b_2(\underline{\eta}))e^{(1-m)v(\underline{\eta})} \\ &\geq \frac{D(\underline{\eta})e^{u(\underline{\eta})}}{A_1(\underline{\eta}) + B_1(\underline{\eta})e^{u(\underline{\eta})} + C_1(\underline{\eta})e^{v(\underline{\eta})} + B_1(\underline{\eta})e^{u((\underline{\eta})}C_1(\underline{\eta})e^{v(\underline{\eta})}} \\ &\geq \frac{D(\underline{\eta})e^{U_2}}{A_1(\underline{\eta}) + B_1(\underline{\eta})e^{U_1} + C_1(\underline{\eta})e^{H_3} + B_1(\underline{\eta})e^{U_1}C_1(\underline{\eta})e^{H_3}}. \end{aligned}$$

So

$$\begin{aligned}v(\underline{\eta}) \geq \frac{1}{1-m} \ln \Big(D(\underline{\eta}) e^{U_2} / \Big((r_2(\underline{\eta}) + b_1(\underline{\eta}) + b_2(\underline{\eta})) A_1(\underline{\eta}) \\ + B_1(\underline{\eta}) e^{U_1} + C_1(\underline{\eta}) e^{H_3} + B_1(\underline{\eta}) e^{U_1} C_1(\underline{\eta}) e^{H_3} \Big) \Big) := S_2.\end{aligned} \tag{3.15}$$

Donating $S_3 = \min\{0, S_2\}$, we obtain $U_2 \leq u(t) \leq U_1$ and $S_3 \leq v(t) \leq H_3$. Thus

$$|u|_0 = \max\{|U_1|, |U_2|\} := U_*, \quad |v|_0 = \max\{|S_3|, |H_3|\} := H_*.$$

So $|u(t)| + |v(t)| \leq U_* + H_* := R_1$.

Suppose $(u, v)^T$ is a constant solution of system (3.1), then

$$\begin{aligned}&r_1(t) - a_1(t)e^u - a_2(t)e^u \\ &- \frac{C(t)e^{mv}}{A_1(t) + B_1(t)e^u + C_1(t)e^u + B_1(t)e^u C_1(t)e^v} = 0, \\ &- r_2(t) - b_1(t)e^v - b_2(t)e^v \\ &+ \frac{D(t)e^u e^{(m-1)v}}{A_1(t) + B_1(t)e^u + C_1(t)e^v + B_1(t)e^u C_1(t)e^v} = 0.\end{aligned} \tag{3.16}$$

Integrating two side of above equations on $[0, \omega]$ and applying integral mean theorem, we obtain

$$\begin{aligned}&\bar{r_1} - \bar{a}_1 e^u - \bar{a}_2 e^u \\ &- \frac{C(t_1)e^{mv}}{A_1(t_1) + B_1(t_1)e^u + C_1(t_1)e^u + B_1(t_1)e^u C_1(t_1)e^v} = 0, \\ &- \bar{r_2} - \bar{b}_1 e^v - \bar{b}_2 e^v \\ &+ \frac{D(t_2)e^u e^{(m-1)v}}{A_1(t_2) + B_1(t_2)e^u + C_1(t_2)e^v + B_1(t_2)e^u C_1(t_2)e^v} = 0,\end{aligned} \tag{3.17}$$

where $t_1, t_2 \in [0, \omega]$. Next we consider the equations

$$\begin{aligned}&\bar{r_1} - \bar{a}_1 e^u - \bar{a}_2 e^u \\ &- \mu \frac{C(t_1)e^{mv}}{A_1(t_1) + B_1(t_1)e^u + C_1(t_1)e^u + B_1(t_1)e^u C_1(t_1)e^v} = 0, \\ &- \mu\bar{r_2} - \bar{b}_1 e^v - \bar{b}_2 e^v \\ &+ \frac{D(t_2)e^u e^{(m-1)v}}{A_1(t_2) + B_1(t_2)e^u + C_1(t_2)e^v + B_1(t_2)e^u C_1(t_2)e^v} = 0,\end{aligned} \tag{3.18}$$

where μ is a parameter.

Lemma 3.5. *Suppose $(u, v)^T$ is a solution of (3.18), then there exists a positive number R_2, such that $|u| + |v| \leq R_2$, where R_2 will be calculated as in the following proof.*

Proof. From the first equation in (3.18) we obtain $\bar{r}_1 \geq (\bar{a}_1 + \bar{a}_2)e^u$. Then

$$u \leq \ln \frac{\bar{r_1}}{\bar{a}_1 + \bar{a}_2} := W_1. \tag{3.19}$$

From the second equation in (3.18), one has

$$(\bar{b}_1 + \bar{b}_2)e^v \leq \frac{D(t_2)e^u e^{(m-1)v}}{A_1(t_2) + B_1(t_2)e^u + C_1(t_2)e^v + B_1(t_2)e^u C_1(t_2)e^v}.$$

Then
$$v \le \frac{1}{2-m} \ln \frac{D(t_2)}{B_1(\bar{b}_1(t_2)+\bar{b}_2(t_2))} := V_1. \tag{3.20}$$
If $u \ge 0$, then 0 is the low bound of u. If $u < 0$, from the first equation of (3.18), we obtain
$$\bar{r_1} \le (\bar{a}_1+\bar{a}_2)e^u + \frac{C(t_1)e^{mv}}{A_1(t_1)}.$$
Thus
$$u \ge \ln\{(\bar{r_1} - \frac{C(t_1)e^{mV_1}}{A_1(t_1)})/(\bar{a}_1+\bar{a}_2)\} := W_2.$$
Letting $W_3 = \min\{0, W_2\}$, we obtain
$$u \ge W_3. \tag{3.21}$$
If $v \ge 0$, then 0 is a lower bound of v. If $v < 0$, from the second equation of (3.18), one has
$$\bar{r_2} + (\bar{b}_1+\bar{b}_2)e^v \ge \frac{D(t_2)e^u e^{(m-1)v}}{A_1(t_2)+B_1(t_2)e^u+C_1(t_2)e^v+B_1(t_2)e^uC_1(t_2)e^v}.$$
Then, in view of $(1-m)v \ge (2-m)v$, and $e^{mv} < 1$, we obtain
$$\begin{aligned}
&(\bar{r_2}+\bar{b}_1+\bar{b}_2)e^{(1-m)v} \\
&\ge \frac{D(t_2)e^u}{A_1(t_2)+B_1(t_2)e^u+C_1(t_2)e^v+B_1(t_2)e^uC_1(t_2)e^v} \\
&\ge \frac{D(t_2)e^{W_3}}{A_1(t_2)+B_1(t_2)e^{W_1}+C_1(t_2)e^{V_1}+B_1(t_2)e^{W_1}C_1(t_2)e^{V_1}},
\end{aligned}$$
and
$$\begin{aligned}
v \ge \frac{1}{1-m} \ln \Big(&D(t_2)e^{W_3}/\Big((A_1(t_2)+B_1(t_2)e^{W_1} \\
&+ C_1(t_2)e^{V_1}+B_1(t_2)e^{W_1}C_1(t_2)e^{V_1})(\bar{r_2}+\bar{b}_1+\bar{b}_2)\Big)\Big) := V_2.
\end{aligned}$$
Letting $V_3 = \min\{0, V_2\}$, we have
$$v \ge V_3. \tag{3.22}$$
From (3.19), (3.20), (3.21) and (3.22), we know that
$$W_3 \le u \le W_1, \quad V_3 \le v \le V_1. \tag{3.23}$$
Denoting
$$|u|_0 = \max\{|W_1|, |W_3|\} = O_1, \quad |v|_0 = \max\{|V_1|, |V_3|\} = O_2.$$
We have $|u| + |v| \le O_1 + O_2 := R_2$. □

Theorem 3.6. *System* (1.5) *has at least one positive ω-periodic solution.*

Proof. Suppose that $(x(t), y(t))^T$ is an arbitrary positive solution of (1.5) and let $u(t) = \ln x(t)$, $v(t) = \ln y(t)$, then (1.5) is changed into (3.1). Let
$$X = Y = \{z(t) | z(t) = (u(t), v(t))^T \in C(R, R^2) : z(t+\omega) = z(t)\},$$
be equipped with the norm
$$\|z\| = \max_{t\in[0,\omega]} \{|z|\}.$$

Then X and Y are both Banach spaces with the norm $\|\cdot\|$. Take $z \in X$ and define operators L, P and Q as follows

$$L : \operatorname{Dom} L \cap X \to Y, \quad Lz = \frac{dz}{dt}, \quad P(z) = \frac{1}{\omega}\int_0^\omega z(t)dt, \quad Q(z) = \frac{1}{\omega}\int_0^\omega z(t)dt,$$

where $\operatorname{Dom} L = \{z \in X : z(t) \in C^1(R, R^2)\}$.

Define $N{:}X \to Y$ by

$$N(z) = \begin{pmatrix} r_1(t) - a_1(t)e^{u(t)} - a_2(t)e^{u(t-\tau)} - \widetilde{C} \\ -r_2(t) - b_1(t)e^{v} - b_2(t)e^{v(t-\tau)} + \widetilde{D} \end{pmatrix}$$

where

$$\widetilde{C} = \frac{C(t)e^{mv}}{A_1(t) + B_1(t)e^u + C_1(t)e^u + B_1(t)e^uC_1(t)e^v}$$
$$\widetilde{D} = \frac{D(t)e^u e^{(m-1)v}}{A_1(t) + B_1(t)e^u + C_1(t)e^v + B_1(t)e^uC_1(t)e^v}$$

Then $\ker L = R^2$, $\dim\ker L = \operatorname{codim}\operatorname{Im} L = 2$, and

$$\operatorname{Im} L = \Big\{z \in Y : \int_0^\omega z(t)dt = 0\Big\},$$

is closed in Y, and P, Q are both continuous projections satisfying

$$\operatorname{Im} P = \ker L, \quad \operatorname{Im} L = \ker Q = \operatorname{Im}(I - Q).$$

So L is a Fredholm operation of index zero, which implies that L has a unique inverse. We denote by $K_p : \operatorname{Im} L \to \ker P \cap \operatorname{Dom} L$ the inverse of L. By a straightforward calculation, we obtain

$$K_p(z) = \int_0^t z(s)ds - \frac{1}{\omega}\int_0^\omega \int_0^t z(s)\,ds\,dt.$$

For any $z(t) \in X$, we obtain

$$\begin{aligned} QN(z) &= Q(F_1(t, u(t), v(t)),\ F_2(t, u(t), v(t)))^T \\ &= \Big(\frac{1}{\omega}\int_0^\omega F_1(t, u(t), v(t))dt,\ \frac{1}{\omega}\int_0^\omega F_2(t, u(t), v(t))dt\Big)^T \\ &= (\bar{F}_1, \bar{F}_2)^T, \end{aligned}$$

and

$$K_p(I - Q)Nz = (W_1, W_2)^T,$$

where

$$W_i(t) = \int_0^\omega F_i(s, u(s), v(s))ds - \frac{1}{\omega}\int_0^\omega \int_0^t F_i(s, u(s), v(s))dsdt - \bar{F}_i t + \frac{\omega}{2}\bar{F}.$$

Obviously, it is undemanding to check by the Lebesgue convergence theorem that both QN and $K_p(I - Q)N$ are continuous. Moreover, by using the Arzela-Ascoli Theorem, the operator $K_p(I - Q)N(\overline{\Omega})$ is compact and $QN(\overline{\Omega})$ is bounded for any open set $\Omega \subset X$. So N is L-compact on $\overline{\Omega}$ respect to any bounded open set $\Omega \subset X$.

Particularly, we take $\Omega = \{z(t) | z(t) = (u(t), v(t))^T \in X, \|z\| \leq R\}$, where $R = R_1 + R_2 + \varepsilon$ ($\varepsilon > 0$), R_1 and R_2 are defined in Lemma 3.4 and Lemma 3.5. Now, we check the three conditions in Lemma 3.3.

(i) For each $\lambda \in (0,1)$, $z(t) \in \partial\Omega \cap \operatorname{Dom} L$, we have $Lx \neq \lambda N(x)$. Otherwise, $z(t)$ is a ω-periodic solution of (3.2) and then $\|z\| \le R_1$, will be derived by Lemma 3.4. It is impossible because $\|z\| = R > R_1$ for $z(t) \in \partial\Omega \cap \operatorname{Dom} L$.

(ii) When $z(t) \in \partial\Omega \cap \ker L$, $\frac{dz(t)}{dt} = 0$, i.e., $z(t)$ is a constant vector $(u,v)^T$ with $\|(u,v)^T\| = R_1 + R_2 + \varepsilon$. If $QN(u,v)^T = 0$, then $(u,v)^T$ is a solution of (3.18) for $\mu = 1$. By Lemma 3.5, we have $\|(u,v)^T\| \le R_2$ which contradicts to $\|(u,v)^T\| = R_1 + R_2 + \varepsilon$. Thus, for each $z \in \operatorname{Im} Q$. When $z \in \partial\Omega \cap \ker L$, $QNz \neq 0$.

(iii) Choose $J : \operatorname{Im} Q \to \ker L$ such that $J(z) = z$ for each $z \in \operatorname{Im} Q$. When $z \in \Omega \cap \ker L$, $z(t) = (u,v)^T$ is a constant vector and satisfies

$$
\begin{aligned}
&JQN(u,v)^T \\
&= JQ(F_1(t,u(t),v(t)), F_2(t,u(t),v(t)))^T \\
&= \Big(\frac{1}{\omega}\int_0^\omega F_1(t,u(t),v(t))dt, \frac{1}{\omega}\int_0^\omega F_2(t,u(t),v(t))dt\Big)^T \\
&= \begin{pmatrix} \bar{r_1} - (\bar{a}_1+\bar{a}_2)e^u - \frac{C(t_1)e^{mv}}{A_1(t_1)+B_1(t_1)e^u+C_1(t_1)e^u+B_1(t_1)e^uC_1(t_1)e^v} \\ -\bar{r_2} - (\bar{b}_1+\bar{b}_2)e^v + \frac{D(t_2)e^ue^{(m-1)v}}{A_1(t_2)+B_1(t_2)e^u+C_1(t_2)e^v+B_1(t_2)e^uC_1(t_2)e^v} \end{pmatrix},
\end{aligned}
$$

where t_1, t_2 were defined as in (3.17). We define $\varphi : z \in \Omega \cap \ker L \times [0,1] \to X$ as follows

$$
\begin{aligned}
\varphi(u,v,\mu) = &\begin{pmatrix} \bar{r_1} - (\bar{a}_1+\bar{a}_2)e^u \\ -(\bar{b}_1+\bar{b}_2)e^v + \frac{D(t_2)e^ue^{(m-1)v}}{A_1(t_2)+B_1(t_2)e^u+C_1(t_2)e^v+B_1(t_2)e^uC_1(t_2)e^v} \end{pmatrix} \\
&+ \mu \begin{pmatrix} -\frac{C(t_1)e^{mv}}{A_1(t_1)+B_1(t_1)e^u+C_1(t_1)e^u+B_1(t_1)e^uC_1(t_1)e^v} \\ -\bar{r}_2 \end{pmatrix}.
\end{aligned}
$$

Then $JQN(u,v)^T = \varphi(u,v,1)$. By Lemma 3.4, we see $\varphi(u,v,1) \neq (0,0)^T$. Hence, using the homotopy invariance theorem of topological degree, we obtain

$$
\begin{aligned}
&\deg\{JQN(u,v)^T, \Omega \cap \ker L, (0,0)^T\} \\
&= \deg\{\varphi(u,v,1), \Omega \cap \ker L, (0,0)^T\} \\
&= \deg\{\varphi(u,v,0), \Omega \cap \ker L, (0,0)^T\} \\
&= \deg\Big\{(\bar{r_1} - (\bar{a}_1+\bar{a}_2)e^u, -(\bar{b}_1+\bar{b}_2)e^v \\
&\quad + \frac{D(t_2)e^ue^{(m-1)v}}{A_1(t_2)+B_1(t_2)e^u+C_1(t_2)e^v+B_1(t_2)e^uC_1(t_2)e^v})^T, \Omega \cap \ker L, (0,0)^T\Big\}.
\end{aligned}
$$

Denote

$$
\begin{gathered}
\psi_1(u,v) := \bar{r_1} - (\bar{a}_1+\bar{a}_2)e^u, \\
\psi_2(u,v) := -(\bar{b}_1+\bar{b}_2)e^v + \frac{D(t_2)e^ue^{(m-1)v}}{A_1(t_2)+B_1(t_2)e^u+C_1(t_2)e^v+B_1(t_2)e^uC_1(t_2)e^v},
\end{gathered}
$$

and consider the algebraic equations

$$
\begin{gathered}
\psi_1(u,v) = 0, \\
\psi_2(u,v) = 0.
\end{gathered} \tag{3.24}
$$

From the first equation of (3.24) we obtain its unique $u^* = \ln \frac{\bar{r}_1}{\bar{a}_1+\bar{a}_2}$. Substituting it into second equation of (3.24), we obtain

$$-(\bar{b}_1+\bar{b}_2)e^v + \frac{D(t_2)e^{u^*}e^{(m-1)v}}{A_1(t_2)+B_1(t_2)e^{u^*}+C_1(t_2)e^v+B_1(t_2)e^{u^*}C_1(t_2)e^v} = 0,$$

which is easy checked to have a unique solution v^* on R. So equations (3.24) has unique solution $(u^*, v^*)^T$ on $\Omega \cap \ker L$. For convenience, we denote

$$p(u,v) = A_1(t_2)+B_1(t_2)e^u+C_1(t_2)e^v+B_1(t_2)e^uC_1(t_2)e^v.$$

Then

$$\frac{\partial \psi_1}{\partial u} = -(\bar{a}_1+\bar{a}_2)e^u,$$
$$\frac{\partial \psi_1}{\partial v} = 0,$$
$$\frac{\partial \psi_2}{\partial u} = \frac{D(t_2)e^ue^{(m-1)v}p(u,v) - D(t_2)e^ue^{(m-1)v}[B_1(t_2)e^u+B_1(t_2)e^uC_1(t_2)e^v]}{p^2(u,v)},$$
$$\frac{\partial \psi_2}{\partial v} = M(u,v) - (\bar{b}_1+\bar{b}_2)e^v,$$

where

$$\begin{aligned}M(u,v) = \Big(&(m-1)D(t_2)e^ue^{(m-1)v}p(u,v) - D(t_2)e^ue^{(m-1)v}[C_1(t_2)e^v \\ &+ B_1(t_2)e^uC_1(t_2)e^v]\Big)/p^2(u,v).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\deg\{JQN(u,v)^T, \Omega\cap\ker L, (0,0)^T\} \\ &= \operatorname{sgn}\begin{vmatrix} \frac{\partial\psi_1}{\partial u} & \frac{\partial\psi_1}{\partial v} \\ \frac{\partial\psi_2}{\partial u} & \frac{\partial\psi_2}{\partial v}\end{vmatrix}_{(u^*,v^*)} \\ &= \operatorname{sgn}[-(\bar{a}_1+\bar{a}_2)e^u(M(u,v)-(\bar{b}_1+\bar{b}_2)e^v)] = 1 \neq 0.\end{aligned}$$

All the conditions in Lemma 3.3 have been checked. This implies that (3.1) has at least one ω-periodic solution. Further system (1.5) has at least one ω-periodic solution. □

4. Global attractivity

Definition 4.1. Suppose $(\tilde{x}(t), \tilde{y}(t))^T$ is a positive ω-periodic solution of (1.5), $(x(t), y(t))^T$ is arbitrary positive solution of (1.5), with

$$\lim_{t\to\infty} |x(t)-\tilde{x}(t)| = 0 \quad \text{and} \quad \lim_{t\to\infty} |y(t)-\tilde{y}(t)| = 0,$$

then $(\tilde{x}(t), \tilde{y}(t))^T$ is called a global attractor.

Lemma 4.2 ([28])**.** *If function f is nonnegative, integrable and uniformly continuous on $[0,\infty)$, then $\lim_{t\to\infty} f(t) = 0$.*

From Theorem 2.3, we know that for any positive ε enough small, there exists T, such that, when $t \geq T$, an arbitrary positive solution $(x(t), y(t))^T$ of system (1.5) satisfies

$$K_3 - \varepsilon \leq x(t) \leq K_1 + \varepsilon, \quad K_4 - \varepsilon \leq y(t) \leq K_2 + \varepsilon. \tag{4.1}$$

For convenience, we denote

$$\gamma = \min\{K_3, K_4, e^{U_2}, e^{S_3}\}, \quad \Gamma = \max\{K_1, K_2, e^{U_1}, e^{H_3}\}, \tag{4.2}$$

$$g(t) = g(t, \gamma, \gamma), \quad G(t) = g(t, \Gamma, \Gamma), \tag{4.3}$$

where

$$g(t, x, y) = A_1(t) + B_1(t)x(t) + C_1(t)y(t) + B_1(t)C_1(t)x(t)y(t).$$

Theorem 4.3. *Suppose system* (1.5) *satisfies*

$$\begin{aligned}\sigma_1 = \min_{t\in[0,\omega]} \{&-a_1(t) - \frac{C(t)B_1(t)}{g^2(t)}\gamma^m - \frac{C(t)B_1(t)C_1(t)}{g^2(t)}\gamma^{m+1} + A_2 \\ &+ \frac{(m-2)D(t)C_1(t)}{G^2(t)}\Gamma^m\} > 0,\end{aligned} \tag{4.4}$$

$$\begin{aligned}&\sigma_2 \\ &= \min_{t\in[0,\omega]} \Big\{ - (\frac{mC(t)A_1(t)}{g^2(t)}\gamma^{m-1} + \frac{mC(t)B_1(t)}{g^2(t)}\gamma^m + \frac{(m-1)C(t)C_1(t)}{g^2(t)}\gamma^m \\ &\quad + \frac{C(t)B_1(t)C_1(t)}{g^2(t)}\gamma^{m+1} + \frac{mC(t)B_1(t)C_1(t)}{g^2(t)}\gamma^{m+1}) \\ &\quad - b(t) + \frac{mD(t)A_1(t)}{G^2(t)}\Gamma^{m-1} + \frac{(m-1)D(t)B_1(t)}{G^2(t)}\Gamma^m \\ &\quad + \frac{(m-2)D(t)C_1(t)}{G^2(t)}\Gamma^m + \frac{(m-2)D(t)C_1(t)B_1(t)}{G^2(t)}\Gamma^{m+1} + B_2\Big\} > 0,\end{aligned} \tag{4.5}$$

then system (1.5) *has only one positive ω-periodic solution which is a global attractor.*

Proof. Suppose that $(x(t), y(t))^T$ is an arbitrary periodic solution of (1.5). Theorem 3.6 indicates that (1.5) has at least one positive ω-periodic solution $(\tilde{x}(t), \tilde{y}(t))^T$ satisfying

$$e^{U_2} \le \tilde{x}(t) \le e^{U_1}, \quad e^{S_3} \le \tilde{y}(t) \le e^{H_3}.$$

We choose the Lyapunov function $V(t) = V_1(t) + V_2(t)$, where

$$\begin{aligned}V_1(t) &= |\ln x(t) - \ln \tilde{x}(t)| + A_2 \int_{t-\tau}^t |x(s) - \tilde{x}(s)| ds, \\ V_2(t) &= |\ln y(t) - \ln \tilde{y}(t)| + B_2 \int_{t-\tau}^t |y(s) - \tilde{y}(s)| ds.\end{aligned}$$

Then

$$\begin{aligned}&D^+V_1(t)|_{(1.5)} \\ &= \operatorname{sgn}(x(t) - \tilde{x}(t))(\frac{\dot{x}(t)}{x(t)} - \frac{\dot{\tilde{x}}(t)}{\tilde{x}(t)}) + A_2|x(t) - \tilde{x}(t)| - A_2|x(t-\tau) - \tilde{x}(t)| \\ &= \operatorname{sgn}(x(t) - \tilde{x}(t))[-a_1(t)(x(t) - \tilde{x}(t)) - a_2(t)(x(t-\tau) - \tilde{x}(t-\tau)) \\ &\quad - (\frac{C(t)y^m(t)}{g(t,x,y)} - \frac{C(t)\tilde{y}^m(t)}{g(t,\tilde{x},\tilde{y})})] + A_2|x(t) - \tilde{x}(t)| - A_2|x(t-\tau) - \tilde{x}(t-\tau)|.\end{aligned} \tag{4.6}$$

Since

$$\frac{C(t)y^m(t)}{g(t,x,y)} - \frac{C(t)\tilde{y}^m(t)}{g(t,\tilde{x},\tilde{y})}$$

$$
\begin{aligned}
&= \frac{C(t)}{g(t,x,y)g(t,\tilde{x},\tilde{y})}\Big\{A_1(t)[y^m(t)-\tilde{y}^m(t)]+B_1(t)[\tilde{x}(t)y^m(t)\\
&\quad - x(t)y^m(t)+x(t)y^m(t)-x(t)\tilde{y}^m(t)]+C_1(t)y(t)\tilde{y}(t)[y^{m-1}(t)-\tilde{y}^{m-1}(t)]\\
&\quad + B_1(t)C_1(t)[\tilde{x}(t)\tilde{y}(t)y^m(t)-\tilde{x}(t)y(t)y^m(t)+\tilde{x}(t)y(t)y^m(t)\\
&\quad - \tilde{x}(t)y(t)\tilde{y}^m(t)+\tilde{x}(t)y(t)\tilde{y}^m(t)-x(t)y(t)\tilde{y}^m(t)]\Big\}\\
&= \frac{C(t)}{g(t,x,y)g(t,\tilde{x},\tilde{y})}\Big\{A_1(t)(y^m(t)-\tilde{y}^m(t))+B_1(t)(\tilde{x}(t)-x(t))y^m(t)\\
&\quad + B_1(t)x(t)(y^m(t)-\tilde{y}^m(t))+C_1(t)y(t)\tilde{y}(t)(y^{m-1}(t)-\tilde{y}^{m-1}(t))\\
&\quad + B_1(t)C_1(t)[\tilde{x}(t)y^m(t)(\tilde{y}(t)-y(t))+\tilde{x}(t)y(t)(y^m(t)-\tilde{y}^m(t))\\
&\quad + y(t)\tilde{y}^m(t)(\tilde{x}(t)-x(t))]\Big\}.
\end{aligned}
$$

Substituting this inequality in (4.6), we obtain

$$
\begin{aligned}
&D^+V_1(t)|_{(1.5)}\\
&\le -a_1(t)|x(t)-\tilde{x}(t)|+\frac{C(t)}{g(t,x,y)g(t,\tilde{x},\tilde{y})}\Big\{-A_1(t)|y^m(t)-\tilde{y}^m(t)|\\
&\quad - B_1(t)[y^m(t)|\tilde{x}(t)-x(t)|+x(t)|y^m(t)-\tilde{y}^m(t)|]\\
&\quad - C_1(t)y(t)\tilde{y}(t)|y^{m-1}(t)\ -\tilde{y}^{m-1}(t)|-B_1(t)C_1(t)[\tilde{x}(t)y^m(t)|\tilde{y}(t)-y(t)|\\
&\quad + \tilde{x}(t)y(t)|y^m(t)-\tilde{y}^m(t)|+y(t)\tilde{y}^m(t)|\tilde{x}(t)-x(t)|]\Big\}+A_2|x(t)-\tilde{x}(t)|.
\end{aligned}
\tag{4.7}
$$

Meanwhile,

$$
\begin{aligned}
&D^+V_2(t)|_{(1.5)}\\
&= \operatorname{sgn}(y(t)-\tilde{y}(t))(\frac{\dot{y}(t)}{y(t)}-\frac{\dot{\tilde{y}}(t)}{\tilde{y}(t)})+B_2|y(t)-\tilde{y}(t)|-B_2|y(t-\tau)-\tilde{y}(t-\tau)|\\
&= \operatorname{sgn}(y(t)-\tilde{y}(t))[-b_1(t)(y(t)-\tilde{y}(t))-b_2(t)(y(t-\tau)-\tilde{y}(t-\tau))\\
&\quad + \frac{D(t)x(t)y^{m-1}(t)}{g(t,x,y)}-\frac{D(t)\tilde{x}(t)\tilde{y}^{m-1}(t)}{g(t,\tilde{x},\tilde{y})}]+B_2|y(t)-\tilde{y}(t)|\\
&\quad - B_2|y(t-\tau)-\tilde{y}(t-\tau)|.
\end{aligned}
\tag{4.8}
$$

Since

$$
\begin{aligned}
&\frac{D(t)x(t)y^{m-1}(t)}{g(t,x,y)}-\frac{D(t)\tilde{x}(t)\tilde{y}^{m-1}(t)}{g(t,\tilde{x},\tilde{y})}\\
&= \frac{D(t)}{g(t,x,y)g(t,\tilde{x},\tilde{y})}\Big\{A_1(t)[x(t)(y^m(t)-\tilde{y}^m(t))+\tilde{y}^{m-1}(t)(x(t)-\tilde{x}(t))]\\
&\quad + B_1(t)\tilde{x}(t)x(t)y^{m-1}(t)-\tilde{y}^{m-1}(t)+C_1(t)y(t)\tilde{y}(t)[y^{m-2}(t)(x(t)-\tilde{x}(t))\\
&\quad + \tilde{x}(y^{m-2}(t)-\tilde{y}^{m-2}(t))]\Big\}\\
&\quad + B_1(t)C_1(t)x(t)\tilde{x}(t)y(t)\tilde{y}(t)(y^{m-2}(t)-\tilde{y}^{m-2}(t)).
\end{aligned}
\tag{4.9}
$$

Substituting the above equality in (4.8), we obtain

$$
\begin{aligned}
&D^+V_2(t)|_{(1.5)}\\
&\le -b_1(t)|y(t)-\tilde{y}(t)| + \frac{D(t)}{g(t,x,y)g(t,\tilde{x},\tilde{y})}(A_1(t)[x(t)|y^m(t)-\tilde{y}^m(t)|\\
&\quad + \tilde{y}^{m-1}(t)|x(t)-\tilde{x}(t)|] + B_1(t)\tilde{x}(t)x(t)|y^{m-1}(t)-\tilde{y}^{m-1}(t)|\\
&\quad + C_1(t)y(t)\tilde{y}(t)[y^{m-2}(t)|x(t)-\tilde{x}(t)| + \tilde{x}|y^{m-2}(t)-\tilde{y}^{m-2}(t)|]\\
&\quad + B_1(t)C_1(t)x(t)\tilde{x}(t)y(t)\tilde{y}(t)|y^{m-2}(t)-\tilde{y}^{m-2}(t)| + B_2|y(t)-\tilde{y}(t)|.
\end{aligned} \tag{4.10}
$$

For any $x_1, x_2 \in [a,b] \subset (0,+\infty)$, we have

$$
\begin{gathered}
|\alpha|b^{\alpha-1}|x_1-x_2| \le |x_1^\alpha - x_2^\alpha| \le |\alpha|a^{\alpha-1}|x_1-x_2|, \quad \text{for } \alpha < 1,\\
\alpha a^{\alpha-1}|x_1-x_2| \le |x_1^\alpha - x_2^\alpha| \le |\alpha|b^{\alpha-1}|x_1-x_2|, \quad \text{for } \alpha > 1.
\end{gathered}
$$

Then, from (4.7) and (4.10), in view of (4.1), (4.2) and (4.3), letting $\varepsilon \to 0$, we obtain

$$
\begin{aligned}
&D^+V_1(t)|_{(1.5)}\\
&\le [-a_1(t) - \frac{C(t)B_1(t)}{g^2(t)}\gamma^m - \frac{C(t)B_1(t)C_1(t)}{g^2(t)}\gamma^{m+1} + A_2]|x(t)-\tilde{x}(t)|\\
&\quad - [\frac{mC(t)A_1(t)}{g^2(t)}\gamma^{m-1} + \frac{mC(t)B_1(t)}{g^2(t)}\gamma^m + \frac{(m-1)C(t)C_1(t)}{g^2(t)}\gamma^m\\
&+ \frac{C(t)B_1(t)C_1(t)}{g^2(t)}\gamma^{m+1} + \frac{mC(t)B_1(t)C_1(t)}{g^2(t)}\gamma^{m+1}]|y(t)-\tilde{y}(t)|,
\end{aligned} \tag{4.11}
$$

and

$$
\begin{aligned}
D^+V_2(t)|_{(1.5)} &\le [-b_1(t) + \frac{mD(t)A_1(t)}{G^2(t)}\Gamma^m + \frac{(m-1)D(t)B_1(t)}{G^2(t)}\Gamma^m\\
&\quad + \frac{(m-2)D(t)C_1(t)}{G^2(t)}\Gamma^m + \frac{(m-2)D(t)C_1(t)B_1(t)}{G^2(t)}\Gamma^{m+1}\\
&\quad + B_2]|y(t)-\tilde{y}(t)| + [\frac{(m-2)D(t)C_1(t)}{G^2(t)}\Gamma^m\\
&\quad + A_1(t)\Gamma^{m-1}]|x(t)-\tilde{x}(t)|.
\end{aligned} \tag{4.12}
$$

Summing (4.11), (4.12), (4.4) and (4.5), we obtain, for $t > T > 0$, that

$$
D^+V(t)|_{(1.5)} = D^+V_1(t)|_{(1.5)} + D^+V_2(t)|_{(1.5)} \le -\sigma_1|x(t)-\tilde{x}(t)| - \sigma_2|y(t)-\tilde{y}(t)|.
$$

Integrating the two sides of above inequality, we have

$$
V(t) + \sigma_1\int_t^T |x(s)-\tilde{x}(s)|ds + \sigma_2\int_t^T |y(s)-\tilde{y}(s)|ds \le V(T) < \infty.
$$

From Lemma 4.2, we obtain

$$
\lim_{t\to\infty}|x(t)-\tilde{x}(t)| = 0, \quad \lim_{t\to\infty}|y(t)-\tilde{y}(t)| = 0.
$$

This proves that any positive ω-periodic solution of (1.5) is a global attractor. Next we prove the uniqueness of the positive ω-periodic solution $(\tilde{x}(s), \tilde{y}(t))$. Suppose there is another positive ω-periodic solution $(\tilde{x}^*(s), \tilde{y}^*(t))$. Otherwise, there exists a $\xi \in [0,\omega]$ such that $\tilde{x}(\xi) \ne \tilde{x}^*(\xi)$ or $\tilde{y}(\xi) \ne \tilde{y}^*(\xi)$, then $\varepsilon_0 > 0$. However

$$
\varepsilon_0 = \lim_{t\to\infty}|\tilde{x}(\xi+n\omega) - \tilde{x}^*(\xi+n\omega)| = \lim_{t\to\infty}|\tilde{x}(t)-\tilde{x}^*(t)| = 0.
$$

This is a contradiction. Hence, the positive ω-periodic solution $(\tilde{x}(s), \tilde{y}(t))$ is unique. □

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