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A VARIATIONAL APPROACH FOR SOLVING $p(x)$-BIHARMONIC EQUATIONS WITH NAVIER BOUNDARY CONDITIONS

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Abstract. In this article, we show the existence of at least three weak solutions for $p(x)$-biharmonic equations with Navier boundary conditions. The proof of the main result is based on variational methods. We also provide an example to illustrate our results.

1. Introduction

The aim of this article is to establish the existence of at least three weak solutions for the Navier boundary-value problem

$$\begin{gathered}\Delta^2_{p(x)}u = \lambda f(x,u(x)) + \mu g(x,u(x)), \quad x \in \Omega,\\ u = \Delta u = 0, \quad x \in \partial\Omega\end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with boundary of class C^1, $\lambda > 0$, $\mu \geq 0$, $f, g \in C^0(\Omega \times \mathbb{R})$, $p(\cdot) \in C^0(\Omega)$ with

$$\max\{2, \frac{N}{2}\} < p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x)$$

and $\Delta^2_{p(x)}u := \Delta(|\Delta|^{p(x)-2}\Delta u)$ which is the operator of fourth order called the $p(x)$-biharmonic operator. This operator is a natural generalization of the p-biharmonic operator (where $p > 1$ is a constant).

The operator $\Delta_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian which is a generalization of the p-Laplacian and possesses more complicated nonlinearities than the p-Laplacian, for example, it is inhomogeneous.

Recently, the investigation of differential equations and variational problems with variable exponent has become a new and interesting topic. The study of various mathematical problems with variable exponent has been received considerable attention in recent years. These problems are interesting in applications, for example in nonlinear elasticity theory and in modelling electrorheological fluids (Acerbi and Mingione [1], Diening [11], Halsey [13], Ružička [37], Rajagopal and Ružička [33])

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and from the study of elastic mechanics (Zhikov [42]), and raise many difficult mathematical problems. After this pioneering models, many other applications of differential operators with variable exponents have appeared in a large range of fields, such as image restoration (Chen et al. [9]) and mathematical biology (Fragnelli [12]).

Fourth-order equations can describe the static form change of beam or the sport of rigid body. In [22], Lazer and Mckenna have pointed out that this type of nonlinearity furnishes a model to study travelling waves in suspension bridges. Numerous authors investigated the existence and multiplicity of solutions for the problems involving biharmonic, p-biharmonic and $p(x)$-biharmonic operators. We refer to [2, 4, 8, 10, 14, 16, 17, 18, 19, 21, 23, 24, 26, 27, 28, 38, 39, 40] for advances and references of this area. For example, Li and Tang in [24] by using a three critical points theorem obtained due to Ricceri, established the existence of at least three weak solutions for a class of Navier boundary value problem involving the p-biharmonic

$$\begin{gathered}\Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x,u) + \mu g(x,u), \quad x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega\end{gathered}$$

where $\lambda, \mu \in [0,+\infty)$ and $f : \bar{\Omega}\times\mathbb{R} \to \mathbb{R}$ is a continuous function, and $g : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Yin and Xu in [39] based on a three critical points theorem due to Ricceri, obtained the existence of at least three weak solutions for a class of quasilinear elliptic equations involving the $p(x)$-biharmonic operator with Navier boundary value conditions. Also in [2] by using critical point theory, the existence of infinitely many weak solutions for a class of Navier boundary-value problem depending on two parameters and involving the $p(x)$-biharmonic operator

$$\begin{gathered}\Delta^2_{p(x)} u = \lambda f(x,u(x)) + \mu g(x,u(x)), \quad x \in \Omega, \\ u = \Delta u = 0, \quad x \in \partial\Omega\end{gathered}$$

where λ is a positive parameter, μ is a non-negative parameter, $f, g \in C^0(\Omega \times \mathbb{R})$, was studied. Kong in [19] using variational arguments based on Ekeland's variational principle and some recent theory on the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ studied a $p(x)$-biharmonic nonlinear eigenvalue problem, while in [19] using variational arguments based on the mountain pass lemma and some recent theory on the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ he studied the multiplicity of weak solutions to a fourth order nonlinear elliptic problem with a $p(x)$-biharmonic operator. In [17], based on variational methods and critical point theory, the existence of solutions for the problem (1.1), in the case $\mu = 0$, was investigated. In fact, the existence of two solutions for the problem under some algebraic conditions with the classical Ambrosetti-Rabinowitz condition on the nonlinear term was established. Moreover, by combining two algebraic conditions on the nonlinear term which guarantee the existence of two solutions, applying the mountain pass theorem given by Pucci and Serrin the existence of the third solution for the problem was ensured, while in [16] based on variational methods the existence of at least one weak solution for the same problem was discussed.

We refer the reader to the recent monograph by Molica Bisci, Rădulescu and Servadei [25] for related problems concerning the variational analysis of solutions of some classes of boundary value problems. Also for further studies on this subject, we refer the reader to [3, 7, 31, 32, 34].

Inspired by the above works, in this article, we discuss the existence of at least three weak solutions for (1.1), in which two parameters are involved. Precise estimates of these two parameters λ and μ will be given. No asymptotic condition at infinity is required on the nonlinear term. In Theorem 3.1 we establish the existence of at least three weak solutions for the problem (1.1). We present example 3.2 which illustrates Theorem 3.1. Theorem 3.3 is a consequence of Theorem 3.1. As a consequence of Theorem 3.3, we obtain Theorem 3.5 for the autonomous case and $\mu = 0$. Finally, we present Example 3.6 in which the hypotheses of Theorems 3.5 are fulfilled.

2. Preliminaries

Let X be a nonempty set and $\Phi, \Psi : X \to \mathbb{R}$ be two functions. For all $r, r_1, r_2 > \inf_X \Phi$, $r_2 > r_1$, $r_3 > 0$, we define

$$\begin{gathered}
\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{(\sup_{u \in \Phi^{-1}(-\infty, r)} \Psi(u)) - \Psi(u)}{r - \Phi(u)}, \\
\beta(r_1, r_2) := \inf_{u \in \Phi^{-1}(-\infty, r_1)} \sup_{v \in \Phi^{-1}[r_1, r_2)} \frac{\Psi(v) - \Psi(u)}{\Phi(v) - \Phi(u)}, \\
\gamma(r_2, r_3) := \frac{\sup_{u \in \Phi^{-1}(-\infty, r_2 + r_3)} \Psi(u)}{r_3}, \\
\alpha(r_1, r_2, r_3) := \max\{\varphi(r_1), \varphi(r_2), \gamma(r_2, r_3)\}.
\end{gathered}$$

We shall discuss the existence of at least three solutions to (1.1). Our main tool to prove the results is [5, Theorem 3.3] that we now recall as follows.

Theorem 2.1. *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

- (A1) $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;
- (A2) *for every $u_1, u_2 \in X$ such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has*
$$\inf_{s \in [0,1]} \Psi(s u_1 + (1 - s) u_2) \geq 0.$$

Assume that there are three positive constants r_1, r_2, r_3 with $r_1 < r_2$, such that

- (A3) $\varphi(r_1) < \beta(r_1, r_2)$;
- (A4) $\varphi(r_2) < \beta(r_1, r_2)$;
- (A5) $\gamma(r_2, r_3) < \beta(r_1, r_2)$.

Then, for each $\lambda \in]\frac{1}{\beta(r_1, r_2)}, \frac{1}{\alpha(r_1, r_2, r_3)}[$ the functional $\Phi - \lambda \Psi$ admits three critical points u_1, u_2, u_3 such that $u_1 \in \Phi^{-1}(]-\infty, r_1[)$, $u_2 \in \Phi^{-1}([r_1, r_2[)$ and $u_3 \in \Phi^{-1}(]-\infty, r_2 + r_3[)$.

Theorem 2.1 is a counter-part of a three critical point theorem by Ricceri [35, 36], which extends previous results by Pucci and Serrin [29, 30].

We refer the interested reader to the papers [6, 15, 20] in which Theorem 2.1 has been successfully used to ensure the existence of at least three solutions for boundary value problems.

For the reader's convenience, we recall some background facts concerning the Lebesgue-Sobolev spaces with variable exponent and introduce some notation. For more details, we refer the reader to [31, 32]. Set

$$C_+(\Omega) := \{h : h \in C(\overline{\Omega}) \text{ and } h(x) > 1, \forall x \in \overline{\Omega}\}.$$

For $p(\cdot) \in C_+(\Omega)$, define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ by

$$L^{p(\cdot)}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} dx < \infty\}.$$

We define a norm, the so-called Luxemburg norm, on this space by the formula

$$|u|_{p(\cdot)} = \inf\{\beta > 0 : \int_\Omega |\frac{u(x)}{\beta}|^{p(x)} dx \le 1\}$$

and $(L^{p(\cdot)}(\Omega), |u|_{p(\cdot)})$ becomes a Banach space, and we call it variable exponent Lebesgue space. Define the variable exponent Sobolev space $W^{m,p(\cdot)}(\Omega)$ by

$$W^{m,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) |\ D^\alpha u \in L^{p(\cdot)}(\Omega), |\alpha| \le m\}$$

where

$$D^\alpha u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} u$$

with $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index and $|\alpha| = \sum_{i=1}^N \alpha_i$. The space $W^{m,p(\cdot)}(\Omega)$, equipped with the norm

$$\|u\|_{m,p(\cdot)} := \sum_{|\alpha| \le m} |D^\alpha u|_{p(\cdot)},$$

becomes a separable, reflexive and uniformly convex Banach space. We denote by X^* its dual.

We denote

$$X := W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$$

where $W_0^{m,p(\cdot)}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{m,p(\cdot)}(\Omega)$.

For $u \in X$, we define

$$\|u\| = \inf\{\beta > 0 : \int_\Omega |\frac{\Delta u(x)}{\beta}|^{p(x)} dx \le 1\}.$$

Clearly, we observe that X endowed with the above norm is a separable and reflexive Banach space.

Remark 2.2. From [41], the norm $\|u\|_{2,p(\cdot)}$ is equivalent to the norm $|\Delta u|p(\cdot)$ in the space X. Consequently, the norms $\|u\|_{2,p(\cdot)}$, $\|u\|$ and $|\Delta u|p(\cdot)$ are equivalent. For the rest of this article, we use $\|u\|$ instead of $\|u\|_{2,p(\cdot)}$ on X.

Proposition 2.3 ([34]). *The conjugate space of $L^{p(\cdot)}(\Omega)$ is $L^{q(\cdot)}(\Omega)$ where $q(\cdot)$ is the conjugate function of $p(\cdot)$; i.e.,*

$$\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1.$$

For $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{q(\cdot)}(\Omega)$, we have

$$\Big|\int_\Omega u(x)v(x)dx\Big| \le (\frac{1}{p^-} + \frac{1}{q^-})|u|_{p(\cdot)}|v|_{q(\cdot)} \le 2|u|_{p(\cdot)}|v|_{q(\cdot)}.$$

Proposition 2.4 ([34]). *Let $\rho(u) = \int_\Omega |u|^{p(x)} dx$. For $u, u_n \in L^{p(\cdot)}(\Omega)$, we have*

(1) $|u|_{p(\cdot)} < \ (=;>); 1 \Leftrightarrow \rho(u) < \ (=;>)\ 1$;
(2) $|u|_{p(\cdot)} > 1 \Rightarrow |u|_{p(\cdot)}^{p^-} \le \rho(u) \le |u|_{p(\cdot)}^{p^+}$;
(3) $|u|_{p(\cdot)} < 1 \Rightarrow |u|_{p(\cdot)}^{p^+} \le \rho(u) \le |u|_{p(\cdot)}^{p^-}$;
(4) $|u_n|_{p(\cdot)} \to 0 \Leftrightarrow \rho(u_n) \to 0$;
(5) $|u_n|_{p(\cdot)} \to \infty \Leftrightarrow \rho(u_n) \to \infty$.

From Proposition 2.4, for $u \in L^{p(\cdot)}(\Omega)$ the following inequalities hold:

$$\|u\|^{p^-} \le \int_\Omega |\Delta u|^{p(x)} dx \le \|u\|^{p^+}, \quad \text{if } \|u\| \ge 1, \tag{2.1}$$

$$\|u\|^{p^+} \le \int_\Omega |\Delta u|^{p(x)} dx \le \|u\|^{p^-}, \quad \text{if } \|u\| \le 1. \tag{2.2}$$

Proposition 2.5 ([38]). *If $\Omega \subset \mathbb{R}^N$ is a bounded domain, then the embedding $X \hookrightarrow C^0(\Omega)$ is compact whenever $\frac{N}{2} < p^-$.*

From Proposition 2.5, there exists a positive constant c depending on $p(\cdot)$, N and Ω such that

$$\|u\|_\infty = \sup_{x\in\overline{\Omega}} |u(x)| \le c\|u\|, \quad \forall u \in X. \tag{2.3}$$

Corresponding to f and g we introduce the functions $F : \Omega \times \mathbb{R} \to \mathbb{R}$ and $G : \Omega \times \mathbb{R} \to \mathbb{R}$, as follows

$$F(x,t) := \int_0^t f(x,\xi) d\xi \quad \text{for } (x,t) \in \Omega \times \mathbb{R},$$

$$G(x,t) := \int_0^t f(x,\xi) d\xi \quad \text{for } (x,t) \in \Omega \times \mathbb{R}.$$

We say that a function $u \in X$ is a weak solution of (1.1) if

$$\int_\Omega |\Delta u(x)|^{p(x)-2}\Delta u(x)\Delta v(x) dx - \lambda \int_\Omega f(x,u(x))v(x) dx - \mu \int_\Omega g(x,u(x))v(x) dx = 0$$

holds for all $v \in X$.

In the sequel meas(Ω) denotes the Lebesgue measure of the set Ω.

3. Main results

In this section, we formulate our main results on the existence of at least three weak solutions for problem (1.1). For our convenience, set

$$G^\theta := \int_\Omega \max_{|\xi|\le\theta} G(x,\xi) dx \quad \text{for } \theta > 0,$$

$$G_\eta := \operatorname{meas}(\Omega) \inf_{\overline{\Omega}\times[0,\eta]} G(x,t) \quad \text{for } \eta > 0.$$

If g is sign-changing, then clearly $G^\theta \ge 0$ and $G_\eta \le 0$.

Fix $x^0 \in \Omega$ and choose s_1, s_2 with $0 < s_1 < s_2$, such that $B(x^0, s_2) \subseteq \Omega$ where $B(x,s)$ stands for the open ball in $\mathbb{R}^N$ of radius s and center x. Let

$$\begin{aligned} \sigma := &\frac{2c^{p^-}\pi^{\frac{N}{2}}(s_2^N - s_1^N)}{N\Gamma(\frac{N}{2})} \\ &\times \max\Big\{\big[\frac{12(N+2)^2(s_1+s_2)}{(s_2-s_1)^3}\big]^{p^-}, \big[\frac{12(N+2)^2(s_1+s_2)}{(s_2-s_1)^3}\big]^{p^+}\Big\} \end{aligned}$$

and

$$\rho := \frac{2c^{p^-}\pi^{\frac{N}{2}}(s_2^N - s_1^N)}{N\Gamma(\frac{N}{2})} \times \min\Big\{\big[\frac{12(N+2)^2(s_1+s_2)}{(s_2-s_1)^3}\big]^{p^-}, \big[\frac{12(N+2)^2(s_1+s_2)}{(s_2-s_1)^3}\big]^{p^+}\Big\}.$$

Fixing four positive constants θ_1, θ_2, θ_3 and $\eta \geq 1$, we put

$$\begin{aligned}\delta_{\lambda,g} := \min\Big\{&\frac{1}{p^+c^{p^-}}\min\Big\{\frac{\theta_1^{p^-} - \lambda p^+c^{p^-}\int_\Omega F(x,\theta_1)dx}{G^{\theta_1}}\\ &, \frac{\theta_2^{p^-} - \lambda p^+c^{p^-}\int_\Omega F(x,\theta_2)dx}{G^{\theta_2}}, \frac{(\theta_3^{p^-} - \theta_2^{p^-}) - \lambda p^+c^{p^-}\int_\Omega F(x,\theta_3)dx}{G^{\theta_3}}\Big\}\\ &, \frac{\frac{\sigma\eta^{p^+}}{p^-c^{p^-}} - \lambda\big(\int_{B(x^0,s_1)} F(x,\eta)dx - \int_\Omega F(x,\theta_1)dx\big)}{G_\eta - G^{\theta_1}}\Big\}.\end{aligned} \tag{3.1}$$

Theorem 3.1. *Assume that there exist positive constants* θ_1, θ_2, θ_3 *and* $\eta \geq 1$ *with* $\theta_1 < \rho^{\frac{1}{p^-}}\eta$, $\eta < \min\{(\frac{p^+}{\sigma p^-})^{\frac{1}{p^+}}\theta_2^{p^-/p^+}, \theta_2\}$ *and* $\theta_2 < \theta_3$ *such that*

- (A6) $f(x,t) \geq 0$ *for each* $(x,t) \in \Omega \times [-\theta_3, \theta_3]$;
- (A7)
$$\begin{aligned}&\max\Big\{\frac{\int_\Omega F(x,\theta_1)dx}{\theta_1^{p^-}}, \frac{\int_\Omega F(x,\theta_2)dx}{\theta_2^{p^-}}, \frac{\int_\Omega F(x,\theta_3)dx}{\theta_3^{p^-} - \theta_2^{p^-}}\Big\}\\ &< \frac{p^-}{p^+\sigma}\frac{\int_{B(x^0,s_1)} F(x,\eta)dx - \int_\Omega F(x,\theta_1)dx}{\eta^{p^+}}.\end{aligned}$$

Then, for every

$$\begin{aligned}\lambda \in \Lambda := \Big(&\frac{\frac{\sigma\eta^{p^+}}{p^-c^{p^-}}}{\int_{B(x^0,s_1)} F(x,\eta)dx - \int_\Omega F(x,\theta_1)dx},\\ &\frac{1}{p^+c^{p^-}}\min\Big\{\frac{\theta_1^{p^-}}{\int_\Omega F(x,\theta_1)dx}, \frac{\theta_2^{p^-}}{\int_\Omega F(x,\theta_2)dx}, \frac{\theta_3^{p^-} - \theta_2^{p^-}}{\int_\Omega F(x,\theta_3)dx}\Big\}\Big)\end{aligned}$$

and for every non-negative continuous function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, *there exists* $\delta_{\lambda,g} > 0$ *given by* (3.1) *such that, for each* $\mu \in [0, \delta_{\lambda,g})$, *problem* (1.1) *has at least three weak solutions* u_1, u_2 *and* u_3 *such that* $\max_{x\in\Omega}|u_1(x)| < \theta_1$, $\max_{x\in\Omega}|u_2(x)| < \theta_2$ *and* $\max_{x\in\Omega}|u_3(x)| < \theta_3$.

Proof. Our goal is to apply Theorem 2.1 to problem (1.1). We consider the auxiliary problem

$$\begin{gathered}\Delta^2_{p(x)}u = \lambda\hat{f}(x,u(x)) + \mu g(x,u(x)), \quad x \in \Omega,\\ u = \Delta u = 0, \quad x \in \partial\Omega\end{gathered} \tag{3.2}$$

where $\hat{f} \in C^0(\Omega \times \mathbb{R})$ defined setting

$$\hat{f}(x,\xi) = \begin{cases} f(x,0), & \text{if } \xi < -\theta_3,\\ f(x,\xi), & \text{if } -\theta_3 \leq \xi \leq \theta_3,\\ f(x,\theta_3), & \text{if } \xi > \theta_3.\end{cases}$$

If a weak solution of (3.2) satisfies the condition $-\theta_3 \le u(x) \le \theta_3$ for every $x \in \Omega$, then, clearly it turns to be also a weak solution of (1.1). Therefore, for our goal, it is sufficient to show that our conclusion holds for (1.1). Consider the functionals Φ, Ψ for every $u \in X$, defined by

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} |\Delta u(x)|^{p(x)} dx, \tag{3.3}$$

$$\Psi(u) = \int_\Omega F(x, u(x)) dx + \frac{\mu}{\lambda} \int_\Omega G(x, u(x)) dx. \tag{3.4}$$

Let us prove that the functionals Φ and Ψ satisfy the required conditions in Theorem 2.1. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_\Omega f(x, u(x)) v(x) dx + \frac{\mu}{\lambda} \int_\Omega g(x, u(x)) v(x) dx$$

for every $v \in X$, as well as it is sequentially weakly upper semicontinuous. Recalling (2.1), we have

$$\Phi(u) \ge \frac{1}{p^+} \|u\|^{p^-},$$

for all $u \in X$ with $\|u\| > 1$, which implies Φ is coercive. Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \int_\Omega |\Delta u(x)|^{p(x)-2} \Delta u(x) \Delta v(x) dx$$

for every $v \in X$. Also, $\Phi' : X \to X^*$ is a compact operator (see [38, Lemma 3.1]). Furthermore, Φ is sequentially weakly lower semicontinuous. Therefore, we observe that the regularity assumptions on Φ and Ψ, as requested of Theorem 2.1, are satisfied. Define w by setting

$$w(x) := \begin{cases} 0, & x \in \overline{\Omega} \setminus B(x^0, s_2), \\ \frac{\eta[3(l^4 - s_2^4) - 4(s_1+s_2)(l^3 - s_2^3) + 6 s_1 s_2 (l^2 - s_2^2)]}{(s_2 - s_1)^3 (s_1 + s_2)}, & x \in B(x^0, s_2) \setminus B(x^0, s_1), \\ \eta, & x \in B(x^0, s_1) \end{cases}$$

where $l = \operatorname{dist}(x, x^0) = \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}$. Then

$$\frac{\partial w(x)}{\partial x_i} = \begin{cases} 0, \quad \text{if } x \in \overline{\Omega} \setminus B(x^0, s_2) \cup B(x^0, s_1), \\ \frac{12\eta[l^2 (x_i - x_i^0) - l(s_1+s_2)(x_i - x_i^0) + s_1 s_2 (x_i - x_i^0)]}{(s_2 - s_1)^3 (s_1 + s_2)}, \\ \quad \text{if } x \in B(x^0, s_2) \setminus B(x^0, s_1), \end{cases}$$

$$\frac{\partial^2 w(x)}{\partial x_i^2} = \begin{cases} 0, \quad \text{if } x \in \overline{\Omega} \setminus B(x^0, s_2) \cup B(x^0, s_1), \\ \frac{12\eta[s_1 s_2 + (2l - s_1 - s_2)(x_i - x_i^0)^2 / l - (s_1 + s_2 - l) l]}{(s_2 - s_1)^3 (s_1 + s_2)}, \\ \quad \text{if } x \in B(x^0, s_2) \setminus B(x^0, s_1), \end{cases}$$

$$\sum_{i=1}^N \frac{\partial^2 w(x)}{\partial x_i^2} = \begin{cases} 0, \quad \text{if } x \in \overline{\Omega} \setminus B(x^0, s_2) \cup B(x^0, s_1), \\ \frac{12\eta[(N+2) l^2 - (N+1)(s_1+s_2) l + N s_1 s_2]}{(s_2 - s_1)^3 (s_1 + s_2)}, \\ \quad \text{if } x \in B(x^0, s_2) \setminus B(x^0, s_1). \end{cases}$$

So, one has

$$\frac{\rho\eta^{p^-}}{p^+c^{p^-}} \le \Phi(w) = \int_{B(x^0,s_2)\setminus B(x^0,s_1)} \frac{1}{p(x)}|\Delta w(x)|^{p(x)}dx \le \frac{\sigma\eta^{p^+}}{p^-c^{p^-}}.$$

On the other hand, bearing (A6) in mind and since g is non-negative, from the definition of Ψ, we infer

$$\Psi(w) = \int_\Omega \Big[F(x,w(x)) + \frac{\mu}{\lambda}G(x,w(x))\Big]\,dx \ge \int_{B(x^0,s_1)} F(x,\eta)dx.$$

Choose $r_1 = \frac{1}{p^+}\big(\frac{\theta_1}{c}\big)^{p^-}$, $r_2 = \frac{1}{p^+}\big(\frac{\theta_2}{c}\big)^{p^-}$ and $r_3 = \frac{1}{p^+}\big(\frac{\theta_3^{p^-}-\theta_2^{p^-}}{c^{p^-}}\big)$. From the conditions $\theta_1 < \rho^{\frac{1}{p^-}}\eta$, $\eta < (\frac{p^+}{\sigma p^-})^{\frac{1}{p^+}}\theta_2^{p^-/p^+}$ and $\theta_2 < \theta_3$, we achieve $r_1 < \Phi(w) < r_2$ and $r_3 > 0$. For all $u \in X$ with $\Phi(u) < r_1$, taking (2.1) and (2.2) into account, one has

$$\|u\| \le \max\big\{(p^+r_1)^{\frac{1}{p^+}}, (p^+r_1)^{\frac{1}{p^-}}\big\}.$$

So, thanks to the embedding $X \hookrightarrow C^0(\overline{\Omega})$ (see (2.3)), one has $\|u\|_\infty < \theta_1$. From the definition of r_1, it follows that

$$\Phi^{-1}(-\infty,r_1) = \{u \in X; \Phi(u) < r_1\} \subseteq \{u \in X; |u| \le \theta_1\}.$$

Hence, by using assumption (A6), one has

$$\sup_{u\in\Phi^{-1}(-\infty,r_1)} \int_\Omega F(x,u(x))dx \le \int_\Omega \sup_{|t|\le\theta_1} F(x,t)dx \le \int_\Omega F(x,\theta_1)dx.$$

As above, we can obtain that

$$\sup_{u\in\Phi^{-1}(-\infty,r_2)} \int_\Omega F(x,u(x))dx \le \int_\Omega F(x,\theta_2)dx,$$

$$\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)} \int_\Omega F(x,u(x))dx \le \int_\Omega F(x,\theta_3)dx.$$

Therefore, since $0 \in \Phi^{-1}(-\infty,r_1)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned}
\varphi(r_1) &= \inf_{u\in\Phi^{-1}(-\infty,r_1)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)) - \Psi(u)}{r_1 - \Phi(u)} \\
&\le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\Psi(u)}{r_1} \\
&= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)} \int_\Omega [F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]dx}{r_1} \\
&\le \frac{\int_\Omega F(x,\theta_1)dx + \frac{\mu}{\lambda}G^{\theta_1}}{\frac{1}{p^+}\big(\frac{\theta_1}{c}\big)^{p^-}},
\end{aligned}$$

$$\begin{aligned}
\varphi(r_2) \le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\Psi(u)}{r_2} &= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)} \int_\Omega [F(x,u(x)) + \frac{\mu}{\lambda}G(x,u(x))]dx}{r_2} \\
&\le \frac{\int_\Omega F(x,\theta_2)dx + \frac{\mu}{\lambda}G^{\theta_2}}{\frac{1}{p^+}\big(\frac{\theta_2}{c}\big)^{p^-}}
\end{aligned}$$

and

$$\gamma(r_2,r_3) \le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\Psi(u)}{r_3}$$

$$
\begin{aligned}
&= \frac{\sup_{u\in\Phi^{-1}(-\infty,r_2+r_3)}\int_\Omega [F(x,u(x))+\frac{\mu}{\lambda}G(x,u(x))]dx}{r_3}\\
&\le \frac{\int_\Omega F(x,\theta_3)dx+\frac{\mu}{\lambda}G^{\theta_3}}{\frac{1}{p^+}\Big(\frac{\theta_3^{p^-}-\theta_2^{p^-}}{c^{p^-}}\Big)}.
\end{aligned}
$$

On the other hand, for each $u\in\Phi^{-1}(-\infty,r_1)$ one has

$$
\begin{aligned}
\beta(r_1,r_2) &\ge \frac{\int_{B(x^0,s_1)}F(x,\eta)dx-\int_\Omega F(x,\theta_1)dx+\frac{\mu}{\lambda}(G_\eta-G^{\theta_1})}{\Phi(w)-\Phi(u)}\\
&\ge \frac{\int_{B(x^0,s_1)}F(x,\eta)dx-\int_\Omega F(x,\theta_1)dx+\frac{\mu}{\lambda}(G_\eta-G^{\theta_1})}{\frac{\sigma\eta^{p^+}}{p^-c^{p^-}}}.
\end{aligned}
$$

From (A7) we obtain $\alpha(r_1,r_2,r_3)<\beta(r_1,r_2)$. Finally, we verify that $\Phi-\lambda\Psi$ satisfies assumption (A2) of Theorem 2.1. Let u_1 and u_2 be two local minima for $\Phi-\lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi-\lambda\Psi$, and so, they are weak solutions of (1.1). Since we assumed f is nonnegative and since g is non-negative, for fixed $\lambda>0$ and $\mu\ge 0$ we have $(\lambda f+\mu g)(x,su_1+(1-s)u_2)\ge 0$ for all $s\in[0,1]$, and consequently, $\Psi(su_1+(1-s)u_2)\ge 0$, for every $s\in[0,1]$. Hence, Theorem 2.1 implies that for every

$$
\begin{aligned}
\lambda\in\Big(&\frac{\frac{\sigma\eta^{p^+}}{p^-c^{p^-}}}{\int_{B(x^0,s_1)}F(x,\eta)dx-\int_\Omega F(x,\theta_1)dx},\\
&\frac{1}{p^+c^{p^-}}\min\Big\{\frac{\theta_1^{p^-}}{\int_\Omega F(x,\theta_1)dx},\frac{\theta_2^{p^-}}{\int_\Omega F(x,\theta_2)dx},\frac{\theta_3^{p^-}-\theta_2^{p^-}}{\int_\Omega F(x,\theta_3)dx}\Big\}\Big)
\end{aligned}
$$

and $\mu\in[0,\delta_{\lambda,g})$, the functional $\Phi-\lambda\Psi$ has three critical points u_i, $i=1,2,3$, in X such that $\Phi(u_1)<r_1$, $\Phi(u_2)<r_2$ and $\Phi(u_3)<r_2+r_3$, that is, $\max_{x\in\Omega}|u_1(x)|<\theta_1$, $\max_{x\in\Omega}|u_2(x)|<\theta_2$ and $\max_{x\in\Omega}|u_3(x)|<\theta_3$. Then, taking into account the fact that the solutions of problem (1.1) are exactly critical points of the functional $\Phi-\lambda\Psi$ we have the desired conclusion. □

The following example illustrates the result of Theorem 3.1.

Example 3.2. Let $\Omega=\{(x,y)\in\mathbb{R}^2: x^2+y^2\le 2\}$. Consider the problem

$$
\begin{cases}
\Delta^2_{p(x,y)}u=\lambda f(u)+\mu g(u), & (x,y)\in\Omega,\\
u=\Delta u=0, \quad (x,y)\in\partial\Omega
\end{cases}
\tag{3.5}
$$

where $p(x,y)=x^2+y^2+2$ for all $(x,y)\in\Omega$ and

$$
f(t)=\begin{cases} 5t^4, & \text{if } t\le 1,\\ \frac{5}{\sqrt{t}}, & \text{if } t>1.\end{cases}
$$

By the expression of f we have

$$
F(t)=\begin{cases} t^5, & \text{if } t\le 1,\\ 10\sqrt{t}-9, & \text{if } t>1.\end{cases}
$$

Direct calculations give $p^-=2$ and $p^+=4$. By choosing $x_0=0$, $s_1=1$ and $s_2=2$, we obtain $\sigma=3^9\times 2^{24}\pi c^2$ and $\rho=3^5\times 2^{12}\pi c^2$. We consider two cases for

c. First, suppose that $c \leq 1$. Choosing $\eta = 1$, $\theta_1 = 10^{-8}c$, $\theta_2 = \frac{10^{12}}{\sqrt{2}}$ and $\theta_3 = 10^{12}$ we see that

$$\begin{aligned}
&\max\Big\{\frac{\operatorname{meas}(\Omega)F(\theta_1)}{\theta_1^2}, \frac{\operatorname{meas}(\Omega)F(\theta_2)}{\theta_2^2}, \frac{\operatorname{meas}(\Omega)F(\theta_3)}{\theta_3^2-\theta_2^2}\Big\}\\
&= \frac{8\times 10^7\pi - 72\pi}{10^{24}}\\
&< \frac{1}{3^9\times 2^{25}\pi c^2}(\pi - 4\times 10^{-24}c^3\pi)\\
&= \frac{p^-}{p^+\sigma}\frac{\int_{B(x^0,s_1)}F(x,\eta)dx - \int_\Omega F(x,\theta_1)dx}{\eta^{p^+}},
\end{aligned}$$

which means the assumption (A7) is satisfied. It is easy to see that other assumptions of Theorem 3.1 are also fulfilled. Therefore, in this case, it follows that for every

$$\lambda \in \Big(\frac{3^9\times 2^{23}\pi}{\pi - 4\times 10^{-24}c^3\pi}, \frac{10^{24}}{32\times 10^7\pi c^2 - 288\pi c^2}\Big)$$

and for every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\hat{\delta} > 0$ such that for each $\mu \in [0, \hat{\delta})$, then problem (3.5) has at least three weak solutions u_1, u_2 and u_3 such that $\max_{x\in\Omega}|u_1(x)| < 10^{-8}c$, $\max_{x\in\Omega}|u_2(x)| < \frac{10^{12}}{\sqrt{2}}$ and $\max_{x\in\Omega}|u_3(x)| < 10^{12}$.

Now, suppose that $c > 1$. Choosing $\eta = 1$, $\theta_1 = \frac{10^{-8}}{c}$, $\theta_2 = \frac{10^{12}}{\sqrt{2}}c^{3/2}$ and $\theta_3 = 10^{12}c^{3/2}$, we have

$$\begin{aligned}
&\max\Big\{\frac{\operatorname{meas}(\Omega)F(\theta_1)}{\theta_1^2}, \frac{\operatorname{meas}(\Omega)F(\theta_2)}{\theta_2^2}, \frac{\operatorname{meas}(\Omega)F(\theta_3)}{\theta_3^2-\theta_2^2}\Big\}\\
&= \frac{8\times 10^7\pi c^{\frac{3}{4}} - 72\pi}{10^{24}c^3}\\
&< \frac{1}{3^9\times 2^{25}\pi c^2}(\pi - \frac{4\times 10^{-24}\pi}{c^3})\\
&= \frac{p^-}{p^+\sigma}\frac{\int_{B(x^0,s_1)}F(x,\eta)dx - \int_\Omega F(x,\theta_1)dx}{\eta^{p^+}},
\end{aligned}$$

which means the assumption (A7) is fulfilled. Clearly, other assumptions of Theorem 3.1 in this case are satisfied too. Then, in this case, it follows for every

$$\lambda \in \Big(\frac{3^9\times 2^{23}\pi}{\pi - \frac{4\times 10^{-24}\pi}{c^3}}, \frac{10^{24}c^3}{32\times 10^7\pi c^{\frac{11}{4}} - 288\pi c^2}\Big)$$

and for every non-negative continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\overline{\delta} > 0$ such that for each $\mu \in [0, \overline{\delta})$, the problem (3.5) has at least three weak solutions u_1, u_2 and u_3 such that $\max_{x\in\Omega}|u_1(x)| < \frac{10^{-8}}{c}$, $\max_{x\in\Omega}|u_2(x)| < \frac{10^{12}}{\sqrt{2}}c^{3/2}$ and $\max_{x\in\Omega}|u_3(x)| < 10^{12}c^{3/2}$.

For given positive constants θ_1, θ_4 and $\eta \geq 1$, we set

$$\begin{aligned}\delta'_{\lambda,g} := \min\Big\{&\frac{1}{p^+c^{p^-}}\min\Big\{\frac{\theta_1^{p^-} - p^+c^{p^-}\lambda\int_\Omega F(x,\theta_1)dx}{G^{\theta_1}},\\ &\frac{\theta_4^{p^-} - 2p^+c^{p^-}\lambda\int_\Omega F(x,\frac{1}{\sqrt[p^-]{2}}\theta_4)dx}{2G^{\frac{1}{\sqrt[p^-]{2}}\theta_4}}, \frac{\theta_4^{p^-} - 2p^+c^{p^-}\lambda\int_\Omega F(x,\theta_4)dx}{2G^{\theta_4}}\Big\},\\ &\frac{\frac{\sigma\eta^{p^+}}{p^-c^{p^-}} - \lambda\big(\int_{B(x^0,s_1)} F(x,\eta)dx - \int_\Omega F(x,\theta_1)dx\big)}{G_\eta - G^{\theta_1}}\Big\}.\end{aligned} \tag{3.6}$$

Now, we deduce the following straightforward consequence of Theorem 3.1.

Theorem 3.3. *Assume that there exist positive constants θ_1, θ_4 and $\eta \geq 1$ with $\theta_1 < \min\{\eta^{p^+/p^-}, \rho^{\frac{1}{p^-}}\eta\}$ and $\eta < \min\{(\frac{p^+}{2\sigma p^-})^{\frac{1}{p^+}}\theta_4^{p^-/p^+}, \theta_4\}$ such that*

(A8) *$f(x,t) \geq 0$ for each $(x,t) \in \Omega \times [-\theta_4, \theta_4]$;*

(A9)
$$\max\Big\{\frac{\int_\Omega F(x,\theta_1)dx}{\theta_1^{p^-}}, \frac{2\int_\Omega F(x,\theta_4)dx}{\theta_4^{p^-}}\Big\} < \frac{p^-}{p^+\sigma + p^-}\frac{\int_{B(x^0,s_1)} F(x,\eta)dx}{\eta^{p^+}}.$$

Then, for every

$$\begin{aligned}\lambda \in \Lambda' := \Big(&\frac{(p^+\sigma + p^-)\eta^{p^+}}{p^-p^+c^{p^-}\int_{B(x^0,s_1)} F(x,\eta)dx},\\ &\frac{1}{p^+c^{p^-}}\min\Big\{\frac{\theta_1^{p^-}}{\int_\Omega F(x,\theta_1)dx}, \frac{\theta_4^{p^-}}{2\int_\Omega F(x,\theta_4)dx}\Big\}\Big)\end{aligned}$$

and for every non-negative continuous function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, there exists $\delta'_{\lambda,g} > 0$ given by (3.6) *such that, for each $\mu \in [0, \delta'_{\lambda,g})$, problem* (1.1) *has at least three weak solutions u_1, u_2 and u_3 such that $\max_{x\in\Omega}|u_1(x)| < \theta_1$, $\max_{x\in\Omega}|u_2(x)| < \frac{1}{\sqrt[p^-]{2}}\theta_4$ and $\max_{x\in\Omega}|u_3(x)| < \theta_4$.*

Proof. Choose $\theta_2 = \frac{1}{\sqrt[p^-]{2}}\theta_4$ and $\theta_3 = \theta_4$. So, from (A9) one has

$$\begin{aligned}\frac{\int_\Omega F(x,\theta_2)dx}{\theta_2^{p^-}} &= \frac{2\int_\Omega F(x,\frac{1}{\sqrt[p^-]{2}}\theta_4)dx}{\theta_4^{p^-}} \leq \frac{2\int_\Omega F(x,\theta_4)dx}{\theta_4^{p^-}}\\ &< \frac{p^-}{p^+\sigma + p^-}\frac{\int_{B(x^0,s_1)} F(x,\eta)dx}{\eta^{p^+}}\end{aligned} \tag{3.7}$$

and

$$\frac{\int_\Omega F(x,\theta_3)dx}{\theta_3^{p^-} - \theta_2^{p^-}} = \frac{2\int_\Omega F(x,\theta_4)dx}{\theta_4^{p^-}} < \frac{p^-}{p^+\sigma + p^-}\frac{\int_{B(x^0,s_1)} F(x,\eta)dx}{\eta^{p^+}}. \tag{3.8}$$

Moreover, since $\theta_1 < \eta^{p^+/p^-}$, from (A9) we have

$$\begin{aligned}&\frac{p^-}{p^+\sigma}\frac{\int_{B(x^0,s_1)} F(x,\eta)dx - \int_\Omega F(x,\theta_1)dx}{\eta^{p^+}}\\ &> \frac{p^-}{p^+\sigma}\frac{\int_{B(x^0,s_1)} F(x,\eta)dx}{\eta^{p^+}} - \frac{p^-}{p^+\sigma}\frac{\int_\Omega F(x,\theta_1)dx}{\theta_1^{p^-}}\end{aligned}$$

$$
\begin{aligned}
&> \frac{p^-}{p^+\sigma}\frac{\int_{B(x^0,s_1)}F(x,\eta)dx}{\eta^{p^+}} - \frac{(p^-)^2}{p^+\sigma(p^+\sigma+p^-)}\frac{\int_{B(x^0,s_1)}F(x,\eta)dx}{\eta^{p^+}}\\
&= \frac{p^-}{p^+\sigma+p^-}\frac{\int_{B(x^0,s_1)}F(x,\eta)dx}{\eta^{p^+}}.
\end{aligned}
$$

Hence, from (A9), (3.7) and (3.8), it is easy to observe that the assumption (A7) of Theorem 3.1 is satisfied, and it follows the conclusion. □

Remark 3.4. We observe that, in our results, no asymptotic conditions on f and g are needed and only algebraic conditions on f are imposed to guarantee the existence of solutions. Moreover, in the conclusions of the above results, one of the three solutions may be trivial since the values of $f(x,0)$ and $g(x,0)$ for $x \in \Omega$ are not determined.

Here, we want to point out a simple consequence of Theorem 3.3 when f does not depend upon x and $\mu = 0$. To be precise, consider the problem

$$
\begin{gathered}
\Delta^2_{p(x)}u = \lambda f(u(x)), \quad x \in \Omega,\\
u = \Delta u = 0, \quad x \in \partial\Omega
\end{gathered}
\tag{3.9}
$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continues function. Put

$$
F(t) = \int_0^t f(\xi)d\xi \quad \text{for } t \in \mathbb{R}.
$$

Theorem 3.5. *Let f be a non-negative and nonzero function such that*

$$
\lim_{t\to 0^+}\frac{f(t)}{|t|^{p^- -1}} = \lim_{t\to +\infty}\frac{f(t)}{|t|^{p^- -1}} = 0.
\tag{3.10}
$$

Then, for every $\lambda > \lambda^*$ where

$$
\lambda^* = \inf\Big\{\frac{(p^+\sigma+p^-)\eta^{p^+}}{p^-p^+c^{p^-}\,\mathrm{meas}(B(x^0,s_1))F(\eta)} : \eta \geq 1, F(\eta) > 0\Big\}
$$

problem (3.9) *has at least two non-trivial weak solutions.*

Proof. Fix $\lambda > \lambda^*$ and let $\eta \geq 1$ such that $F(\eta) > 0$ and

$$
\lambda > \frac{(p^+\sigma+p^-)\eta^{p^+}}{p^-p^+c^{p^-}\,\mathrm{meas}(B(x^0,s_1))F(\eta)}.
$$

From (3.10) there is $\theta_1 > 0$ such that

$$
\theta_1 < \min\{\eta^{p^+/p^-}, \rho^{\frac{1}{p^-}}\eta\} \text{ and } \frac{F(\theta_1)}{\theta_1^{p^-}} < \frac{1}{\lambda\,\mathrm{meas}(\Omega)p^+c^{p^-}},
$$

and $\theta_4 > 0$ such that

$$
\eta < \min\Big\{(\frac{p^+}{2\sigma p^-})^{\frac{1}{p^+}}\theta_4^{p^-/p^+}, \theta_4\Big\}, \quad \frac{F(\theta_4)}{\theta_4^{p^-}} < \frac{1}{2\lambda\,\mathrm{meas}(\Omega)p^+c^{p^-}}.
$$

Therefore, all assumptions of Theorem 3.3 are fulfilled and it ensures the conclusion. □

Finally, we present an example in which the hypotheses of Theorem 3.5 are satisfied.

Example 3.6. We consider the problem

$$\begin{gathered} \Delta^2_{p(x,y)} u = \lambda f(u), \quad (x,y) \in \Omega, \\ u = \Delta u = 0, \quad (x,y) \in \partial\Omega \end{gathered} \tag{3.11}$$

where $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2\}$, $p(x,y) = x^2 + y^2 + 2$ for $(x,y) \in \Omega$ and

$$f(t) = \begin{cases} 4t^3, & \text{if } t \leq 1, \\ \frac{4}{\sqrt{t}}, & \text{if } t > 1. \end{cases}$$

A direct calculation shows that

$$F(t) = \begin{cases} t^4, & \text{if } t \leq 1, \\ 8\sqrt{t} - 7, & \text{if } t > 1. \end{cases}$$

By simple calculations, we obtain $p^- = 2$ and $p^+ = 4$. Choosing $x_0 = 0$, $s_1 = 1$, $s_2 = 2$ and $\eta = 1$, we observe that all assumptions of Theorem 3.5 are fulfilled. Therefore, it follows that for every

$$\lambda > \frac{2^{26} \times 3^9 \pi c^2 + 2}{8\pi c^2},$$

problem (3.11) has at least two distinct non-trivial weak solutions.

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