Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 47, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

BOUNDEDNESS AND SQUARE INTEGRABILITY OF SOLUTIONS OF NONLINEAR FOURTH-ORDER DIFFERENTIAL EQUATIONS WITH BOUNDED DELAY

ERDAL KORKMAZ, CEMIL TUNÇ

Communicated by Mokhtar Kirane

Abstract. In this article, we give sufficient conditions for the boundedness, uniformly asymptotic stability and square integrability of the solutions to a fourth-order non-autonomous differential equation with bounded delay by using Lyapunov's second method.

1. Introduction

Ordinary differential equations have been studied for more than 300 years since the seventeenth century after the concepts of differentiation and integration were formulated by Newton and Leibniz. By means of ordinary differential equations, researchers can explain many natural phenomena like gravity, projectiles, wave, vibration, nuclear physics, and so on. In addition, in Newtonian mechanics, the system's state variable changes over time, and the law that governs the change of the system's state is normally described by an ordinary differential equation. The question concerning the stability of ordinary differential equations has been originally raised by the general problem of the stability of motion (Lyapunov [22]).

However, thereafter along with the development of technology, it is seen that the ordinary differential equations cannot respond to the needs arising in sciences and engineering. For example, in many applications, it can be seen that physical or biological background of modeling system shows that the change rate of the system's current status often depends not only on the current state but also on the history of the system. This usually leads to so-called retarded functional differential equations (Smith [33]).

To the best of our knowledge, the study of qualitative properties of functional differential equations of higher order has been developed at a high rate in the last four decades. Functional differential equations of higher order can serve as excellent tools for description of mathematical modeling of systems and processes in economy, stochastic processes, biomathematics, population dynamics, medicine, information theory, physics, chemistry, aerodynamics and many fields of engineering like atomic energy, control theory, mechanics, etc., Therefore, the investigation of

2010 *Mathematics Subject Classification.* 34D20, 34C11.
Key words and phrases. Stability; boundedness; Lyapunov functional; fourth order; delay differential equations; square integrability.

Submitted January 3, 2017. Published February 16, 2017.

the qualitative properties of solutions of functional differential equations of higher order, stability, boundedness, oscillation, integrability etc. of solutions play an important role in many real world phenomena related to the sciences and engineering technique fields. In fact, we would not like to give here the details of the applications related to functional differential equations of higher order here.

In particular, for more results on the stability, boundedness, convergence, etc. of ordinary or functional equations differential equations of fourth order, see the book of Reissig et al. [30] as a good survey for the works done by 1974 and the papers of Burton [6], Cartwright [7], Ezeilo [11, 12, 13, 14], Harrow [15, 16], Tunç [36, 37, 38, 39, 40, 41, 42], Remili et al. [25, 26, 27, 28, 29], Wu [44] and others and theirs references. These information indicate the importance of investigating the qualitative properties, of solutions of retarded functional differential equations of fourth order.

In this article, we study the uniformly asymptotic stability of the solutions for $p(t,x,x',x'',x''')\equiv 0$ and also square integrable and boundedness of solutions to the fourth order nonlinear differential equation with delay

$$\begin{aligned} &x^{(4)}+a(t)(g(x(t))x''(t))'+b(t)(q(x(t))x'(t))' \\ &+c(t)f(x(t))x'(t)+d(t)h(x(t-r(t)))=p(t,x,x',x'',x'''). \end{aligned} \tag{1.1}$$

For convenience, we let

$$\theta_1(t)=g'(x(t))x'(t),\quad \theta_2(t)=q'(x(t))x'(t),\quad \theta_3(t)=f'(x(t))x'(t).$$

We write (1.1) in the system form

$$\begin{aligned} x' &= y,\\ y' &= z,\\ z' &= w,\\ w' &= -a(t)g(x)w-(b(t)q(x)+a(t)\theta_1)z-(b(t)\theta_2+c(t)f(x))y \\ &\quad -d(t)h(x)+d(t)\int_{t-r(t)}^{t}h'(x)y\,d\eta+p(t,x,y,z,w), \end{aligned} \tag{1.2}$$

where r is a bounded delay, $0\le r(t)\le\psi$, $r'(t)\le\xi$, $0<\xi<1$, ξ and ψ some positive constants, ψ which will be determined later, the functions a,b,c,d are continuously differentiable functions and the functions f,h,g,q,p are continuous functions depending only on the arguments shown. Also derivatives $g'(x),q'(x),f'(x)$ and $h'(x)$ exist and are continuous. The continuity of the functions a,b,c,d,p,g,g',q,q',f,h guarantees the existence of the solutions of equation (1.1). If the right hand side of the system (1.2) satisfies a Lipchitz condition in $x(t),y(t),z(t),w(t)$ and $x(t-r)$,and exists of solutions of system (1.2) , then it is unique solution of system (1.2).

Assume that there are positive constants a_0, b_0, c_0, d_0, f_0, g_0, q_0, a_1, b_1, c_1, d_1, f_1, g_1, q_1, m, M, δ, η_1 such that the following assumptions hold:

- (A1) $0<a_0\le a(t)\le a_1$, $0<b_0\le b(t)\le b_1$, $0<c_0\le c(t)\le c_1$, $0<d_0\le d(t)\le d_1$ for $t\ge 0$;
- (A2) $0<f_0\le f(x)\le f_1$, $0<g_0\le g(x)\le g_1$, $0<q_0\le q(x)\le q_1$ for $x\in R$ and $0<m<\min\{f_0,g_0,1\}$, $M>\max\{f_1,g_1,1\}$;
- (A3) $\frac{h(x)}{x}\ge\delta>0$ for $x\ne 0$, $h(0)=0$;
- (A4) $\int_0^{\infty}(|a'(t)|+|b'(t)|+|c'(t)|+|d'(t)|)dt<\eta_1$;
- (A5) $|p(t,x,y,z,w)|\le|e(t)|$.

Motivated by the results of references, we obtain some new results on the uniformly asymptotic stability and boundedness of the solutions by means of the Lyapunov's functional approach. Our results differ from that obtained in the literature (see, the references in this article and the references therein). By this way, we mean that this paper has a contribution to the subject in the literature, and it may be useful for researchers working on the qualitative behaviors of solutions of functional differential equations of higher order. In view of all the mentioned information, it can be checked the novelty and originality of the current paper.

2. Preliminaries

We also consider the functional differential equation

$$\dot{x} = f(t, x_t), \quad x_t(\theta) = x(t+\theta), \quad -r \le \theta \le 0, \quad t \ge 0. \tag{2.1}$$

where $f : I \times C_H \to \mathbb{R}^n$ is a continuous mapping, $f(t,0) = 0$, $C_H := \{\phi \in (C[-r,0], \mathbb{R}^n) : \|\phi\|t \le H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(t,\phi)| < L(H_1)$ when $\|\phi\|t < H_1$.

Lemma 2.1 ([6]). *Let $V(t,\phi) : I \times C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipchitz condition, $V(t,0) = 0$, and wedges W_i such that:*

- (i) $W_1(\|\phi\|t) \le V(t,\phi) \le W_2(\|\phi\|t)$;
- (ii) $V'_{(2.1)}(t,\phi) \le -W_3(\|\phi\|t)$.

Then, the zero solution of (2.1) is uniformly asymptotically stable.

3. Main results

Lemma 3.1 ([19]). *Let $h(0) = 0$, $xh(x) > 0$ $(x \ne 0)$ and $\delta(t) - h'(x) \ge 0$ $(\delta(t) > 0)$, then $2\delta(t)H(x) \ge h^2(x)$, where $H(x) = \int_0^x h(s)ds$.*

Theorem 3.2. *In addition to the basic assumptions imposed on the functions a, b, c, d, p, f, h, g, q suppose that there are positive constants h_0, h_1, δ_0, δ_1, η_2, η_3 such that the following conditions are satisfied:*

- (i) $h_0 - \frac{a_0 m \delta_0}{d_1} \le h'(x) \le \frac{h_0}{2}$ *for* $x \in R$;
- (ii) $\delta_1 = \frac{d_1 h_0 a_1 M}{c_0 m} + \frac{c_1 M + \delta_0}{a_0 m} < b_0 q_0$;
- (iii) $\int_{-\infty}^{+\infty} (|g'(s)| + |q'(s)| + |f'(s)|)ds < \eta_2$;
- (iv) $\int_0^\infty |e(t)|dt < \eta_3$.

Then any solution $x(t)$ of (1.1) and its derivatives $x'(t), x''(t)$ $x'''(t)$ are bounded and satisfy

$$\int_0^\infty (x'^2(s) + x''^2(s) + x'''^2(s))ds < \infty,$$

provided that

$$\psi < \frac{(1-\xi)}{d_1 h_1} \min\Big\{\frac{\varepsilon c_0 f_0}{\alpha + \beta(2-\xi) + 1}, \frac{2[b_0 q_0 - \delta_1 - \varepsilon M(a_1 + c_1)]}{(1-\xi)}, \frac{2\varepsilon}{\alpha(1-\xi)}\Big\}.$$

Proof. We define a Lyapunov functional

$$W = W(t,x,y,z,w) = e^{\frac{-1}{\eta}\int_0^t \gamma(s)ds} V, \tag{3.1}$$

where

$$\gamma(t) = |a'(t)| + |b'(t)| + |c'(t)| + |d'(t)| + |\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)|,$$

and

$$\begin{aligned}2V &= 2\beta d(t)H(x) + c(t)f(x)y^2 + \alpha b(t)q(x)z^2 + a(t)g(x)z^2 + 2\beta a(t)g(x)yz \\ &\quad + [\beta b(t)q(x) - \alpha h_0 d(t)]y^2 - \beta z^2 + \alpha w^2 + 2d(t)h(x)y + 2\alpha d(t)h(x)z \\ &\quad + 2\alpha c(t)f(x)yz + 2\beta yw + 2zw + \sigma \int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\gamma)d\gamma ds\end{aligned}$$

with $H(x) = \int_0^x h(s)ds$, $\alpha = \frac{1}{a_0 m} + \varepsilon$, $\beta = \frac{d_1 h_0}{c_0 m} + \varepsilon$, ε and η are positive constants to be determined later in the proof. We can rearrange $2V$ as

$$\begin{aligned}2V &= a(t)g(x)\Big[\frac{w}{a(t)g(x)} + z + \beta y\Big]^2 + c(t)f(x)\Big[\frac{d(t)h(x)}{c(t)f(x)} + y + \alpha z\Big]^2 \\ &\quad + \frac{d^2(t)h^2(x)}{c(t)f(x)} + 2\varepsilon d(t)H(x) + \sigma\int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\gamma)d\gamma ds + V_1 + V_2 + V_3\,,\end{aligned}$$

where

$$\begin{gathered}V_1 = 2d(t)\int_0^x h(s)\Big[\frac{d_1 h_0}{c_0 m} - 2\frac{d(t)}{c(t)f(x)}h'(s)\Big]ds, \\ V_2 = \big[\alpha b(t)q(x) - \beta - \alpha^2 c(t)f(x)\big]z^2, \\ V_3 = \big[\beta b(t)q(x) - \alpha h_0 d(t) - \beta^2 a(t)g(x)\big]y^2 + \big[\alpha - \frac{1}{a(t)g(x)}\big]w^2.\end{gathered}$$

Let

$$\varepsilon < \min\Big\{\frac{1}{a_0 m}, \frac{d_1 h_0}{c_0 m}, \frac{b_0 q_0 - \delta_1}{M(a_1 + c_1)}\Big\} \tag{3.2}$$

then

$$\frac{1}{a_0 m} < \alpha < \frac{2}{a_0 m}, \quad \frac{d_1 h_0}{c_0 m} < \beta < 2\frac{d_1 h_0}{c_0 m}. \tag{3.3}$$

By using conditions (A1)–(A3), (i)–(ii) and inequalities (3.2), (3.3) we obtain

$$\begin{aligned}V_1 &\geq 4d(t)\frac{d_1}{c_0 m}\int_0^x h(s)[\frac{h_0}{2} - h'(s)]ds \geq 0, \\ V_2 &= (\alpha(b(t)q(x) - \beta a(t) - \alpha c(t)f(x)) + \beta(\alpha a(t) - 1))z^2 \\ &\geq \alpha\Big(b_0 q_0 - \frac{d_1 h_0 a_1}{c_0 m} - \frac{c_1 M}{a_0 m} - \varepsilon(a_1 + c_1 M)\Big)z^2 + \beta(\frac{1}{m} - 1)z^2 \\ &\geq \alpha(b_0 q_0 - \delta_1 - \varepsilon M(a_1 + c_1))z^2 \geq 0,\end{aligned}$$

and

$$\begin{aligned}V_3 &\geq \beta\Big(b_0 q_0 - \frac{\alpha}{\beta}h_0 d_1 - \beta a_1 M\Big)y^2 + (\alpha - \frac{1}{a_0 m})w^2 \\ &\geq \beta\Big(b_0 q_0 - \frac{c_0}{a_0} - a_1\frac{d_1 h_0 M}{c_0 m} - \varepsilon(c_0 m + a_1 M)\Big)y^2 + \varepsilon w^2 \\ &\geq \beta(b_0 q_0 - \delta_1 - \varepsilon M(c_1 + a_1))y^2 + \varepsilon w^2 \geq 0.\end{aligned}$$

Thus, it is clear from the above inequalities that there exists positive constant D_0 such that

$$2V \geq D_0(y^2 + z^2 + w^2 + H(x)). \tag{3.4}$$

From Lemma 3.1, (A3) and (i), it follows that there is a positive constant D_1 such that

$$2V \geq D_1(x^2 + y^2 + z^2 + w^2) \tag{3.5}$$

In this way V is positive definite. From (A1)–(A3), it is clear that there is a positive constant U_1 such that

$$V \le U_1(x^2 + y^2 + z^2 + w^2). \tag{3.6}$$

From (iii), we have

$$\begin{aligned}
&\int_0^t (|\theta_1(s)| + |\theta_2(s)| + |\theta_3(s)|)ds \\
&= \int_{\alpha_1(t)}^{\alpha_2(t)} (|g'(u)| + |q'(u)| + |f'(u)|)du \\
&\le \int_{-\infty}^{+\infty} (|g'(u)| + |q'(u)| + |f'(u)|)du \quad < \eta_2 < \infty
\end{aligned} \tag{3.7}$$

where $\alpha_1(t) = \min\{x(0), x(t)\}$ and $\alpha_2(t) = \max\{x(0), x(t)\}$. From inequalities (3.2), (3.6) and (3.7), it follows that

$$W \ge D_2(x^2 + y^2 + z^2 + w^2) \tag{3.8}$$

where $D_2 = \frac{D_1}{2} e^{-\frac{\eta_1+\eta_2}{\eta}}$. Also, it is easy to see that there is a positive constant U_2 such that

$$W \le U_2(x^2 + y^2 + z^2 + w^2) \tag{3.9}$$

for all x, y, z, w and all $t \ge 0$.

Now, we show that $\dot{W}$ is negative definite function. The derivative of the function V along any solution $(x(t), y(t), z(t), w(t))$ of system (1.2), with respect to t is after simplifying

$$2\dot{V}_{(1.2)} = -2\varepsilon c(t)f(x)y^2 + V_4 + V_5 + V_6 + V_7 + V_8 + V_9 + 2(\beta y + z + \alpha w)p(t, x, y, z, w)$$

where

$$\begin{gathered}
V_4 = -2\Big(\frac{d_1 h_0}{c_0 m} c(t)f(x) - d(t)h'(x)\Big)y^2 - 2\alpha d(t)(h_0 - h'(x))yz, \\
V_5 = -2(b(t)q(x) - \alpha c(t)f(x) - \beta a(t)g(x))z^2, \\
V_6 = -2(\alpha a(t)g(x) - 1)w^2,
\end{gathered}$$

$$\begin{aligned}
V_7 &= 2\alpha d(t)w \int_{t-r(t)}^t h'(x(\eta))x'(\eta)d\eta + 2\beta d(t)y(t) \int_{t-r(t)}^t h'(x(\eta))x'(\eta)d\eta \\
&\quad + 2d(t)z(t) \int_{t-r(t)}^t h'(x(\eta))x'(\eta)d\eta + \sigma r(t)y^2(t) - \sigma(1 - r'(t)) \int_{t-r(t)}^t y^2(\eta)d\eta,
\end{aligned}$$

$$\begin{aligned}
V_8 &= -a(t)\theta_1(z^2 + 2\alpha zw) - b(t)\theta_2(\alpha z^2 + 2\alpha zw + \beta y^2 + 2yz) \\
&\quad + c(t)\theta_3(y^2 + 2\alpha yz), \\
V_9 &= d'(t)[2\beta H(x) - \alpha h_0 y^2 + 2h(x)y + 2\alpha h(x)z] \\
&\quad + c'(t)[f(x)y^2 + 2\alpha f(x)yz] + b'(t)[\alpha q(x)z^2 + \beta q(x)y^2] \\
&\quad + a'(t)[g(x)z^2 + 2\beta g(x)yz].
\end{aligned}$$

By regarding conditions (A1), (A2), (i), (ii) and inequalities (3.3), (3.4), we have

$$\begin{aligned}
V_4 &\le -2[d(t)h_0 - d(t)h'(x)]y^2 - 2\alpha d(t)[h_0 - h'(x)]yz \\
&\le -2d(t)[h_0 - h'(x)]y^2 - 2\alpha d(t)[h_0 - h'(x)]yz \\
&\le 2d(t)[h_0 - h'(x)][(y + \frac{\alpha}{2}z)^2 - (\frac{\alpha}{2}z)^2]
\end{aligned}$$

$$\leq \frac{\alpha^2}{2}d(t)[h_0 - h'(x)]z^2.$$

In this case,

$$\begin{aligned}
V_4 + V_5 &\leq -2\big[b(t)q(x) - \alpha c(t)f(x) - \beta a(t)g(x) - \frac{\alpha^2}{4}d(t)[h_0 - h'(x)]\big]z^2 \\
&\leq -2\big[b_0q_0 - (\frac{1}{a_0m} + \varepsilon)c_1M - (\frac{d_1h_0}{c_0m} + \varepsilon)a_1M - \frac{\alpha^2}{4}(a_0m\delta_0)\big]z^2 \\
&\leq -2\big[b_0q_0 - \frac{M}{a_0m}c_1 - \frac{d_1h_0a_1M}{c_0m} - \frac{\delta_0}{a_0m} - \varepsilon M(a_1 + c_1)\big]z^2 \\
&\leq -2[b_0q_0 - \delta_1 - \varepsilon M(a_1 + c_1)]z^2 \leq 0,
\end{aligned}$$

and

$$V_6 \leq -2[\alpha a_0m - 1]w^2 = -2\varepsilon w^2 \leq 0.$$

By taking $h_1 = \max\{|\frac{d_1h_0 - a_0m\delta_0}{d_1}|, |\frac{h_0}{2}|\}$, we have

$$V_7 \leq d_1h_1r(t)(\alpha w^2 + \beta y^2 + z^2) + \sigma r(t)y^2 + [d_1h_1(\alpha+\beta+1) - \sigma(1-\xi)]\int_{t-r(t)}^{t} y^2(s)ds$$

If we choose $\sigma = \frac{d_1h_1(\alpha+\beta+1)}{(1-\xi)}$, we obtain

$$V_7 \leq \frac{d_1h_1}{(1-\xi)}r(t)[\alpha(1-\xi)w^2 + (\alpha + \beta(2-\xi) + 1)y^2 + (1-\xi)z^2].$$

Thus, there exists a positive constant D_3 such that

$$-\varepsilon c(t)f(x)y^2 + V_4 + V_5 + V_6 + V_7 \leq -2D_3(y^2 + z^2 + w^2).$$

From (3.4), and the Cauchy Schwartz inequality, we obtain

$$\begin{aligned}
V_8 &\leq a(t)|\theta_1|(z^2 + \alpha(z^2 + w^2)) + b(t)|\theta_2|(\alpha z^2 + \alpha(z^2 + w^2) + \beta y^2 + y^2 + z^2) \\
&\quad + c(t)|\theta_3|(y^2 + \alpha(y^2 + z^2)) \\
&\leq \lambda_1(|\theta_1| + |\theta_2| + |\theta_3|)(y^2 + z^2 + w^2 + H(x)) \\
&\leq 2\frac{\lambda_1}{D_0}(|\theta_1| + |\theta_2| + |\theta_3|)V,
\end{aligned}$$

where $\lambda_1 = \max\{a_1(1+\alpha), b_1(1+2\alpha+\beta), c_1(1+\alpha)\}$. Using condition (iii) and Lemma 3.1, we can write

$$h^2(x) \leq h_0H(x),$$

hereby,

$$\begin{aligned}
|V_9| &\leq |d'(t)|[2\beta H(x) + \alpha h_0y^2 + h^2(x) + y^2 + \alpha(h^2(x) + z^2)] \\
&\quad + |c'(t)|[y^2 + \alpha(y^2 + z^2)] + |b'(t)|[\alpha z^2 + \beta y^2] \\
&\quad + |a'(t)|[z^2 + 2\beta(y^2 + z^2)] \\
&\leq \lambda_2[|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|](y^2 + z^2 + w^2 + H(x)) \\
&\leq 2\frac{\lambda_2}{D_0}[|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|]V,
\end{aligned}$$

such that $\lambda_2 = \max\{2\beta+(\alpha+1)h_0, \alpha h_0+1, \alpha+1\}$. By taking $\frac{1}{\eta} = \frac{1}{D_0}\max\{\lambda_1, \lambda_2\}$, we obtain

$$\begin{aligned}\dot{V}_{(1.2)} &\le -D_3(y^2+z^2+w^2) + (\beta y + z + \alpha w)p(t,x,y,z,w)\\ &\quad + \frac{1}{\eta}(|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)| + |\theta_1| + |\theta_2| + |\theta_3|)V.\end{aligned} \tag{3.10}$$

From (A4), (A5),(iii), (3.7), (3.8), (3.10) and the Cauchy Schwartz inequality, we obtain

$$\begin{aligned}\dot{W}_{(1.2)} &= \Big(\dot{V}_{(1.2)} - \frac{1}{\eta}\gamma(t)V\Big)e^{-\frac{1}{\eta}\int_0^t \gamma(s)ds}\\ &\le (-D_3(y^2+z^2+w^2) + (\beta y + z + \alpha w)p(t,x,y,z,w))e^{-\frac{1}{\eta}\int_0^t \gamma(s)ds} \quad (3.11)\\ &\le (\beta|y| + |z| + \alpha|w|)|p(t,x,y,z,w)|\\ &\le D_4(|y|+|z|+|w|)|e(t)|\\ &\le D_4(3+y^2+z^2+w^2)|e(t)|\\ &\le D_4\big(3 + \frac{1}{D_2}W\big)|e(t)|\\ &\le 3D_4|e(t)| + \frac{D_4}{D_2}W|e(t)|,\end{aligned} \tag{3.12}$$

where $D_4 = \max\{\alpha, \beta, 1\}$. Integrating (3.12) from 0 to t and using the condition (iv) and the Gronwall inequality, we have

$$\begin{aligned}W &\le W(0,x(0),y(0),z(0),w(0)) + 3D_4\eta_3\\ &\quad + \frac{D_4}{D_2}\int_0^t W(s,x(s),y(s),z(s),w(s))|e(s)|ds\\ &\le (W(0,x(0),y(0),z(0),w(0)) + 3D_4\eta_3)e^{\frac{D_4}{D_2}\int_0^t |e(s)|ds}\\ &\le (W(0,x(0),y(0),z(0),w(0)) + 3D_4\eta_3)e^{\frac{D_4}{D_2}\eta_3} = K_1 < \infty\end{aligned} \tag{3.13}$$

Because of inequalities (3.8) and (3.13), we write

$$(x^2+y^2+z^2+w^2) \le \frac{1}{D_2}W \le K_2, \tag{3.14}$$

where $K_2 = \frac{K_1}{D_2}$. Clearly (3.14) implies

$$|x(t)| \le \sqrt{K_2}, \quad |y(t)| \le \sqrt{K_2}, \quad |z(t)| \le \sqrt{K_2}, \quad |w(t)| \le \sqrt{K_2} \quad \text{for } t \ge 0.$$

Hence

$$|x(t)| \le \sqrt{K_2}, \quad |x'(t)| \le \sqrt{K_2}, \quad |x''(t)| \le \sqrt{K_2}, |x'''(t)| \le \sqrt{K_2} \tag{3.15}$$

for $t \ge 0$. Now, we proof the square integrability of solutions and their derivatives. We define

$$F_t = F(t,x(t),y(t),z(t),w(t)) = W + \rho\int_0^t (y^2(s)+z^2(s)+w^2(s))ds,$$

where $\rho > 0$. It is easy to see that F_t is positive definite, since $W = W(t,x,y,z,w)$ is already positive definite. Using the following estimate

$$e^{-\frac{\eta_1+\eta_2}{\eta}} \le e^{-\frac{1}{\eta}\int_0^t \gamma(s)ds} \le 1$$

by (3.12) we have

$$\begin{aligned}\dot{F}_{t(1.2)} &\le -D_3(y^2(t)+z^2(t)+w^2(t))e^{-\frac{\eta_1+\eta_2}{\eta}} \\ &\quad + D_4(|y(t)|+|z(t)|+|w(t)|)|e(t)| \\ &\quad + \rho(y^2(t)+z^2(t)+w^2(t))\end{aligned} \tag{3.16}$$

By choosing $\rho = D_3 e^{-\frac{\eta_1+\eta_2}{\eta}}$ we obtain

$$\begin{aligned}\dot{F}_{t(1.2)} &\le D_4(3+y^2(t)+z^2(t)+w^2(t))|e(t)| \\ &\le D_4(3+\frac{1}{D_2}W)|e(t)| \\ &\le 3D_4|e(t)| + \frac{D_4}{D_2}F_t|e(t)|.\end{aligned} \tag{3.17}$$

Integrating from 0 to t and using again the Gronwall inequality and the condition (iv), we obtain

$$\begin{aligned}F_t &\le F_0 + 3D_4\eta_3 + \frac{D_4}{D_2}\int_0^t F_s|e(s)|ds \\ &\le (F_0+3D_4\eta_3)e^{\frac{D_4}{D_2}\int_0^t|e(s)|ds} \\ &\le (F_0+3D_4\eta_3)e^{\frac{D_4}{D_2}\eta_3} = K_3 < \infty\end{aligned} \tag{3.18}$$

Therefore,

$$\int_0^\infty y^2(s)ds < K_3, \quad \int_0^\infty z^2(s)ds < K_3, \quad \int_0^\infty w^2(s)ds < K_3,$$

which implies

$$\int_0^\infty [x'(s)]^2 ds < K_3, \quad \int_0^\infty [x''(s)]^2 ds < K_3, \quad \int_0^\infty [x'''(s)]^2 ds < K_3. \tag{3.19}$$

which completes the proof. □

Remark 3.3. If $p(t,x,y,z,w) \equiv 0$, similarly to the above proof, the inequality (3.11) becomes

$$\begin{aligned}\dot{W}_{(1.2)} &= \Big(\dot{V}_{(1.2)} - \frac{1}{\eta}\gamma(t)V\Big)e^{-\frac{1}{\eta}\int_0^t\gamma(s)ds} \\ &\le -D_3(y^2+z^2+w^2)e^{-\frac{1}{\eta}\int_0^t\gamma(s)ds} \\ &\le -\mu(y^2+z^2+w^2),\end{aligned}$$

where $\mu = D_3 e^{-\frac{\eta_1+\eta_2}{\eta}}$. It can also be observed that the only solution of system (1.2) for which $\dot{W}_{(1.2)}(t,x,y,z,w) = 0$ is the solution $x = y = z = w = 0$. The above discussion guarantees that the trivial solution of equation (1.1) is uniformly asymptotically stable, and the same conclusion as in the proof of theorem can be drawn for square integrability of solutions of equation (1.1).

Example 3.4. We consider the fourth-order nonlinear differential equation with delay

$$
\begin{aligned}
&x^{(4)} + (e^{-2t}\sin 3t + 2)\Big(\big(\frac{5x + 2e^{x} + 2e^{-x}}{e^{x} + e^{-x}}\big)x''\Big)' \\
&+ \Big(\frac{\sin 2t + 11t^2 + 11}{t^2 + 1}\Big)\Big(\big(\frac{\sin x + 9e^{x} + 9e^{-x}}{e^{x} + e^{-x}}\big)x'\Big)' \\
&+ \big(e^{-t}\sin t + 3\big)\big(\frac{x\cos x + x^4 + 1}{x^4 + 1}\big)x' \\
&+ \big(\frac{\sin^2 t + t^2 + 1}{5t^2 + 5}\big)\Big(\frac{x(t - \frac{1}{e^t + 15})}{x^2(t - \frac{1}{e^t + 15}) + 1}\Big) \\
&= \frac{2\sin t}{t^2 + 1 + (xx'x'')^2 + (x''')^2}
\end{aligned}
\tag{3.20}
$$

by taking

$$
g(x) = \frac{5x + 2e^{x} + 2e^{-x}}{e^{x} + e^{-x}}, \quad q(x) = \frac{\sin x + 9e^{x} + 9e^{-x}}{e^{x} + e^{-x}}, \quad f(x) = \frac{x\cos x + x^4 + 1}{x^4 + 1},
$$
$$
h(x) = \frac{x}{x^2 + 1}, a(t) = e^{-2t}\sin 3t + 2, \quad b(t) = \frac{\sin 2t + 11t^2 + 11}{t^2 + 1},
$$
$$
c(t) = e^{-t}\sin t + 3, \quad d(t) = \frac{\sin^2 t + t^2 + 1}{5t^2 + 5}, \quad r(t) = \frac{1}{e^t + 15},
$$
$$
p(t, x, x'x'', x''') = \frac{2\sin t}{t^2 + 1 + (xx'x'')^2 + (x''')^2}.
$$

We obtain $g_0 = 0.33$, $g_1 = 3.7$, $f_0 = 0.5$, $f_1 = 1.5$, $q_0 = 8.5$, $q_1 = 9.5$, $a_0 = 1$, $a_1 = 3$, $b_0 = 10$, $b_1 = 12$, $c_0 = 2$, $c_1 = 4$, $d_0 = 0.2$, $d_1 = 0.3$, $m = 0.3$, $M = 3.8$, $h_0 = 2$, $\alpha = \frac{23}{6}$, $\beta = \frac{3}{2}$, $\delta_0 = \frac{17}{8}$ and $\delta_1 = 69.15$. Also we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty} |g'(x)|dx \\
&= 5\int_{-\infty}^{\infty} \Big|\frac{1}{e^{x} + e^{-x}} + x\frac{e^{-x} - e^{x}}{(e^{x} + e^{-x})^2}\Big|dx \\
&\le 5\int_{-\infty}^{0} \Big|\frac{1}{e^{x} + e^{-x}} - x\frac{e^{-x} - e^{x}}{(e^{x} + e^{-x})^2}\Big|dx + 5\int_{0}^{\infty} \Big|\frac{1}{e^{x} + e^{-x}} - x\frac{e^{-x} - e^{x}}{(e^{x} + e^{-x})^2}\Big|dx \\
&= 5\pi,
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} |q'(x)|dx &= \int_{-\infty}^{\infty} \Big|\frac{(e^{x} + e^{-x})\cos x - (e^{x} - e^{-x})\sin x}{(e^{x} + e^{-x})^2}\Big|dx \\
&\le \int_{-\infty}^{\infty} \Big|\frac{1}{e^{x} + e^{-x}} + x\frac{e^{x} - e^{-x}}{(e^{x} + e^{-x})^2}\Big|dx = \pi,
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\infty}^{\infty} |f'(x)|dx &= \int_{-\infty}^{\infty} \Big|\frac{\cos x}{x^4 + 1} - 4x^4\frac{\cos x}{(x^4 + 1)^2} + -x\frac{\sin x}{x^4 + 1}\Big|dx \\
&\le \int_{-\infty}^{\infty} \Big|\frac{5}{x^4 + 1} + \frac{x^2}{x^4 + 1}\Big|dx = 6\sqrt{2}\pi,
\end{aligned}
$$

$$
\int_{0}^{\infty} |p(t, x, x', x'', x''')|dt = \int_{0}^{\infty} \Big|\frac{2\sin t}{t^2 + 1 + (xx'x'')^2 + (x''')^2}\Big|dt
$$

$$
\begin{aligned}
&\le \int_0^\infty \Big|\frac{2\sin t}{t^2+1}\Big| dt \\
&\le \int_0^\infty \frac{2}{t^2+1} dt = \pi,
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty |a'(t)|dt &= \int_0^\infty \Big| -2e^{-2t}\sin 3t + 3e^{-2t}\cos 3t\Big| dt \\
&\le \int_0^\infty 5e^{-2t} dt = \frac{5}{2},
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty |b'(t)|dt &= \int_0^\infty \Big|\frac{2\cos 2t}{t^2+1} - 2t\frac{\sin 2t}{(t^2+1)^2}\Big| dt \\
&\le \int_0^\infty \frac{3}{t^2+1} dt = \frac{3\pi}{2},
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty |c'(t)|dt &= \int_0^\infty | -e^{-t}\sin t + e^{-t}\cos t| dt \\
&\le \int_0^\infty 2e^{-t} dt = 2,
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty |d'(t)|dt &= \int_0^\infty \Big|\frac{2\sin t\cos t}{5t^2+5} - 2t\frac{\sin^2 t}{(5t^2+5)^2}\Big| dt \\
&\le \frac{11}{25}\int_0^\infty \frac{1}{t^2+1} dt = \frac{11\pi}{50}.
\end{aligned}
$$

Consequently

$$
\begin{gathered}
\int_{-\infty}^{+\infty} (|g'(s)| + |q'(s)| + |f'(s)|) ds < \infty, \\
\int_0^\infty (|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|) dt < \infty.
\end{gathered}
$$

Thus all the assumptions of Theorem 3.2 hold, this shows that every solutions of equation (3.20) are bounded and derivatives of solutions are square integrable.

Conclusion. A class of nonlinear retarded functional differential equations of fourth order is considered. Sufficient conditions are established guaranteeing the uniformly asymptotic stability of the solutions for $p(t, x, x', x'', x''') \equiv 0$ and also square integrable and boundedness of solutions of equation (1.1) with delay. In the proofs of the main results, we benefit from the Lyapunov functional approach. The results obtained essentially improve, include and complement the results in the literature. An example is furnished to illustrate the hypotheses by MATLAB-Simulink.

The asymptotically stability of the null solution for the mentioned differential equation is shown by the following graph.

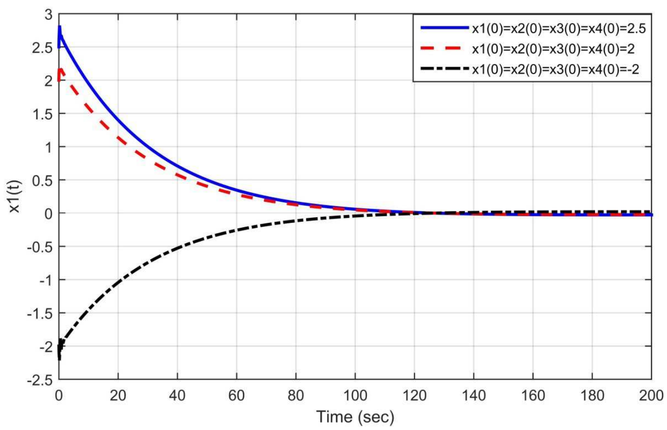

Figure 1. Trajectory of $x_1(t)$ for Example.

The boundedness of all the solutions for the mentioned differential equation is shown by the following graph.

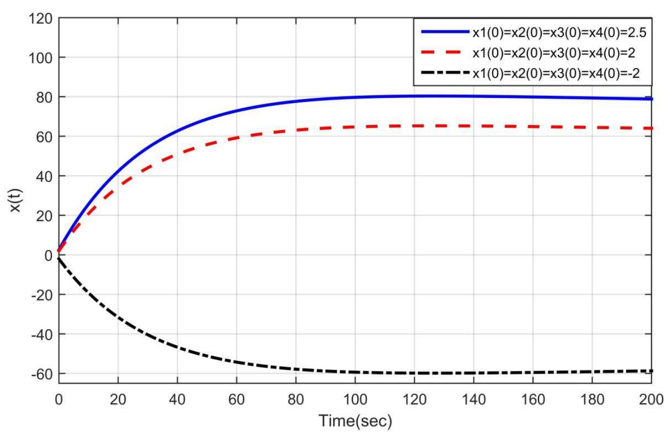

Figure 2. Trajectory of $x(t)$ for Example.

References

[1] Abou-El-Ela, A. M. A.; Sadek, A. I.; *A stability result for certain fourth order differentialequations.* Ann. Differential Equations 6 (1) (1990), 1–9.

[2] Adesina, O. A.; Ogundare, B. S.; *Some new stability and boundedness results on a certainfourth order nonlinear differential equation.* Nonlinear Studies 19 (3) (2012), 359–369.

[3] Andres, J.; Vlcek, V.; *On the existence of square integrable solutions and their derivatives to fourth and fifth order differential equations.* Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica 28 (1) (1989), 65–86.

[4] Barbashin, E. A.; *The construction of Liapunov function. Differ. Uravn.*, 4 (1968), 2127–2158.

[5] Bereketoglu, H.; *Asymptotic stability in a fourth order delay differential equation*. Dynam. Systems Appl., 7 (1) (1998) ,105–115.

[6] Burton, T. A.; *Stability and periodic solutions of ordinary and functional differential equations*. Mathematics in science and engineering, Volume 178, Academic Press, Inc, (1985).

[7] Cartwright, M. L.; *On the stability of solutions of certain differential equations of the fourth order*. Quart. J. Mech. Appl. Math., 9 (1956), 185–194.

[8] Chin, P. S. M.; *Stability results for the solutions of certain fourth-order autonomous differential equations*. Internat. J. Control., 49 (4) (1989), 1163–1173.

[9] Chukwu, E. N.; *On the stability of a nonhomogeneous differential equation of the fourth order*. Ann. Mat. Pura Appl., 92 (4) (1972), 1–11.

[10] Elsgolts, L.; *Introduction to the Theory of Differential Equations with Deviating Arguments*. Translated from the Russian by Robert J. McLaughlin Holden-Day, Inc. San Francisco, Calif.-London-Amsterdam, 1966.

[11] Ezeilo, J. O. C.; *A stability result for solutions of a certain fourth order differential equation*. J. London Math. Soc. 37 (1962), 28–32.

[12] Ezeilo, J. O. C.; *On the boundedness and the stability of solutions of some differential equations of the fourth order*. J. Math. Anal. Appl., 5 (1962), 136–146.

[13] Ezeilo, J. O. C.; *Stability results for the solutions of some third and fourth order differential equations*. Ann. Mat. Pura Appl., 66 (4) (1964), 233–249.

[14] Ezeilo, J. O. C.; Tejumola, H. O.; *On the boundedness and the stability properties of solutions of certain fourth order differential equations*. Ann. Mat. Pura Appl., 95 (4) (1973), 131–145.

[15] Harrow, M.; *A Stability result for Solutions of a Certain Fourth Order Homogeneous Differential Equations*, J. London Math. Soc. 42, pp. 51-56, (1967).

[16] Harrow, M.; *Further results on the boundedness and the stability of solutions of some differential equations of the fourth order*. SIAM J. Math. Anal., 1 (1970), 189–194.

[17] Hu, Chao Yang; *The stability in the large for certain fourth order differential equations*. Ann. Differential Equations, 8 (4) (1992), 422–428.

[18] Hara, T.; *On the asymptotic behavior of the solutions of some third and fourth order nonautonomous differential equations*. Publ. RIMS, Kyoto Univ., 9 (1974), 649–673.

[19] Hara, T.; *On the asymptotic behavior of solutions of some third order ordinary differential equations*. Proc. Japan Acad., 47 (1971).

[20] Korkmaz, E.; Tunç, C.; *On some qualitative behaviors of certain differential equations of fourth order with multiple retardations*. Journal of Applied Analysis and Computation, Volume 6, Number 2, (2016), 336–349.

[21] Krasovskii, N. N.; *On the stability in the large of the solution of a nonlinear system of differential equations* (in Russian). Prikl. Mat. Meh. 18 (1954) 735–737.

[22] Lyapunov, A. M.; *The general problem of the stability of motion*. Translated from Edouard Davaux's French translation (1907) of the 1892 Russian original and edited by A. T. Fuller. Taylor & Francis, Ltd., London, 1992.

[23] Omeike, P. S. M.; *Boundedness Of Solutions To Fourth-Order Differential Equation With Oscillatory Restoring And Forcing Terms*. Electronic Journal of Differential Equations, 2007, (104) (2007), 1–8.

[24] Rauch, L. L.; *Oscillations of a third order nonlinear autonomous system*. Contributions to the Theory of Nonlinear Oscillations. Ann. Math. Stud., 20 (1950), 39–88.

[25] Remili, M.; Oudjedi, D. L.; *Uniform stability and boundedness of a kind of third order delay differential equations*. Bull. Comput. Appl. Math., 2 (1) (2014), 25–35.

[26] Remili, M.; Oudjedi, D. L.; *Stability and boundedness of the solutions of non autonomous third order differential equations with delay*. Cta Univ. Palacki. Olomuc., Fac. Rer. Nat., Mathematica 53 (2) (2014), 139–147.

[27] Remili, M.; Beldjerd, D.; *On the asymptotic behavior of the solutions of third order delay differential equations*. Rend. Circ. Mat. Palermo, 63 (3) (2014), 447–455.

[28] Remili, M.; Rahmane, M.; *Sufficient conditions for the boundedness and square integrability of solutions of fourth-order differential equations*. Proyecciones Journal of Mathematics., Vol 35, No 1, (2016) pp. 41-61 .

[29] Remili, M.; Rahmane, M.; *Boundedness and square integrability of solutions of nonlinear fourth-order differential equations*. Nonlinear Dynamics and Systems Theory, 16 (2) (2016), 192-205.

[30] Reissig, R.; Sansone, G.; Conti, R.; *Non-linear Differential Equations of Higher Order. Translated from the German.* Noordhoff International Publishing, Leyden, 1974.
[31] Shair, A.; *Asymptotic properties of linear fourth order differential equations.* American mathematical society, 59 (1) (1976), 45–51.
[32] Sinha, A. S. C.; *On stability of solutions of some third and fourth order delay-differential equations*, Information and Control, 23 (1973), 165-172.
[33] Smith, H.; *An introduction to delay differential equations with applications to the life sciences.* Texts in Applied Mathematics, 57. Springer, New York, 2011
[34] Tejumola, H. O.; *Further results on the boundedness and the stability of certain fourth order differential equations.* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 52 (8) (1972), 16–23.
[35] Tiryaki, A.; Tunç, C.; *Constructing Liapunov functions for some fourth-order autonomous differential equations.* Indian J. Pure Applied Math, 26 (3) (1995), 225–232.
[36] Tunç, C.; *Some stability results for the solutions of certain fourth order delay differential equations.* Differential Equations and Applications, 4 (2004), 165–174.
[37] Tunç, C.; *Stability and boundedness of solutions to certain fourth order differential equations.* Electronic Journal of Differential Equations, (35) (2006), 1–10.
[38] Tunç, C.; *Some remarks on the stability and boundedness of solutions of certain differential equations of fourth-order.* Computational and Applied Mathematics, 26 (1) (2007), 1–17.
[39] Tunç, C.; *On the stability and the boundedness properties of solutions of certain fourth order differential equations.* Istanbul U niv. Fen Fak. Mat. Derg., 54 (1997), 161–173.
[40] Tunç, C.; *A note on the stability and boundedness results of solutions of certain fourth order differential equations.* Appl. Math. Comput., 155 (3) (2004), 837–843.
[41] Tunç, C.; *Some stability and boundedness results for the solutions of certain fourth order differential equations.* Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math., 44 (2005), 161–171.
[42] Tunç, C.; *Stability and boundedness of solutions to certain fourth order differential equations.* Electron. J. Differential Equations, (35) (2006), 10 pp.
[43] Yoshizawa, T.; *Stability Theory by Liapunovs Second Method*, The Mathematical Society of Japan, Tokyo, (1966).
[44] Wu, X.; Xiong, K.; *Remarks on stability results for the solutions of certain fourth-order autonomous differential equations.* Internat. J. Control., 69 (2) (1998), 353–360.

Erdal Korkmaz
Department of Mathematics, Faculty of Arts and Sciences, Mus Alparslan University, 49100, Muş, Turkey
E-mail address: `korkmazerdal36@hotmail.com`

Cemil Tunç
Department of Mathematics, Faculty of Sciences, Yüzüncü Yil University, 65080, Van, Turkey
E-mail address: `cemtunc@yahoo.com`