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EXISTENCE OF SOLUTIONS TO ASYMPTOTICALLY PERIODIC SCHRÖDINGER EQUATIONS

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Abstract. We show the existence of a nonzero solution for the semilinear Schrödinger equation $-\Delta u + V(x)u = f(x,u)$. The potential V is periodic and 0 belongs to a gap of $\sigma(-\Delta + V)$. The function f is superlinear and asymptotically periodic with respect to x variable. In the proof we apply a new critical point theorem for strongly indefinite functionals proved in [3].

1. Introduction

We consider the existence of nonzero solutions for the semilinear Schrödinger equation

$$-\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $V \in C(\mathbb{R}^N, \mathbb{R})$ the nonlinearity $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfy the following assumptions:

(A1) $V(x) = V(x_1, \dots, x_N)$ is 1-periodic in $x_1, \dots, x_N$;

(A2) if $\sigma(-\Delta + V)$ denotes the spectrum of the operator $-\Delta + V$, then $0 \notin \sigma(-\Delta + V)$ and $\sigma(-\Delta + V) \cap (-\infty, 0) \neq \emptyset$,

(A3) there exist $c_1, c_2 > 0$ and $p \in (2, 2^*)$ such that

$$|f(x,t)| \le c_1|t| + c_2|t|^{p-1}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R};$$

(A4) $f(x,t)t \ge 0$, for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$;

(A5) $f(x,t) = o(|t|)$, as $t \to 0$, uniformly in $x \in \mathbb{R}^N$;

(A6) it holds

$$\lim_{|t|\to\infty} \frac{F(x,t)}{t^2} = \infty, \quad \text{uniformly in } x \in \mathbb{R}^N,$$

where $F(x,t) := \int_0^t f(x,\tau)d\tau$.

We denote by $\mathfrak{F}$ the class of all functions $h \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N, \mathbb{R})$ such that, for every $\varepsilon > 0$, the set $\{x \in \mathbb{R}^N : |h(x)| \ge \varepsilon\}$ has finite Lebesgue measure, and we assume that

(A7) there exist $p_\infty \in (2, 2^*)$, $\varphi \in \mathfrak{F}$ and $f_\infty \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, 1-periodic in $x_1, \dots, x_N$, such that, for all for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$,

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(i) $f_\infty(x,t)t \ge 0$ and $f_\infty(x,t)/|t|$ is not decreasing in $\mathbb{R}\setminus\{0\}$;
(ii) $F(x,t) \ge F_\infty(x,t) := \int_0^t f_\infty(x,\tau)d\tau$;
(iii) $|f(x,t) - f_\infty(x,t)| \le \varphi(x)|t|^{p_\infty - 1}$.

(A8) there exists $\theta_0 \in (0,1)$ such that

$$\frac{1-\theta^2}{2} t f(x,t) \ge F(x,t) - F(x,\theta t), \quad \forall \theta \in [0,\theta_0],\ (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Our first result can be stated as follows.

Theorem 1.1. *Suppose that* (A1)–(A8) *are satisfied. Then problem* (1.1) *has a nonzero solution.*

In the proof we apply a version of the Linking Theorem due to Li and Szulkin [4] to obtain a Cerami sequence for the associated functional. Thanks to (A8), the same argument employed by Tang in [8] provides the boundedness of this sequence. If f is periodic is sufficient to guarantee that, up to translations, the weak limit of the sequence is a nonzero solution. In our case we do not have periodicity and therefore the strategy of [8] fails. To overcome this difficult we use a a local version of the Linking Theorem proved in [3].

The same idea can be used to replace condition (A8) by another one introduced by Ding and Lee in [2] (see also [10] for a weaker condition). More specifically, we assume that

(A4') $F(x,t) \ge 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$;

(A8') there exist $\tau > \max\{1, N/2\}$ and positive constants r, a_1, R_1 such that

$$q(r) := \inf\{\widehat{F}(x,t) : x \in \mathbb{R}^N \text{ and } |t| \ge r\} > 0,$$
$$|f(x,t)|^\tau \le a_1 |t|^\tau \widehat{F}(x,t), \quad \text{for all } x \in \mathbb{R}^N,\ |t| \ge R_1,$$

where $\widehat{F}(x,t) := \frac{1}{2} f(x,t)t - F(x,t)$.

Theorem 1.2. *Suppose that* (A1), (A2), (A4'), (A5)–(A7), (A8') *are satified. Then problem* (1.1) *has a nonzero solution.*

In this article we denote $B_R(y) := \{x \in \mathbb{R}^N : |x-y| < R\}$ and $|A|$ for the Lebesgue measure of a set $A \subset \mathbb{R}^N$. We write $\int_A u$ instead of $\int_A u(x)dx$. We also omit the set A whenever $A = \mathbb{R}^N$. Also we write $|\cdot|_p$ for the norm in $L^p(\mathbb{R}^N)$.

2. Variational setting

We denote by S the selfadjoint operator $-\Delta + V$ acting on $L^2(\mathbb{R}^N)$ with domain $\mathcal{D}(S) := H^2(\mathbb{R}^N)$. Under the conditions (A1) and (A2), we have the orthogonal decomposition $L^2(\mathbb{R}^N) = L^- \oplus L^+$, with the subspaces L^+ and L^- being such that S is negative in L^- and positive in L^+. If we consider the Hilbert space $H := \mathcal{D}(|S|^{1/2})$ with the inner product $(u,v) := (|S|^{1/2}u, |S|^{1/2}v)_{L^2}$, and the corresponding norm $\|u\| := ||S|^{1/2}u|_2$, it follows from (A1) and (A2) that $H = H^1(\mathbb{R}^N)$ and the above norm is equivalent to the usual norm of this space. Hence, we obtain the decomposition

$$H = H^+ \oplus H^-,\ H^\pm = H \cap L^\pm,$$

which is orthogonal with respect to $(\cdot,\cdot)_{L^2}$ and $(\cdot,\cdot)$.

Let $(e_k) \subset H$ be a total orthonormal sequence in H^-. We introduce a new topology on H by setting

$$\|u\|_\tau := \max\Big\{\|u^+\|, \sum_{k=1}^{\infty} \frac{1}{2^k}|\langle u^-, e_k\rangle|\Big\}. \tag{2.1}$$

The above norm induces a topology in H which we call τ-topology. Given a set $M \subset H$, an homotopy $h : [0,1] \times M \to H$ is said to be admissible if

- (i) h is τ-continuous, that is, if $t_n \to t$ and $u_n \xrightarrow{\tau} u$ then $h(t_n, u_n) \xrightarrow{\tau} h(t,u)$;
- (ii) for each $(t,u) \in [0,1] \times M$ there is a neighborhood U of (t,u) in the product topology of $[0,1]$ and (H,τ) such that the set $\{w - h(t,w) : (t,w) \in U \cap ([0,1] \times M)\}$ is contained in a finite dimensional subspace of H.

When $I \in C^1(E,\mathbb{R})$ the symbol Γ denotes the class of maps

$$\begin{aligned}\Gamma := \big\{h \in C([0,1] \times M, H) : h \text{ is admissible, } h(0,\cdot) = \mathrm{Id}_M,\\ I(h(t,u)) \le \max\{I(u), -1\} \text{ for all } (t,u) \in [0,1] \times M\big\}.\end{aligned}$$

The first part of the following abstract result can be found in [4, Theorem 2.1] while the last one was proved in [3, Theorem 2.3].

Theorem 2.1. *Suppose that $I \in C^1(H,\mathbb{R})$ satisfies*

(A9) *The functional I can be written as*

$$I(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - J(u),$$

with $J \in C^1(H,\mathbb{R})$ bounded from below, weakly sequentially lower semicontinuous and J' is weakly sequentially continuous;

(A10) *there exist $u_0 \in H^+\backslash\{0\}$, $\alpha > 0$ and $R > r > 0$ such that*

$$\inf_{N_r} I \ge \alpha, \quad \sup_{\partial M} I \le 0,$$

where $N_r := \{u \in H^+ : \|u\| = r\}$,

$$M_{R,u_0} = M := \{u = u^- + \rho u_0 : u^- \in H^-,\ \|u\| \le R,\ \rho \ge 0\},$$

and ∂M denotes the boundary of M relative to $\mathbb{R}u_0 \oplus H^-$.

If

$$c := \inf_{h\in\Gamma} \sup_{u\in M} I(h(1,u)),$$

then there exists $(u_n) \subset H$ such that

$$I(u_n) \to c \ge \alpha, \quad (1 + \|u_n\|)\|I'(u_n)\|_{H^*} \to 0. \tag{2.2}$$

If there exists $h_0 \in \Gamma$ such that

$$c = \inf_{h\in\Gamma} \sup_{u\in M} I(h(1,u)) = \sup_{u\in M} I(h_0(1,u)),$$

then I possesses a nonzero critical point $u_0 \in h_0(1,M)$ such that $I(u_0) = c$.

We intend to apply the above result to obtain solutions for our equation. To define the functional we notice that for a given $\varepsilon > 0$, we can use (A3) and (A5) to obtain $C_\varepsilon > 0$ such that

$$|f(x,t)| \le \varepsilon|t| + C_\varepsilon|t|^{p-1}, \quad |F(x,t)| \le \varepsilon|t|^2 + C_\varepsilon|t|^p, \tag{2.3}$$

for any $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. The same inequality holds under conditions (A5), $\widehat{(f_5)}$ and (A7)(ii) (see [3, Lemma 4.1]). Therefore, in the setting of our main theorems, we can easily conclude that the functional $I : H \to \mathbb{R}$ given by

$$I(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int F(x,u),$$

for any $u = u^+ + u^-$, with $u^\pm \in H^\pm$, is well defined. Moreover, it belongs to $C^1(H,\mathbb{R})$ and its critical points are the weak solutions of (1.1).

To define the linking subsets we consider the periodic limit problem

$$-\Delta u + V(x)u = f_\infty(x,u), \quad x \in \mathbb{R}^N.$$

Under our conditions we can use [8, Theorem 1.2] to conclude that it has a ground state solution $u_\infty \in H^1(\mathbb{R}^N)$. More precisely, if

$$I_\infty(u) := \frac{1}{2}\|u^+\|^2 - \frac{1}{2}\|u^-\|^2 - \int F_\infty(x,u),$$

we have

$$I_\infty(u_\infty) = \inf\{I_\infty(u) : u \in H \setminus \{0\},\, I'_\infty(u) = 0\} > 0. \tag{2.4}$$

We set $u_0 := u_\infty^+$ and consider

$$M := \{u = u^- + \rho u_0 : u^- \in H^-,\ \|u\| \le R,\ \rho \ge 0\}, \quad N_r := \{u \in H^+ : \|u\| = r\}.$$

As proved in [7, Proposition 39 and Theorem 40] and [9, Corollary 2.4], we have

$$\sup_M I_\infty(u) \le I_\infty(u_\infty). \tag{2.5}$$

We finish this section by stating two technical convergence results whose proofs can be found in [5, Lemmas 5.1 and 5.2], respectively.

Lemma 2.2. *Suppose that* (A7) *holds. Let $(u_n) \subset H^1(\mathbb{R}^N)$ be a bounded sequence and $v_n(x) := v(x - y_n)$, where $v \in H^1(\mathbb{R}^N)$ and $(y_n) \subset \mathbb{R}^N$. If $|y_n| \to \infty$, then $[f_\infty(x,u_n) - f(x,u_n)]v_n \to 0$, strongly in $L^1(\mathbb{R}^N)$, as $n \to \infty$.*

Lemma 2.3. *Suppose that $h \in \mathfrak{F}$ and $s \in [2, 2^*)$. If $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$, then $\int h(x)|v_n|^s \to \int h(x)|v|^s$, as $n \to \infty$.*

3. Proofs of main results

In this section we prove Theorems 1.1 and 1.2.

Lemma 3.1. *Under the hypothesis of our main theorems the functional I satisfies the geometric conditions* (A9) *and* (A10).

Proof. Conditions (A5), (A8') and (A7)(ii) imply (A3). Thus, the inequalities in (2.3) holds under the assumptions of our main theorems and we can easily conclude that I satisfies (A9). Since $N_r \subset H^+$, for any $u \in N_r$, it holds $I(u) = (1/2)\|u^+\|^2 - \int F(x,u)$. Hence, it follows from (2.3) that $\inf_{N_r} I \ge \alpha > 0$ for some $r, \alpha > 0$. For $R > r$ large we need to verify that $\sup_{\partial M} I \le 0$. We fix $u = u^- + \rho u_0 \in \partial M_R$. If $\|u\| \le R$ and $\rho = 0$, we have $u = u^- \in H^-$ and therefore $I(u) \le 0$, since (A4) implies that $F \ge 0$. Thus, it remains to consider $\|u\| = R$ and $\rho > 0$. Arguing by contradiction, we suppose that for some sequence (u_n) such that $u_n = u_n^- + \rho_n u_0$, $\rho_n > 0$, $\|u_n\| = R_n \to \infty$ we have that $I(u_n) > 0$. Then

$$\frac{I(u_n)}{\|u_n\|^2} = \frac{1}{2}\Big(\frac{\rho_n^2\|u_0\|^2}{\|u_n\|^2} - \frac{\|u_n^-\|^2}{\|u_n\|^2}\Big) - \int \frac{F(x,u_n)}{\|u_n\|^2} > 0.$$

Since $F \geq 0$, we must have $\rho_n\|u_0\| \geq \|u_n^-\|$. From

$$\frac{\rho_n^2\|u_0\|^2}{\|u_n\|^2} + \frac{\|u_n^-\|^2}{\|u_n\|^2} = 1,$$

it follows that $\frac{1}{\sqrt{2}\|u_0\|} \leq \frac{\rho_n}{\|u_n\|} \leq \frac{1}{\|u_0\|}$ and $u_n^-/\|u_n\|$ is bounded. Thus, up to a subsequence, we have

$$\frac{\rho_n}{\|u_n\|} \to \rho > 0, \quad \frac{u_n^-}{\|u_n\|} \rightharpoonup v \in H^-, \quad \frac{u_n^-}{\|u_n\|} \to v \quad \text{a.e. for } x \in \mathbb{R}^N.$$

This and $\|u_n\| \to \infty$ imply that $\rho_n \to \infty$. Thus, we have

$$\lim |u_n(x)| = \infty \ \text{ a.e. in } \Omega = \{x \in \mathbb{R}^N : \rho u_0(x) + v(x) \neq 0\}.$$

Taking the lim sup in the inequality

$$0 < \frac{I(u_n)}{\|u_n\|^2} \leq \frac{1}{2}\Big(\frac{\rho_n^2\|u_0\|^2}{\|u_n\|^2} - \frac{\|u_n^-\|^2}{\|u_n\|^2}\Big) - \int_\Omega \frac{F(x,u_n)}{u_n^2}\frac{u_n^2}{\|u_n\|^2}\,\mathrm{d}x,$$

using Fatou's Lemma and (A6), we conclude that

$$0 \leq \frac{1}{2}\left(\rho^2\|u_0\|^2 - \|v\|^2\right) - \int_\Omega \liminf_{n\to\infty} \frac{F(x,u_n)}{u_n^2}(\rho u_0 + v)^2\,\mathrm{d}x = -\infty,$$

which is a contradiction. □

We are ready to obtain a solution for equation (1.1).

Proof of the main results. By Lemma 3.1 and the first part of Theorem 2.1 we can obtain $(u_n) \subset H$ such that

$$I(u_n) \to c \geq \alpha > 0, \quad (1 + \|u_n\|)I'(u_n) \to 0, \quad \text{as } n \to \infty.$$

Under condition (A8), arguing along the same lines as in [8, Lemma 3.4] we can prove that this sequence is bounded. As proved in [3, Lemma 4.3], the same holds if f satisfies (A4') and (A8'). We omit the (rather long) details in both cases. Since (u_n) is bounded in H, up to a subsequence, we have that $u_n \rightharpoonup u$ weakly in H. By using (A3), (A5) and standard calculations we can show that $I'(u) = 0$. If $u \neq 0$ we are done. So, we need only to consider only the case $u = 0$.

We claim that there exist a sequence $(y_n) \subset \mathbb{R}^N$, $R > 0$, and $\beta > 0$ such that $|y_n| \to \infty$ as $n \to \infty$, and

$$\limsup_{n\to\infty} \int_{B_R(y_n)} |u_n|^2\,\mathrm{d}x \geq \beta > 0. \tag{3.1}$$

Indeed, if this is not the case, from a result due to Lions [6] it follows that $|u_n|_s \to 0$ for any $s \in (2, 2^*)$. Hence, the first inequality in (2.3) implies that $\int F(x,u_n) \to 0$ as $n \to +\infty$. The same holds with $\int f(x,u_n)u_n$. On the other hand

$$c = \lim_{n\to\infty}\left(I(u_n) - \frac{1}{2}I'(u_n)u_n\right) = \lim_{n\to\infty}\int\left(\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right) = 0$$

which contradicts $c > 0$.

Without loss of generality we may assume that $(y_n) \subset \mathbb{Z}^N$ (see [1]). Writing $\widetilde{u}_n(x) := u_n(x + y_n)$ and observing that $\|\widetilde{u}_n\|_{H^1} = \|u_n\|_{H^1}$, up to subsequence we have $\widetilde{u}_n \rightharpoonup \widetilde{u}$ in H, $\widetilde{u}_n \to \widetilde{u}$ in $L^2_{loc}(\mathbb{R}^N)$ and for almost every $x \in \mathbb{R}^N$. It follows from (3.1) that $\widetilde{u} \neq 0$.

We fix $\eta \in C_0^\infty(\mathbb{R}^N)$ and define, for each $n \in \mathbb{N}$, the translation $\eta_n(x) := \eta(x-y_n)$. Using (2.3), the Lebesgue Theorem and the periodicity of f_∞ we get

$$I'_\infty(\widetilde{u}_n)\eta = I'_\infty(u_n)\eta_n = I'_\infty(\widetilde{u})\eta + o_n(1),$$

where $o_n(1)$ stands for a quantity approaching zero as $n \to +\infty$. Hence, we need only to show that $I'_\infty(u_n)\eta_n = o_n(1)$. However, Lemma 2.2 provides

$$I'_\infty(u_n)\eta_n = I'(u_n)\eta_n - \int [f(x,u_n) - f_\infty(x,u_n)]\eta_n = I'(u_n)\eta_n + o_n(1).$$

Since $I'(u_n)\eta_n \to 0$ it follows that $I'_\infty(\widetilde{u}) = 0$.

We claim that $\liminf_{n\to\infty} \int \widehat{F}(x,\widetilde{u}_n) \ge \int \widehat{F}_\infty(x,\widetilde{u})$. Indeed, from (A7) we obtain

$$|\widehat{F}(x,u_n) - \widehat{F}_\infty(x,u_n)| \le \Big(\frac{1}{2} + \frac{1}{p_\infty}\Big) h(x)|u_n|^{p_\infty}.$$

Thus, by Lemma 2.3, Fatou's lemma and periodicity of $\widehat{F}_\infty$,

$$\liminf_{n\to\infty} \int \widehat{F}(x,u_n) = \liminf_{n\to\infty} \int \widehat{F}_\infty(x,\widetilde{u}_n) \ge \int \widehat{F}_\infty(x,\widetilde{u}).$$

In view of the above considerations we obtain

$$\begin{aligned} c &= \lim_{n\to\infty} \Big(I(u_n) - \frac{1}{2} I'(u_n)u_n\Big) \\ &= \liminf_{n\to\infty} \int \widehat{F}(x,u_n) \\ &\ge \int \widehat{F}_\infty(x,\widetilde{u}) = I_\infty(\widetilde{u}) - \frac{1}{2} I'_\infty(\widetilde{u})\widetilde{u} = I_\infty(\widetilde{u}), \end{aligned}$$

and therefore $I_\infty(\widetilde{u}) \le c$. Hence, using the definition of c, (A7) and ((2.5) we obtain

$$c \le \sup_{u\in M} I(u) \le \sup_{u\in M} I_\infty(u) \le I_\infty(u_\infty) \le I_\infty(\widetilde{u}) \le c.$$

Thus, if we define $h_0 : [0,1] \times M \to H$ by $h_0(t,u) := u$ for any $(t,u) \in [0,1] \times M$, the above inequality implies $\sup_{u\in M} I(h_0(u,1)) = c$. It follows from the last statement of Theorem 2.1 that I has a nonzero critical point. □

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