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OPTIMAL CONTROL FOR SYSTEMS GOVERNED BY PARABOLIC EQUATIONS WITHOUT INITIAL CONDITIONS WITH CONTROLS IN THE COEFFICIENTS

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Abstract. We consider an optimal control problem for systems described by a Fourier problem for parabolic equations. We prove the existence of solutions, and obtain necessary conditions of the optimal control in the case of final observation when the control functions occur in the coefficients.

1. Introduction

Optimal control of determined systems governed by partial differential equations (PDEs) is currently of much interest. Optimal control problems for PDEs are most completely studied for the case in which the control functions occur either on the right-hand sides of the state equations, or the boundary or initial conditions. So far, problems in which control functions occur in the coefficients of the state equations are less studied. A simple model of such type problem is the following.

Let Ω be a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary Γ, $T > 0$, $Q := \Omega \times (0, T)$, $\Sigma := \partial\Omega \times (0, T)$. A state of controlled system for given control $v \in U := L^\infty(Q)$ is defined by a weak solution $y = y(v) = y(x, t; v)$, $(x, t) \in Q$, from the space $L^2(0, T; H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega))$, of the problem

$$y_t - \Delta y + vy = f \in L^2(Q), \quad y\big|_\Sigma = 0, \quad y\big|_{t=0} = y_0 \in L^2(\Omega).$$

The cost functional is

$$J(v) := \|y(\cdot, T; v) - z_0(\cdot)\|^2_{L^2(\Omega)} + \mu\|v\|^2_{L^\infty(Q)} \quad \forall v \in U,$$

where $\mu > 0$, $z_0 \in L^2(\Omega)$ are given. An optimal control problem is to find a function $u \in U_\partial := \{v \in U : v \geq 0 \text{ a. e. on } Q\}$ such that

$$J(u) = \inf_{v \in U_\partial} J(v).$$

This problem is nonlinear, since the dependence between the state and the control is nonlinear.

The direct generalization of this problem is given as only one among many other problems which were considered in monograph [19]. Other various generalizations

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of this problem were investigated in many papers, including [1, 2, 4, 5, 9, 11, 15, 17, 21, 22, 25, 26] where the state of controlled system is described by the initial-boundary value problems for parabolic equations.

In [1, 22, 25, 26] the state of controlled system is described by linear parabolic equations and systems, while in [1, 22] control functions appears as coefficients at lower derivatives, and in [25, 26] the control functions are coefficients at higher derivatives. In [22] the existence and uniqueness of optimal control in the case of final observation was shown and a necessary optimality condition in the form of the generalized rule of Lagrange multipliers was obtained. In [1] the authors proved the existence of at least one optimal control for system governed by a system of general parabolic equations with degenerate discontinuous parabolicity coefficient. In papers [25, 26] the authors consider cost function in general form, and as special case it includes different kinds of specific practical optimization problems. The well-posedness of the problem statement is investigated and a necessary optimality condition in the form of the generalized principle of Lagrange multiplies is established in this papers.

In [2, 9, 11, 15, 17, 21] the authors investigate optimal control of systems governed by nonlinear PDEs. In particular, in [2] the problem of allocating resources to maximize the net benefit in the conservation of a single species is studied. The population model is an equation with density dependent growth and spatial-temporal resource control coefficient. The existence of an optimal control and the uniqueness and the characterization of the optimal control are established. Numerical simulations illustrate several cases with Dirichlet and Neumann boundary conditions. In [9] the problem of optimal control of a Kirchhoff plate is considered. A bilinear control is used as a force to make the plate close to a desired profile taking into the account, a quadratic cost of control. The authors prove the existence of an optimal control and characterize it uniquely through the solution of an optimality system. In [12] the optimal control problem is converted to an optimization problem which is solved using a penalty function technique. The existence and uniqueness theorems are investigated. The derivation of formula for the gradient of the modified function is explained by solving the adjoint problem. Paper [17] presents analytical and numerical solutions of an optimal control problem for quasilinear parabolic equations. The existence and uniqueness of the solution are shown. The derivation of formula for the gradient of the modified cost function by solving the conjugated boundary value problem is explained. In [18] the authors consider the optimal control of the degenerate parabolic equation governing a diffusive population with logistic growth terms. The optimal control is characterized in terms of the solution of the optimality system, which is the state equation coupled with the adjoint equation. Uniqueness for the solutions of the optimality system is valid for a sufficiently small time interval due to the opposite time orientations of the two equations involved. In [21] optimal control for semilinear parabolic equations without Cesari-type conditions is investigated.

In this article, we study an optimal control problem (see (4.1), (3.2), (3.4), (4.2), (4.3) below) for systems whose states are described by problems without initial conditions or, other words, Fourier problems for parabolic equations. The model example of considered optimal control problem is a problem which differs from the previous one (see beginning of this section) by the following facts: the initial moment is $-\infty$ and, correspondingly, the state equation and control functions are considered in the domain $Q = \Omega \times (-\infty, T)$, a boundary condition is given on the surface $\Sigma = \partial\Omega \times (-\infty, T)$, while the initial condition is replaced by the condition

$$\lim_{t\to-\infty} \|y(\cdot, t)\|_{L^2(\Omega)} = 0.$$

The problem without initial conditions for evolution equations describes processes that started a long time ago and initial conditions do not affect on them in the actual time moment. Such problem were investigated in the works of many mathematicians (see [3, 7, 24] and bibliography there).

As we know among numerous works devoted to the optimal control problems for PDEs, only in [4, 5] the state of controlled system is described by the solution of Fourier problem for parabolic equations. In the current paper, unlike the above two, we consider optimal control problem in case when the control functions occur in the coefficients of the state equation. The main result of this paper is existence of the solution of this problem.

The outline of this article is as follows. In Section 2, we give notation, definitions of function spaces and auxiliary results. In Section 3, we prove existence and uniqueness of the solutions for the state equations. Furthermore, we construct a priori estimates for the weak solutions of the state equations. In Section 4, we formulate the optimal control problem. Finally, the existence and necessary conditions of the optimal control are presented in Section 5.

2. Preliminaries

Let n be a natural number, $\mathbb{R}^n$ be the linear space of ordered collections $x = (x_1, \dots, x_n)$ of real numbers with the norm $|x| := (|x_1|^2 + \dots + |x_n|^2)^{1/2}$. Suppose that Ω is a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary Γ. Set $S := (-\infty, 0]$, $Q := \Omega \times S$, $\Sigma := \Gamma \times S$.

Denote by $L^\infty_{\rm loc}(\overline{Q})$ the linear space of measurable functions on Q such that their restrictions to any bounded measurable set $Q' \subset Q$ belong to the space $L^\infty(Q')$.

Let X be an arbitrary Hilbert space with the scalar product $(\cdot, \cdot)_X$ and the norm $\|\cdot\|_X$. Denote by $L^2_{\rm loc}(S; X)$ the linear space of measurable functions defined on S with values in X, whose restrictions to any segment $[a, b] \subset S$ belong to the space $L^2(a, b; X)$.

Let $\omega \in \mathbb{R}$, $\alpha \in C(S)$ be such that $\alpha(t) > 0$ for all $t \in S$, $\gamma = \alpha$ or $\gamma = 1/\alpha$, and let X be as above. Put by definition

$$L^2_{\omega,\gamma}(S; X) := \Big\{ f \in L^2_{\rm loc}(S; X) : \int_S \gamma(t) e^{2\omega \int_0^t \alpha(s)ds} \|f(t)\|^2_X dt < \infty \Big\}.$$

This space is a Hilbert space with respect to the scalar product

$$(f, g)_{L^2_{\omega,\gamma}(S;X)} = \int_S \gamma(t)\, e^{2\omega \int_0^t \alpha(s)\, ds} (f(t), g(t))_X\, dt$$

and the norm

$$\|f\|_{L^2_{\omega,\gamma}(S;X)} := \Big(\int_S \gamma(t)\, e^{2\omega \int_0^t \alpha(s)\, ds} \|f(t)\|^2_X\, dt \Big)^{1/2}.$$

For an interval I, we denote by $C^1_c(I)$ the linear space of continuously differentiable functions on I with compact supports (if $I = (t_1, t_2)$, then we will write $C^1_c(t_1, t_2)$ instead of $C^1_c((t_1, t_2))$).

Let $H^1(\Omega) := \{v \in L^2(\Omega) : v_{x_i} \in L^2(\Omega)\ (i = \overline{1, n})\}$ be a Sobolev space, which is a Hilbert space with respect to the scalar product $(v, w)_{H^1(\Omega)} := \int_\Omega \big\{\nabla v \nabla w + vw\big\}\, dx$

and the corresponding norm $\|v\|_{H^1(\Omega)} := \big(\int_\Omega \{|\nabla v|^2 + |v|^2\}\,dx\big)^{1/2}$, where $\nabla v = (v_{x_1},\dots,v_{x_n})$, $|\nabla v|^2 = \sum_{i=1}^n |v_{x_i}|^2$. Under $H_0^1(\Omega)$ we mean the closure in $H^1(\Omega)$ of the space $C_c^\infty(\Omega)$ consisting of infinitely differentiable functions on Ω with compact supports. Denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$, that is, the space of all continuous linear functionals on $H_0^1(\Omega)$.

We suppose (after appropriate identification of functionals), that the space $L^2(\Omega)$ is a subspace of $H^{-1}(\Omega)$. Identifying spaces $L^2(\Omega)$ and $\big(L^2(\Omega)\big)'$, we obtain continuous and dense embeddings

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega). \tag{2.1}$$

Note, that in this case $\langle g, v\rangle_{H_0^1(\Omega)} = (g, v)$ for every $v \in H_0^1(\Omega), g \in L^2(\Omega)$, where $(\cdot,\cdot)$ is the scalar product on $L^2(\Omega)$ and $\langle\cdot,\cdot\rangle_{H_0^1(\Omega)}$ is the scalar product for the duality $H^{-1}(\Omega)$, $H_0^1(\Omega)$. Therefore, further we use the notation $(\cdot,\cdot)$ instead of $\langle\cdot,\cdot\rangle_{H_0^1(\Omega)}$.

We define

$$K := \inf_{v\in H_0^1(\Omega),\ v\neq 0} \frac{\int_\Omega |\nabla v|^2\,dx}{\int_\Omega |v|^2\,dx}. \tag{2.2}$$

It is well known that the constant K is finite and coincides with the first eigenvalue of the eigenvalue problem

$$-\Delta v = \lambda v, \quad v|_{\partial\Omega} = 0. \tag{2.3}$$

From (2.2) it clearly follows the Friedrichs inequality

$$\int_\Omega |\nabla v|^2\,dx \ge K \int_\Omega |v|^2\,dx \quad \forall v \in H_0^1(\Omega). \tag{2.4}$$

Further, an important role will be played by the following statement, which is a well-known result (see, e.g. [10, Theorem 3, p. 287]), but we reformulate it according to our needs.

Lemma 2.1. *Suppose that a function $z \in L^2(t_1, t_2; H_0^1(\Omega))$, with $t_1 < t_2$, satisfies*

$$\int_{t_1}^{t_2}\int_\Omega \big\{ -z\psi\varphi' + (g_0\psi + \sum_{i=1}^n g_i\psi_{x_i})\varphi\big\}\,dx\,dt = 0, \tag{2.5}$$

for $\psi \in H_0^1(\Omega)$, $\varphi \in C_c^1(t_1,t_2)$, where $g_i \in L^2(\Omega\times(t_1,t_2))$ $(i = \overline{0,n})$. Then

(1) the derivative z_t of the function z in the sense $D'(t_1,t_2;H^{-1}(\Omega))$ (the distributions space) belongs to $L^2(t_1,t_2;H^{-1}(\Omega))$, furthermore for a.e. $t \in (t_1,t_2)$,

$$z_t(\cdot,t) = -g_0(\cdot,t) + \sum_{i=1}^n \big(g_i(\cdot,t)\big)_{x_i} \quad \text{in } H^{-1}(\Omega), \tag{2.6}$$

$$\frac12\frac{d}{dt}\|z(\cdot,t)\|^2_{L^2(\Omega)} = (z_t(\cdot,t), z(\cdot,t)), \tag{2.7}$$

$$\int_{t_1}^{t_2} \|z_t(\cdot,t)\|^2_{H^{-1}(\Omega)}\,dt \le \sum_{i=0}^n \|g_i\|^2_{L^2(\Omega\times(t_1,t_2))}; \tag{2.8}$$

(2) the function z belongs to the space $C([t_1,t_2];L^2(\Omega))$ and for all $\tau_1,\tau_2\in[t_1,t_2]$ $(\tau_1<\tau_2)$ and for every $\theta\in C^1([t_1,t_2])$, $q\in L^2(t_1,t_2;H_0^1(\Omega))$ we have

$$\begin{aligned}&\frac{1}{2}\theta(t)\int_\Omega |z(x,t)|^2\,dx\Big|_{t=\tau_1}^{t=\tau_2}-\frac{1}{2}\int_{\tau_1}^{\tau_2}\int_\Omega |z|^2\theta'\,dx\,dt\\&+\int_{\tau_1}^{\tau_2}\int_\Omega\big\{g_0z+\sum_{i=1}^n g_iz_{x_i}\big\}\theta\,dx\,dt=0,\end{aligned}\tag{2.9}$$

$$\int_{\tau_1}^{\tau_2}\big(z_t(\cdot,t),q(\cdot,t)\big)\,dt+\int_{\tau_1}^{\tau_2}\int_\Omega g_0q\,dx\,dt+\sum_{i=1}^n\int_{\tau_1}^{\tau_2}\int_\Omega g_iq_{x_i}\,dx\,dt=0.\tag{2.10}$$

Proof. As it has already been mentioned, this lemma follows directly from the well-known result. But for clarity we re-present schematically some points of the proof. The first statement is: Since the spaces $L^2(t_1,t_2;H_0^1(\Omega))$, $L^2(t_1,t_2;H^{-1}(\Omega))$ can be identified with subspaces of the space of distributions $D'(t_1,t_2;H^{-1}(\Omega))$, then it allows us to speak about derivatives of functions from $L^2(t_1,t_2;H_0^1(\Omega))$ in the sense $D'(t_1,t_2;H^{-1}(\Omega))$ and their belonging to the space $L^2(t_1,t_2;H^{-1}(\Omega))$.

Let us rewrite equality (2.5) in the form

$$-\int_{t_1}^{t_2}\int_\Omega z\psi\varphi'\,dx\,dt=-\int_{t_1}^{t_2}\int_\Omega(g_0\psi+\sum_{i=1}^n g_i\psi_{x_i})\varphi\,dx\,dt,\tag{2.11}$$

for $\psi\in H_0^1(\Omega)$, $\varphi\in C_c^1(t_1,t_2)$. According to the definition of the derivative of distributions from $D'(t_1,t_2;H^{-1}(\Omega))$, (2.11) implies existence of z_t and its belonging to the space $L^2(t_1,t_2;H^{-1}(\Omega))$, then according to [10, Theorem 3, p. 287] identity (2.7) holds. From (2.11) for almost all $t\in(t_1,t_2)$ we have

$$\big(z_t(\cdot,t),\psi(\cdot)\big)=-\int_\Omega\big[g_0(x,t)\psi(x)+\sum_{i=1}^n g_i(x,t)\psi_{x_i}(x)\big]\,dx,\tag{2.12}$$

that is, (2.6) holds.

From (2.12), using the Cauchy-Schwarz inequality, for almost all $t\in(t_1,t_2)$ we obtain

$$\begin{aligned}&\big|\big(z_t(\cdot,t),\psi(\cdot)\big)\big|\\&\le\|g_0(\cdot,t)\|_{L^2(\Omega)}\|\psi(\cdot)\|_{L^2(\Omega)}+\sum_{i=1}^n\|g_i(\cdot,t)\|_{L^2(\Omega)}\|\psi_{x_i}(\cdot)\|_{L^2(\Omega)}\\&\le\Big(\sum_{i=0}^n\|g_i(\cdot,t)\|^2_{L^2(\Omega)}\Big)^{1/2}\|\psi(\cdot)\|_{H^1(\Omega)}.\end{aligned}\tag{2.13}$$

From (2.13) it follows that for almost all $t\in(t_1,t_2)$ the following estimate is valid

$$\|z_t(\cdot,t)\|^2_{H^{-1}(\Omega)}\le\sum_{i=0}^n\|g_i(\cdot,t)\|^2_{L^2(\Omega)},$$

which easily implies (2.8).

Let us prove the second statement of Lemma 2.1. The fact that the function z belongs to the space $C([t_1,t_2];L^2(\Omega))$ follows directly from [10, Theorem 3, p. 287] according to the first statement.

Since for a.e. $t \in S$ the function $q(\cdot,t) \in H_0^1(\Omega)$, we can take $\psi(\cdot) = q(\cdot,t)$ in (2.12) and obtain

$$\big(z_t(\cdot,t), q(\cdot,t)\big) = -\int_\Omega \Big[g_0(x,t)q(x,t) + \sum_{i=1}^n g_i(x,t)q_{x_i}(x,t)\Big]\,dx, \quad t \in S. \tag{2.14}$$

Integrating this inequality by t over (τ_1,τ_2) for arbitrary $\tau_1,\tau_2 \in S$, we obtain (2.10).

Taking $q(\cdot,t) = \theta(t)z(\cdot,t)$, $t \in S$, in (2.10) and integrating over (τ_1,τ_2), we obtain

$$\int_{\tau_1}^{\tau_2} \theta(t)\big(z_t(\cdot,t), z(\cdot,t)\big)\,dt + \int_{\tau_1}^{\tau_2}\int_\Omega \Big\{g_0 z + \sum_{i=1}^n g_i z_{x_i}\Big\}\theta\,dx\,dt = 0. \tag{2.15}$$

Using (2.7) and integration by parts, we have

$$\begin{aligned}\int_{\tau_1}^{\tau_2} \theta(t)\big(z_t(\cdot,t), z(\cdot,t)\big)\,dt &= \frac{1}{2}\int_{\tau_1}^{\tau_2} \theta(t)\frac{d}{dt}\|z(\cdot,t), z(\cdot,t\|^2_{L^2(\Omega)}\,dt\\ &= \frac{1}{2}\theta(t)\|z(\cdot,t)\|^2_{L^2(\Omega)}\Big|_{t=\tau_1}^{t=\tau_2} - \frac{1}{2}\int_{\tau_1}^{\tau_2} \theta'\|z(\cdot,t)\|_{L^2(\Omega)}\,dt,\end{aligned}$$

which, together with (2.15), gives (2.9). □

3. Well-posedness of the problem without initial conditions for linear parabolic equations

Consider the equation

$$y_t - \sum_{i,j=1}^n \big(a_{ij}(x,t)y_{x_i}\big)_{x_j} + a_0(x,t)y = f(x,t), \quad (x,t) \in Q, \tag{3.1}$$

where $y : \overline{Q} \to \mathbb{R}$ is an unknown function and data-in satisfies conditions:

- (A1) $a_0, a_{ij} \in L^\infty_{\rm loc}(\overline{Q})$, $a_{ij} = a_{ji}$ $(i,j = \overline{1,n})$, $a_0(x,t) \ge 0$ for a. e. $(x,t) \in Q$, there exists a function $\alpha \in C(S)$ such that $\alpha(t) > 0$ for all $t \in S$ and $\sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \ge \alpha(t)|\xi|^2$ for every $\xi \in \mathbb{R}^n$ and for a. e. $(x,t) \in Q$;
- (A2) $f \in L^2_{\rm loc}(S; L^2(\Omega))$.

Additionally, we impose the boundary condition

$$y\big|_\Sigma = 0 \tag{3.2}$$

on a solution of equation (3.1).

Definition 3.1. A weak solution of problem (3.1), (3.2) is a function y which belongs to $L^2_{\rm loc}(S; H_0^1(\Omega)) \cap C(S; L^2(\Omega))$ and satisfies

$$\begin{aligned}&\iint_Q \Big\{-y\psi\varphi' + \sum_{i,j=1}^n a_{ij}y_{x_i}\psi_{x_j}\varphi + a_0 y\psi\varphi\Big\}\,dx\,dt\\ &= \iint_Q f\psi\varphi\,dx\,dt, \quad \psi \in H_0^1(\Omega),\ \varphi \in C_c^1(-\infty,0).\end{aligned} \tag{3.3}$$

In other words: a weak solution of problem (3.1), (3.2) is the function y which belongs to $L^2_{\rm loc}(S; H_0^1(\Omega)) \cap C(S; L^2(\Omega))$ with $y_t \in L^2_{\rm loc}(S; H^{-1}(\Omega))$, and satisfies

$$y_t - \sum_{i,j=1}^n (a_{ij}y_{x_i})_{x_j} + a_0 y = f \quad \text{in } L^2_{\rm loc}(S; H^{-1}(\Omega)).$$

Remark 3.2. There may exist many weak solutions of problem (3.1), (3.2). E.g., the functions $y_c(x,t) = cv(x)e^{-Kt}$, $(x,t) \in \overline{Q}$ $(c \in \mathbb{R})$, where v is an eigenfunction of problem (2.3) corresponding to the first eigenvalue, are weak solutions of problem (3.1), (3.2) when $a_{ij} = \delta_{ij}$, $a_0 = 0$ and $f = 0$, where δ_{ij} is Kronecker's delta $(i,j = \overline{1,n})$. Therefore, to ensure uniqueness of the weak solution of (3.1) satisfying condition (3.2), we have to impose some additional conditions on solutions, for instance, some restrictions on their behavior as $t \to -\infty$.

We will consider the problem of finding the weak solution of (3.1), (3.2) satisfying the analogue of the initial condition

$$\lim_{t\to-\infty} e^{\omega \int_0^t \alpha(s)ds} \|y(\cdot,t)\|_{L^2(\Omega)} = 0, \tag{3.4}$$

where $\omega \in \mathbb{R}$ is given.

We will briefly call this problem by problem (3.1), (3.2), (3.4), and the function y is called the solution of problem (3.1), (3.2), (3.4).

Theorem 3.3. *Suppose that condition* (A1) *holds, K is a constant defined by* (2.2). *The following two statements hold:*

(1) If $\omega \le K$ then (3.1), (3.2), (3.4) *has at most one weak solution.*

(2) If $\omega < K$ and

$$f \in L^2_{\omega,1/\alpha}(S; L^2(\Omega)), \tag{3.5}$$

then there exists a unique weak solution of (3.1), (3.2), (3.4), *it belongs to the space $L^2_{\omega,\alpha}(S; H^1_0(\Omega))$ and the following estimates are satisifed*

$$e^{\omega \int_0^\tau \alpha(s)\,ds} \|y(\cdot,\tau)\|_{L^2(\Omega)} \le C_1 \|f\|_{L^2_{\omega,1/\alpha}(S_\tau;L^2(\Omega))}, \quad \tau \in S, \tag{3.6}$$

$$\|y\|_{L^2_{\omega,\alpha}(S_\tau;H^1_0(\Omega))} \le C_2 \|f\|_{L^2_{\omega,1/\alpha}(S_\tau;L^2(\Omega))}, \quad \tau \in S, \tag{3.7}$$

where $S_\tau := (-\infty,\tau]$ $(\tau \in (-\infty,0]$, $S_0 = S)$, C_1, C_2 are positive constants depending only on K and ω.

Remark 3.4. In the particular case of equation (3.1), which was considered in Remark 3.2, we have $\alpha(t) = 1$, therefore condition (3.4) takes on the form:

$$e^{\omega t} \|y(\cdot,t)\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to -\infty.$$

Obviously in this case for the nonzero solutions of (3.1), (3.2), (3.4), indicated in Remark 3.2, we have $\lim_{t\to-\infty} e^{Kt}\|y_c(\cdot,t)\|_{L^2(\Omega)} = C$, where C is a nonzero constant; $\lim_{t\to-\infty} e^{\omega t}\|y_c(\cdot,t)\|_{L^2(\Omega)} = +\infty$, if $\omega < K$; $\lim_{t\to-\infty} e^{\omega t}\|y_c(\cdot,t)\|_{L^2(\Omega)} = 0$, if $\omega > K$. This means that the condition $\omega \le K$ is essential for ensuring the uniqueness of the weak solution of (3.1), (3.2), (3.4), i.e., it cannot be simplified.

Proof of Theorem 3.3. In the proof we use the same technique as in the proofs of corresponding results in [4,5]. Nevertheless, we present the proof, because it is important for us to obtain more precise estimates of the solution of (3.1), (3.2), (3.4) and to track how this solution depends on the coefficient (which serves as a control in the following sections).

Let us prove the first statement of Theorem 3.3. Assume the opposite. Let y_1, y_2 be two weak solutions of (3.1), (3.2), (3.4). Substituting them one by one into integral identity (3.3) and subtracting the obtained equalities, for the difference $z := y_1 - y_2$ we obtain

$$-\iint_Q z\psi\varphi'\,dx\,dt + \iint_Q \Big(\sum_{i,j=1}^n a_{ij} z_{x_i} \psi_{x_j} + a_0 z\psi\Big)\varphi\,dx\,dt = 0, \tag{3.8}$$

for all $\psi \in H_0^1(\Omega)$, $\varphi \in C_c^1(-\infty,0)$.

From (3.4) it follows that

$$e^{2\omega \int_0^t \alpha(s)ds} \int_\Omega |z(x,t)|^2\,dx \to 0 \quad \text{as } t \to -\infty. \tag{3.9}$$

According to Lemma 2.1 with $\theta(t) = 2e^{2\omega \int_0^t \alpha(s)\,ds}$, $t \in \mathbb{R}$, (3.8) implies that

$$\begin{aligned}
&e^{2\omega \int_0^{\tau_2} \alpha(s)\,ds} \int_\Omega |z(x,\tau_2)|^2\,dx - e^{2\omega \int_0^{\tau_1} \alpha(s)\,ds} \int_\Omega |z(x,\tau_1)|^2\,dx \\
&- 2\omega \int_{\tau_1}^{\tau_2} \int_\Omega \alpha(t) e^{2\omega \int_0^t \alpha(s)\,ds} |z|^2\,dx\,dt \\
&+ 2\int_{\tau_1}^{\tau_2} \int_\Omega e^{2\omega \int_0^t \alpha(s)\,ds} \Big[\sum_{i,j=1}^n a_{ij} z_{x_i} z_{x_j} + a_0|z|^2 \Big]\,dx\,dt = 0,
\end{aligned}$$

where $\tau_1, \tau_2 \in S$ $(\tau_1 < \tau_2)$ are arbitrary numbers.

Taking into account condition (A1) and inequality (2.4), we obtain

$$\begin{aligned}
&e^{2\omega \int_0^{\tau_2} \alpha(s)\,ds} \int_\Omega |z(x,\tau_2)|^2\,dx - e^{2\omega \int_0^{\tau_1} \alpha(s)\,ds} \int_\Omega |z(x,\tau_1)|^2\,dx \\
&+ 2(K-\omega) \int_{\tau_1}^{\tau_2} \int_\Omega \alpha(t) e^{2\omega \int_0^t \alpha(s)\,ds} |z|^2\,dx\,dt \le 0.
\end{aligned} \tag{3.10}$$

Since $\omega \le K$, from (3.10) we obtain

$$e^{2\omega \int_0^{\tau_2} \alpha(s)\,ds} \int_\Omega |z(x,\tau_2)|^2\,dx \le e^{2\omega \int_0^{\tau_1} \alpha(s)\,ds} \int_\Omega |z(x,\tau_1)|^2\,dx. \tag{3.11}$$

In (3.11) fix τ_2 and let τ_1 to $-\infty$. According to condition (3.9) we obtain the equality

$$e^{2\omega \int_0^{\tau_2} \alpha(s)\,ds} \int_\Omega |z(x,\tau_2)|^2\,dx = 0.$$

Since $\tau_2 \in S$ is an arbitrary number, we have $z(x,t) = 0$ for a. e. $(x,t) \in Q$, that is, $y_1(x,t) = y_2(x,t) = 0$ for a. e. $(x,t) \in Q$. The resulting contradiction proves the first statement.

Let us prove the second statement. First we determine a priori estimates of a weak solution of (3.1), (3.2), (3.4). According to Lemma 2.1, condition (3.3) implies

$$\begin{aligned}
&\frac{1}{2}\theta(\tau_2) \int_\Omega |y(x,\tau_2)|^2\,dx - \frac{1}{2}\theta(\tau_1) \int_\Omega |y(x,\tau_1)|^2\,dx \\
&- \frac{1}{2} \int_{\tau_1}^{\tau_2} \int_\Omega |y|^2 \theta'\,dx\,dt + \int_{\tau_1}^{\tau_2} \int_\Omega \Big[\sum_{i,j=1}^n a_{ij} y_{x_i} y_{x_j} + a_0|y|^2 \Big] \theta\,dx\,dt \\
&= \int_{\tau_1}^{\tau_2} \int_\Omega f y \theta\,dx\,dt,
\end{aligned} \tag{3.12}$$

where $\theta \in C^1(S)$ is an arbitrary function, $\tau_1, \tau_2 \in S$ $(\tau_1 < \tau_2)$ are arbitrary numbers. Further assume that $\theta(t) \ge 0$ for all $t \in S$.

Using the Cauchy inequality with "ε":

$$ab \le \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \quad a, b \in \mathbb{R},\ \varepsilon > 0,$$

we estimate the right side of (3.12) as follows:

$$\Big|\int_{\tau_1}^{\tau_2}\int_\Omega fy\theta\,dx\,dt\Big| \le \frac{\varepsilon}{2}\int_{\tau_1}^{\tau_2}\int_\Omega \alpha|y|^2\theta\,dx\,dt + \frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\int_\Omega [\alpha]^{-1}|f|^2\theta\,dx\,dt, \quad (3.13)$$

where $\varepsilon > 0$ is arbitrary.

From condition (A1) we obtain

$$\int_{\tau_1}^{\tau_2}\int_\Omega \big[\sum_{i,j=1}^{n} a_{ij}y_{x_i}y_{x_j} + a_0|y|^2\big]\theta\,dx\,dt \ge \int_{\tau_1}^{\tau_2}\int_\Omega \alpha|\nabla y|^2\theta\,dx\,dt, \quad (3.14)$$

where $\nabla y := (y_{x_1},\dots,y_{x_n})$ is the gradient of y.

According to (3.13) and (3.14), equality (3.12) implies

$$\begin{aligned}
&\frac{1}{2}\theta(\tau_2)\int_\Omega |y(x,\tau_2)|^2\,dx - \frac{1}{2}\theta(\tau_1)\int_\Omega |y(x,\tau_1)|^2\,dx\\
&-\frac{1}{2}\int_{\tau_1}^{\tau_2}\int_\Omega |y|^2\theta'\,dx\,dt + \int_{\tau_1}^{\tau_2}\int_\Omega \alpha|\nabla y|^2\theta\,dx\,dt\\
&\le \frac{\varepsilon}{2}\int_{\tau_1}^{\tau_2}\int_\Omega \alpha|y|^2\theta\,dx\,dt + \frac{1}{2\varepsilon}\int_{\tau_1}^{\tau_2}\int_\Omega [\alpha]^{-1}|f|^2\theta\,dx\,dt,
\end{aligned}$$

where $\varepsilon > 0$ is arbitrary.

Taking $\theta(t) = 2e^{2\omega\int_0^t \alpha(s)\,ds}$ with $t \in S$, we obtain

$$\begin{aligned}
&e^{2\omega\int_0^{\tau_2}\alpha(s)\,ds}\int_\Omega |y(x,\tau_2)|^2dx - e^{2\omega\int_0^{\tau_1}\alpha(s)\,ds}\int_\Omega |y(x,\tau_1)|^2dx\\
&-2\omega\int_{\tau_1}^{\tau_2}\int_\Omega \alpha(t)e^{2\omega\int_0^t\alpha(s)\,ds}|y|^2dx\,dt + 2\int_{\tau_1}^{\tau_2}\int_\Omega \alpha(t)e^{2\omega\int_0^t\alpha(s)\,ds}|\nabla y|^2\,dx\,dt\\
&\le \varepsilon\int_{\tau_1}^{\tau_2}\int_\Omega \alpha(t)e^{2\omega\int_0^t\alpha(s)\,ds}|y|^2\,dx\,dt + \frac{1}{\varepsilon}\int_{\tau_1}^{\tau_2}\int_\Omega [\alpha(t)]^{-1}e^{2\omega\int_0^t\alpha(s)\,ds}|f|^2\,dx\,dt.
\end{aligned}$$

By the above inequality and using (2.4), we obtain

$$\begin{aligned}
&e^{2\omega\int_0^{\tau_2}\alpha(s)\,ds}\int_\Omega |y(x,\tau_2)|^2\,dx - e^{2\omega\int_0^{\tau_1}\alpha(s)\,ds}\int_\Omega |y(x,\tau_1)|^2\,dx\\
&+\chi(K,\omega,\varepsilon)\int_{\tau_1}^{\tau_2}\int_\Omega \alpha(t)e^{2\omega\int_0^t\alpha(s)\,ds}|\nabla y|^2\,dx\,dt\\
&\le \frac{1}{\varepsilon}\int_{\tau_1}^{\tau_2}\int_\Omega [\alpha(t)]^{-1}e^{2\omega\int_0^t\alpha(s)\,ds}|f|^2\,dx\,dt,
\end{aligned} \quad (3.15)$$

where

$$\chi(K,\omega,\varepsilon) := \begin{cases} \frac{2(K-\omega)-\varepsilon}{K} & \text{if } 0<\omega<K,\\ \frac{2K-\varepsilon}{K} & \text{if } \omega \le 0.\end{cases}$$

Taking $\varepsilon = K$ if $\omega \le 0$, and $\varepsilon = K-\omega$ if $0<\omega<K$ in (3.15), we obtain

$$\begin{aligned}
&e^{2\omega\int_0^{\tau_2}\alpha(s)\,ds}\int_\Omega |y(x,\tau_2)|^2\,dx - e^{2\omega\int_0^{\tau_1}\alpha(s)\,ds}\int_\Omega |y(x,\tau_1)|^2\,dx\\
&+C_3\int_{\tau_1}^{\tau_2}\int_\Omega \alpha(t)e^{2\omega\int_0^t\alpha(s)\,ds}|\nabla y|^2\,dx\,dt\\
&\le C_4\int_{\tau_1}^{\tau_2}\int_\Omega [\alpha(t)]^{-1}e^{2\omega\int_0^t\alpha(s)\,ds}|f|^2\,dx\,dt,
\end{aligned} \quad (3.16)$$

where $C_3 > 0$, $C_4 > 0$ are constants depending only on K and ω.

Taking into account (3.4) and (3.5), we let $\tau_1 \to -\infty$ in (3.16). As a result, adopting $\tau_2 = \tau \in S$, we obtain

$$\begin{aligned} &e^{2\omega \int_0^\tau \alpha(s)\,ds} \int_\Omega |y(x,\tau)|^2\,dx + C_3 \int_{-\infty}^{\tau} \int_\Omega \alpha(t) e^{2\omega \int_0^t \alpha(s)\,ds} |\nabla y|^2\,dx\,dt \\ &\le C_4 \int_{-\infty}^{\tau} \int_\Omega [\alpha(t)]^{-1} e^{2\omega \int_0^t \alpha(s)\,ds} |f|^2\,dx\,dt. \end{aligned} \tag{3.17}$$

Hence, using inequality (2.4), we easily obtain estimates (3.6) and (3.7).

Now let us prove the existence of a weak solution of problem (3.1), (3.2), (3.4). First, for each $m \in N$ we define $Q_m := \Omega \times (-m, 0]$, $f_m(\cdot,t) := f(\cdot,t)$, if $-m < t \le 0$, and $f_m(\cdot,t) := 0$, if $t \le -m$, and consider the problem of finding a function $y_m \in L^2(-m, 0; H_0^1(\Omega)) \cap C([-m, 0]; L^2(\Omega))$ satisfying the initial condition

$$y_m(x, -m) = 0, \quad x \in \Omega, \tag{3.18}$$

(as an element of space $C([-m, 0]; L^2(\Omega)))$ and equation (3.1) in Q_m in the sense of integral identity; that is,

$$\iint_{Q_m} \Big\{ - y_m \psi \varphi' + \sum_{i,j=1}^{n} a_{ij} y_{m,x_i} \psi_{x_j} \varphi + a_0 y_m \psi \varphi \Big\}\,dx\,dt = \iint_{Q_m} f_m \psi \varphi\,dx\,dt,$$

for $\psi \in H_0^1(\Omega)$, $\varphi \in C_c^1(-m, 0)$.

The existence and uniqueness of the solution of this problem easily follows from the known results (see, for example, [16]). For every $m \in \mathbb{N}$ we extend y_m by zero for the entire set Q and keep the same notation y_m for this extension. Note that for each $m \in N$, the function y_m belongs to $L^2(S; H_0^1(\Omega)) \cap C(S; L^2(\Omega))$ and satisfies integral identity (3.3) with f_m substituted for f, i.e.,

$$\iint_{Q} \Big\{ - y_m \psi \varphi' + \sum_{i,j=1}^{n} a_{ij} y_{m,x_i} \psi_{x_j} \varphi + a_0 y_m \psi \varphi \Big\}\,dx\,dt = \iint_{Q} f_m \psi \varphi\,dx\,dt, \tag{3.19}$$

for $\psi \in H_0^1(\Omega)$, $\varphi \in C_c^1(-\infty, 0)$. Consequently, we have shown that y_m is a weak solution of problem (3.1), (3.2), (3.4) with f_m substituted for f. Therefore, for y_m we obtain estimates similar to (3.6), (3.7), in particular, for $\tau \in S$,

$$e^{2\omega \int_0^\tau \alpha(s)\,ds} \|y_m(\cdot,\tau)\|^2_{L^2(\Omega)} \le C_1 \int_{-\infty}^{\tau} [\alpha(t)]^{-1} e^{2\omega \int_0^t \alpha(s)\,ds} \|f(\cdot,t)\|^2_{L^2(\Omega)}\,dt, \tag{3.20}$$

Let us take identity (3.19) with alternating $m = k$ and $m = l$, where k, l are arbitrary positive integers, $l > k$, and then subtract the obtained identities. As a result, we obtain the same identity as (3.19) with $z_{k,l} := y_k - y_l$, $f_{k,l} := f_k - f_l$ instead of y_m and f_m, respectively. Finally taking into account that the function $z_{k,l}$ satisfies conditions (3.2) and (3.4), replacing y with $z_{k,l}$, we see that the function $z_{k,l}$ is a weak solution of the problem, which differs from problem (3.1), (3.2), (3.4) only in that instead of y and f, there are $z_{k,l}$ and $f_{k,l}$, respectively. Thus, for $z_{k,l}$ we have estimates similar to (3.6), (3.7), i.e.

$$\begin{aligned} &e^{2\omega \int_0^\tau \alpha(s)\,ds} \|y_k(\cdot,\tau) - y_l(\cdot,\tau)\|^2_{L^2(\Omega)} \\ &\le C_1 \int_{-l}^{-k} [\alpha(t)]^{-1} e^{2\omega \int_0^t \alpha(s)\,ds} \|f(\cdot,t)\|^2_{L^2(\Omega)}\,dt, \quad \tau \in S, \end{aligned} \tag{3.21}$$

$$\|y_k - y_l\|_{L^2_{\omega,\alpha}(S; H_0^1(\Omega))} \le C_2 \int_{-l}^{-k} [\alpha(t)]^{-1} e^{2\omega \int_0^t \alpha(s)\,ds} \|f(\cdot,t)\|^2_{L^2(\Omega)}\,dt. \tag{3.22}$$

Condition (3.5) implies that the right-hand sides of inequalities (3.21) and (3.22) tend to zero when k and l tend to $+\infty$. This means that the sequence $\{y_m\}_{m=1}^{\infty}$ is a Cauchy sequence in the space $L^2_{\omega,\alpha}(S;H^1_0(\Omega))$ and $C(S;L^2(\Omega))$. Consequently, we obtain the existence of the function $y\in L^2_{\omega,\alpha}(S;H^1_0(\Omega))\cap C(S;L^2(\Omega))$ such that

$$y_m \underset{m\to\infty}{\longrightarrow} y \quad \text{strongly in } L^2_{\omega,\alpha}(S;H^1_0(\Omega)) \text{ and } C(S;L^2(\Omega)). \tag{3.23}$$

Note that (3.23) implies

$$y_m \underset{m\to\infty}{\longrightarrow} y, \quad y_{m,x_i} \underset{m\to\infty}{\longrightarrow} y_{x_i}\ (i=\overline{1,n}) \quad \text{strongly in } L^2_{\rm loc}(S;L^2(\Omega)). \tag{3.24}$$

Let us show that the function y is a weak solution of (3.1), (3.2), (3.4). To do this, first we let $m\to\infty$ in identity (3.19), taking into account (3.24) and the definition of the function f_m. Consequently, we obtain identity (3.3). Now, taking into account (3.23), we let $m\to+\infty$ in (3.20). From the resulting inequality and condition (3.5), we obtain condition (3.4). Hence, we have proven that y is a weak solution of problem (3.1), (3.2), (3.4). □

4. Formulation of the optimal control problem and main result

Let $U:=L^\infty(Q)$ be a space of controls and U_∂ be a convex and closed subset of $\{v\in U: v\ge 0 \text{ a. e. in } Q\}$. We suppose that U_∂ is the set of admissible controls.

We assume that the state of the investigated evolutionary system for a given control $v\in U_\partial$ is described by a weak solution of (3.1), (3.2), (3.4) when $a_0=\widetilde{a}_0+v$, where $\widetilde{a}_0\in L^\infty_{\rm loc}(\overline{Q})$ is a given function such that $\widetilde{a}_0\ge 0$ a. e. in Q. Then, equation (3.1) has the form

$$y_t-\sum_{i,j=1}^{n}\big(a_{ij}(x,t)y_{x_i}\big)_{x_j}+(\widetilde{a}_0(x,t)+v(x,t))y=f(x,t), \quad (x,t)\in Q. \tag{4.1}$$

The specified problem will be called problem (4.1), (3.2), (3.4). The weak solution y of (4.1), (3.2), (3.4) for a given control v, denoted by y, or $y(v)$, or $y(x,t)$, $(x,t)\in Q$, or $y(x,t;v)$, $(x,t)\in Q$. Further, we assume that conditions (A1), (3.5) and the inequality $\omega<K$ hold. From the previous section (see Theorem 3.3), we immediately obtain the existence and uniqueness of the weak solution of problem (4.1), (3.2), (3.4) and its estimates (3.6), (3.7).

We assume that the cost functional has the form

$$J(v)=\|y(\cdot,0;v)-z_0(\cdot)\|^2_{L^2(\Omega)}+\mu\|v\|_{L^\infty(Q)}, \quad v\in U_\partial, \tag{4.2}$$

where $z_0\in L^2(\Omega)$, $\mu\ge 0$ if U_∂ is bounded, and $\mu>0$ otherwise.

We consider the following optimal control problem: find a control $u\in U_\partial$ such that

$$J(u)=\inf_{v\in U_\partial} J(v). \tag{4.3}$$

We call this problem (4.3), and its solutions will be called *optimal controls*.

The main results of this paper are the following.

Theorem 4.1 (Existence of an optimal control)**.** *With the above assumptions in this section, a set of optimal controls of problem* (4.3) *is nonempty and $*$-weakly closed in $L^\infty(Q)$.*

Theorem 4.2 (Necessary conditions of an optimal control)**.** *Let U_∂ be bounded, $\mu = 0$, and*

$$\alpha(t) \geq \alpha_0 = \text{const.} > 0 \quad \textit{for a.e.} \quad t \in S. \tag{4.4}$$

Then an optimal control of problem (4.3) *satisfies the relations*

$$\begin{gathered} y \in L^2_{\omega,\alpha}(S; H^1_0(\Omega)), \quad y_t \in L^2_{\text{loc}}(S; H^{-1}(\Omega)), \\ y_t - \sum_{i,j=1}^n (a_{ij} y_{x_i})_{x_j} + (\widetilde{a}_0 + u) y = f \quad \textit{in } L^2_{\text{loc}}(S; H^{-1}(\Omega)), \\ y\big|_\Sigma = 0, \quad \lim_{t \to -\infty} e^{\omega \int_0^t \alpha(s) ds} \|y(\cdot, t)\|_{L^2(\Omega)} = 0, \end{gathered} \tag{4.5}$$

$$\begin{gathered} p \in L^2_{-\omega, 1/\alpha}(S; H^1_0(\Omega)), \quad p_t \in L^2_{\text{loc}}(S; H^{-1}(\Omega)), \\ -p_t - \sum_{i,j=1}^n (a_{ij} p_{x_i})_{x_j} + (\widetilde{a}_0 + u) p = 0 \quad \text{in} \quad L^2_{\text{loc}}(S; H^{-1}(\Omega)), \\ p\big|_\Sigma = 0, \quad p(\cdot, 0) = y(\cdot, 0) - z_0(\cdot), \end{gathered} \tag{4.6}$$

$$\iint_Q yp(v - u)\, dx\, dt \leq 0 \quad \forall\, v \in U_\partial. \tag{4.7}$$

Since y belongs to $L^2_{\omega,\alpha}(S; H^1_0(\Omega))$, and p belongs to $L^2_{-\omega,1/\alpha}(S; H^1_0(\Omega))$, the product py belongs to $L^1(Q)$, and thus the left-hand side of inequality (4.7) is well-defined.

Problem (4.6) is called an adjoint problem, its solution is called an adjoint state and is introduced in order to characterize an optimal control.

5. Proof of main results

Proof of Theorem 4.1. Since the cost functional J is bounded below, there exists a minimizing sequence $\{v_k\}$ in U_∂: $\lim_{k\to\infty} J(v_k) = \inf_{v \in U_\partial} J(v)$. This and (4.2) imply that the sequence $\{v_k\}$ is bounded in the space $L^\infty(Q)$, that is

$$\operatorname{ess\,sup}_{(x,t)\in Q} |v_k(x,t)| \leq C_5, \tag{5.1}$$

where C_5 is a constant, which does not depend on k.

Since for each $k \in \mathbb{N}$ the function $y_k := y(v_k)$ ($k \in \mathbb{N}$) is a weak solution of (4.1), (3.2), (3.4) for $v = v_k$, the following identity holds:

$$\begin{aligned} &\iint_Q \Big\{ - y_k \psi \varphi' + \sum_{i,j=1}^n a_{ij} y_{k,x_i} \psi_{x_j} \varphi + (\widetilde{a}_0 + v_k) y_k \psi \varphi \Big\}\, dx\, dt \\ &= \iint_Q f \psi \varphi\, dx\, dt, \quad \psi \in H^1_0(\Omega),\ \varphi \in C^1_c(-\infty, 0). \end{aligned} \tag{5.2}$$

According to Theorem 3.3 we have the estimates

$$e^{2\omega \int_0^\tau \alpha(s)\, ds} \|y_k(\cdot,\tau)\|^2_{L^2(\Omega)} \leq C_1 \|f\|_{L^2_{\omega,1/\alpha}(S_\tau; L^2(\Omega))}, \quad \tau \in S, \tag{5.3}$$

$$\|y_k\|_{L^2_{\omega,\alpha}(S_\tau; H^1_0(\Omega))} \leq C_2 \|f\|_{L^2_{\omega,1/\alpha}(S_\tau; L^2(\Omega))}\ . \tag{5.4}$$

Taking into account the first statement of Lemma 2.1, from (5.2) for arbitrary $\tau_1, \tau_2 \in S$ ($\tau_1 < \tau_2$) we obtain

$$\int_{\tau_1}^{\tau_2} \|y_{k,t}\|^2_{H^{-1}(\Omega)}\, dt \leq \int_{\tau_1}^{\tau_2} \int_\Omega \Big(\sum_{j=1}^n \Big| \sum_{i=1}^n a_{ij} y_{k,x_i} \Big|^2 + |(\widetilde{a}_0 + v_k) y_k - f|^2 \Big) dx\, dt. \tag{5.5}$$

By condition (A1), (3.5), (5.1), and (5.4), estimate (5.5) implies

$$\int_{\tau_1}^{\tau_2} \|y_{k,t}(\cdot,t)\|^2_{H^{-1}(\Omega)}\,dt \le C_6, \tag{5.6}$$

where $\tau_1,\tau_2 \in S$ ($\tau_1 < \tau_2$) are arbitrary, $C_6 > 0$ is a constant which depends on τ_1 and τ_2, but does not depend on k.

By the Compactness Lemma (see [20, Proposition 4.2]), and the compactness of the embedding $H_0^1(\Omega) \subset L^2(\Omega)$ (see [19, p. 245]), estimates (5.1), (5.4), (5.6) yield that there exist a subsequence of the sequence $\{v_k, y_k\}$ (which is also denoted by $\{v_k, y_k\}$) and functions $u \in U_\partial$, and $y \in L^2_{\omega,\alpha}(S; H_0^1(\Omega))$ such that

$$v_k \underset{k\to\infty}{\longrightarrow} u \quad *\text{-weakly in} \quad L^\infty(Q), \tag{5.7}$$

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{weakly in} \quad L^2_{\omega,\alpha}(S; H_0^1(\Omega)), \tag{5.8}$$

$$y_k \underset{k\to\infty}{\longrightarrow} y \quad \text{strongly in} \quad L^2_{\rm loc}(S; L^2(\Omega)). \tag{5.9}$$

Note that (5.8) implies

$$y_k \underset{k\to\infty}{\longrightarrow} y, \quad y_{k,x_i} \underset{k\to\infty}{\longrightarrow} y_{x_i}\ (i=\overline{1,n}) \quad \text{weakly in} \quad L^2_{\rm loc}(S; L^2(\Omega)). \tag{5.10}$$

Let us show that (5.7) and (5.9) yield

$$\iint_Q y_k v_k \psi\varphi\,dx\,dt \underset{k\to\infty}{\longrightarrow} \iint_Q yu\psi\varphi\,dx\,dt \quad \forall \psi \in H_0^1(\Omega), \forall\ \varphi \in C_c^1(-\infty,0). \tag{5.11}$$

Indeed, let $g := \psi\varphi$, and $t_1, t_2 \in S$ be such that $\operatorname{supp}\varphi \subset [t_1,t_2]$. Then we have

$$\begin{aligned} \iint_Q y_k v_k g\,dx\,dt &= \int_{t_1}^{t_2}\int_\Omega (y_k v_k - y v_k + y v_k) g\,dx\,dt \\ &= \int_{t_1}^{t_2}\int_\Omega y v_k g\,dx\,dt + \int_{t_1}^{t_2}\int_\Omega (y_k - y) v_k g\,dx\,dt. \end{aligned} \tag{5.12}$$

From (5.1) and (5.9) it follows that

$$\begin{aligned} &\Big|\int_{t_1}^{t_2}\int_\Omega (y_k - y) v_k g\,dx\,dt\Big| \\ &\le \Big(\int_{t_1}^{t_2}\int_\Omega |v_k g|^2\,dx\,dt\Big)^{1/2}\Big(\int_{t_1}^{t_2}\int_\Omega |y_k - y|^2\,dx\,dt\Big)^{1/2} \to 0 \quad \text{as } k\to\infty. \end{aligned} \tag{5.13}$$

Thus, using (5.7) and (5.13), (5.12) implies (5.11).

Using (5.10) and (5.11), and letting $k\to\infty$ in (5.2), we obtain

$$\begin{aligned} &\iint_Q \Big\{ -y\psi\varphi' + \sum_{i,j=1}^n a_{ij} y_{x_i}\psi_{x_j}\varphi + (\widetilde{a}_0 + u) y\psi\varphi\Big\}\,dx\,dt \\ &= \iint_Q f\psi\varphi\,dx\,dt, \quad \psi \in H_0^1(\Omega),\ \varphi \in C_c^1(-\infty,0). \end{aligned} \tag{5.14}$$

According to Lemma 2.1, identity (5.14) implies that $y \in C(S; L^2(\Omega))$ and $y_t \in L^2_{\rm loc}(S; H^{-1}(\Omega))$. Hence, the function $y = y(u)$ is a weak solution of problem (4.1), (3.2). Let us show that y satisfies condition (3.4). First, we prove the convergence

$$\forall \tau \in S: \quad y_k(\cdot,\tau) \underset{k\to\infty}{\longrightarrow} y(\cdot,\tau) \quad \text{strongly in } L^2(\Omega). \tag{5.15}$$

For this purpose, we subtract (5.2) from (5.14). To the resulting identity, we apply Lemma 2.1 with $z = y - y_k$, $g_0 = (\widetilde{a}_0 + u)y - (\widetilde{a}_0 + v_k)y_k$, $g_i = \sum_{j=1}^n a_{ij}(y_{x_j} - y_{k,x_j})$ $(i = \overline{1,n})$, $\theta(t) = 2(t - \tau + 1)$, $\tau_1 = \tau - 1$, $\tau_2 = \tau$, where $\tau \in S$ is arbitrary. Consequently,

$$\begin{aligned} &\int_\Omega |y(x,\tau) - y_k(x,\tau)|^2\,dx - \int_{\tau-1}^{\tau}\int_\Omega |y - y_k|^2\,dx\,dt \\ &+ \int_{\tau-1}^{\tau}\int_\Omega \Big[\sum_{i,j=1}^n a_{ij}(y_{x_i} - y_{k,x_i})(y_{x_j} - y_{k,x_j})\Big]\theta\,dx\,dt \\ &+ \int_{\tau-1}^{\tau}\int_\Omega \big((\widetilde{a}_0 + u)y - (\widetilde{a}_0 + v_k)y_k\big)\big(y - y_k\big)\theta\,dx\,dt = 0. \end{aligned} \tag{5.16}$$

Let us transform the last term on the left side of (5.16) as follows:

$$\begin{aligned} &\int_{\tau-1}^{\tau}\int_\Omega \big((\widetilde{a}_0 + u)y - (\widetilde{a}_0 + v_k)y_k\big)\big(y - y_k\big)\theta\,dx\,dt \\ &= \int_{\tau-1}^{\tau}\int_\Omega \big((\widetilde{a}_0 + u)y - (\widetilde{a}_0 + v_k)(y_k - y + y)\big)\big(y - y_k\big)\theta\,dx\,dt \\ &= \int_{\tau_1}^{\tau_2}\int_\Omega \big[(\widetilde{a}_0 + v_k)|y - y_k|^2 + (u - v_k)y(y - y_k)\big]\theta\,dx\,dt. \end{aligned} \tag{5.17}$$

From (5.16), taking into account (A1) and (5.17), we obtain

$$\begin{aligned} &\int_\Omega |y(x,\tau) - y_k(x,\tau)|^2\,dx + 2\int_{\tau-1}^{\tau}\int_\Omega (\widetilde{a}_0 + v_k)|y - y_k|^2\,dx\,dt \\ &\le \int_{\tau-1}^{\tau}\int_\Omega |y(y - y_k)||u - v_k|\,dx\,dt + \int_{\tau-1}^{\tau}\int_\Omega |y - y_k|^2\,dx\,dt. \end{aligned} \tag{5.18}$$

Using (5.1) and Cauchy-Schwarz inequality, (5.18) yields

$$\begin{aligned} &\int_\Omega |y(x,\tau) - y_k(x,\tau)|^2\,dx \\ &\le C_7\Big(\int_{\tau-1}^{\tau}\int_\Omega |y - y_k|^2\,dx\,dt\Big)^{1/2} + \int_{\tau-1}^{\tau}\int_\Omega |y - y_k|^2\,dx\,dt, \end{aligned} \tag{5.19}$$

where $C_7 > 0$ is a constant which does not depend on k.

From (5.9), according to (5.19), we obtain (5.15). Taking into account (5.15), letting $k \to \infty$ in (5.3), the resulting inequality, according to condition (3.5), implies

$$\lim_{\tau\to-\infty} e^{2\omega\int_0^\tau \alpha(s)\,ds}\int_\Omega |y(x,\tau)|^2\,dx = 0. \tag{5.20}$$

Hence, we have shown that $y = y(u) = y(x,t;u)$, $(x,t) \in Q$, is the state of the controlled system for the control u.

It remains to prove that u is a minimizing element of the functional J. Indeed, (5.15) implies

$$\|y_k(\cdot,0) - z_0(\cdot)\|^2_{L^2(\Omega)} \underset{k\to\infty}{\longrightarrow} \|y(\cdot,0) - z_0(\cdot)\|^2_{L^2(\Omega)}. \tag{5.21}$$

Also, (5.7) and properties of $*$-weakly convergent sequences yield

$$\liminf_{k\to\infty}\|v_k\|_{L^\infty(Q)} \ge \|u\|_{L^\infty(Q)}. \tag{5.22}$$

From (4.2), (5.21) and (5.22), it easily follows that $\lim_{k\to\infty} J(v_k) \ge J(u)$. Thus, we have shown that u is a solution of problem (4.3).

Now let us show that the set of optimal controls of problem (4.3) is $*$-weakly closed. Indeed, let $\{u_k\}$ is a sequence of optimal controls such that $u_k \to u$ $*$-weakly in $L^\infty(Q)$. Similarly as above we show that $\liminf_{k\to\infty} J(u_k) \ge J(u)$. But $J(u_k) = \inf_{v\in U_\partial} J(v)$ $\forall k \in \mathbb{N}$. Then u is an optimal control of (4.3). □

We now turn to the proof of Theorem 4.2. To do this we need some extra statements.

Lemma 5.1. *Under condition* (4.4) *the following continuous embeddings hold*

$$L^2_{\omega,\alpha}(S_\tau; H^1_0(\Omega)) \subset L^2_{\omega,1/\alpha}(S_\tau; H^1_0(\Omega)) \subset L^2_{\omega,1/\alpha}(S_\tau; L^2(\Omega)) \quad \forall \tau \in S,$$

so, there exist positive constants C_8, C_9 such that for arbitrary $z \in L^2_{\omega,\alpha}(S; H^1_0(\Omega))$ and $\tau \in S$ we have

$$\|z\|_{L^2_{\omega,1/\alpha}(S_\tau;L^2(\Omega))} \le C_8 \|z\|_{L^2_{\omega,1/\alpha}(S_\tau;H^1_0(\Omega))} \le C_9 \|z\|_{L^2_{\omega,\alpha}(S_\tau;H^1_0(\Omega))}. \tag{5.23}$$

Proof. The first inequality of (5.23) follows easily from (2.2). According to (4.4) we have $1/\alpha(t) \le 1/\alpha_0 \le \alpha(t)/(\alpha_0)^2$ for a.e. $t \in S$. This yields

$$\int_{-\infty}^{\tau}\int_\Omega [\alpha(t)]^{-1} e^{2\omega\int_0^t \alpha(s)ds} |\nabla z|^2 \,dx\,dt \le [\alpha_0]^{-2} \int_{-\infty}^{\tau}\int_\Omega \alpha(t) e^{2\omega\int_0^t \alpha(s)ds} |\nabla z|^2 \,dx\,dt.$$

So, we obtain (5.23) with $C_9 = C_8[\alpha_0]^{-2}$. □

To proof Theorem 4.2, we need to differentiate the map $v \mapsto J(v)$ with respect to the control v. Since $y = y(v)$ appears in $J(v)$, we first prove the appropriate differentiability of the map $v \mapsto y(v)$ whose derivative is called *sensitivity* (see [2, Section 5]).

Lemma 5.2. *For every $u, v \in U_\partial$ there exists function $\chi = \chi(u,v) = \chi(x,t;u,v) = \chi(x,\tau)$, $(x,t) \in Q$, from $L^2_{\omega,\alpha}(S; H^1_0(\Omega))$ such that $\chi_t \in L^2_{\rm loc}(S; H^{-1}(\Omega))$ (so $\chi \in C(S; L^2(\Omega))$), and*

$$\chi^\varepsilon(u,v) := \frac{y(u+\varepsilon(v-u)) - y(u)}{\varepsilon} \underset{\varepsilon\to 0+}{\longrightarrow} \chi(u,v) \quad \text{weakly in } L^2_{\omega,\alpha}(S; H^1_0(\Omega)), \tag{5.24}$$

$$\chi^\varepsilon(u,v) \underset{\varepsilon\to 0+}{\longrightarrow} \chi(u,v) \quad \text{strongly in } L^2_{\rm loc}(S; L^2(\Omega)), \tag{5.25}$$

$$\forall \tau \in S: \quad \chi^\varepsilon(\cdot,\tau) \underset{\varepsilon\to 0+}{\longrightarrow} \chi(\cdot,\tau) \quad \text{strongly in } L^2(\Omega)\ (\varepsilon \in (0,1)). \tag{5.26}$$

Moreover, sensitivity χ is a weak solution of the problem

$$\chi_t - \sum_{i,j=1}^n (a_{ij}\chi_{x_i})_{x_j} + (\widetilde{a}_0 + u)\chi = (u-v)y, \tag{5.27}$$

$$\chi\big|_\Sigma = 0, \tag{5.28}$$

$$\lim_{t\to-\infty} e^{\omega\int_0^t \alpha(s)ds} \|\chi(\cdot,t)\|_{L^2(\Omega)} = 0. \tag{5.29}$$

Proof. First we denote $w := v - u$, $v^\varepsilon := u + \varepsilon w$, $\varepsilon \in (0,1)$. Since the set U_∂ is convex then for each $\varepsilon \in (0,1)$ an element v^ε belongs to U_∂ for all $u, v \in U_\partial$. It is clear that

$$v^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u \quad \text{strongly in } L^\infty(Q). \tag{5.30}$$

Let the function $y^\varepsilon := y(v^\varepsilon)$ be a weak solution of problem (4.1), (3.2), (3.4) for $v = v^\varepsilon$, where $\varepsilon \in (0,1)$. Theorem 3.3 imply that y^ε exists, it is unique, belongs to $L^2_{\omega,\alpha}(S; H^1_0(\Omega))$ and the following estimates hold

$$e^{\omega \int_0^\tau \alpha(s)\,ds} \|y^\varepsilon(\cdot,\tau)\|_{L^2(\Omega)} \le C_1 \|f\|_{L^2_{\omega,1/\alpha}(S_\tau;L^2(\Omega))}, \quad \tau \in S, \tag{5.31}$$

$$\|y^\varepsilon\|_{L^2_{\omega,\alpha}(S_\tau;H^1_0(\Omega))} \le C_2 \|f\|_{L^2_{\omega,1/\alpha}(S_\tau;L^2(\Omega))}\,, \quad \tau \in S. \tag{5.32}$$

Also by Lemma 5.1 and (5.32) we have

$$\begin{aligned} \|y^\varepsilon\|_{L^2_{\omega,1/\alpha}(S_\tau;L^2(\Omega))} &\le C_9 \|y^\varepsilon\|_{L^2_{\omega,\alpha}(S_\tau;H^1_0(\Omega))} \\ &\le C_2 C_9 \|f\|_{L^2_{\omega,1/\alpha}(S_\tau;L^2(\Omega))}, \quad \tau \in S. \end{aligned} \tag{5.33}$$

Repeating the proof of Theorem 4.1 with v_k being replaced by v^ε and y_k replaced by y^ε we easily obtain convergence similar to (5.8), (5.9), (5.15), i.e.,

$$y^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} y \quad \text{weakly in } L^2_{\omega,\alpha}(S; H^1_0(\Omega)), \tag{5.34}$$

$$y^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} y \quad \text{strongly in } L^2_{\rm loc}(S; L^2(\Omega)), \tag{5.35}$$

$$\forall \tau \in S: \quad y^\varepsilon(\cdot,\tau) \underset{\varepsilon\to 0}{\longrightarrow} y(\cdot,\tau) \quad \text{strongly in } L^2(\Omega), \tag{5.36}$$

where $y := y(u)$ is a solution of (4.1), (3.2), (3.4) for $v = u$, that is, problem (4.5).

Obviously, by the definition of χ^ε, we obtain that χ^ε is the weak solution of the problem

$$\chi^\varepsilon_t - \sum_{i,j=1}^n (a_{ij}\chi^\varepsilon_{x_i})_{x_j} + (\widetilde{a}_0 + u)\chi^\varepsilon = -w y^\varepsilon, \tag{5.37}$$

$$\chi^\varepsilon\big|_\Sigma = 0, \tag{5.38}$$

$$\lim_{t\to-\infty} e^{\omega \int_0^t \alpha(s) ds} \|\chi^\varepsilon(\cdot,t)\|_{L^2(\Omega)} = 0. \tag{5.39}$$

In particular, we have

$$\begin{aligned} &\iint_Q \Big\{ -\chi^\varepsilon \psi \varphi' + \sum_{i,j=1}^n a_{ij}\chi^\varepsilon_{x_i}\psi_{x_j}\varphi + (\widetilde{a}_0 + u)\chi^\varepsilon \psi\varphi \Big\}\,dx\,dt \\ &= -\iint_Q w y^\varepsilon \psi\varphi\,dx\,dt, \quad \psi \in H^1_0(\Omega),\ \varphi \in C^1_c(-\infty,0). \end{aligned} \tag{5.40}$$

Clearly, problem (5.37)-(5.39) coincides with problem (4.1), (3.2), (3.4) when $v = u$ and $f = -wy^\varepsilon$. Hence, taking into account Theorem 3.3 we obtain that χ^ε belongs to $L^2_{\omega,\alpha}(S; H^1_0(\Omega))$, χ^ε_t belongs to $L^2_{\rm loc}(S; H^{-1}(\Omega))$, and satisfies the following estimates

$$e^{\omega \int_0^\tau \alpha(s)\,ds} \|\chi^\varepsilon(\cdot,\tau)\|_{L^2(\Omega)} \le C_1 \|wy^\varepsilon\|_{L^2_{\omega,1/\alpha}(S_\tau;L^2(\Omega))}, \quad \tau \in S,$$

$$\|\chi^\varepsilon\|_{L^2_{\omega,\alpha}(S_\tau;H^1_0(\Omega))} \le C_2 \|wy^\varepsilon\|_{L^2_{\omega,1/\alpha}(S_\tau;L^2(\Omega))}, \quad \tau \in S.$$

Estimate (5.33) implies that

$$\|wy^\varepsilon\|_{L^2_{\omega,1/\alpha}(S_\tau;L^2(\Omega))} \le C_2 C_9 \|w\|_{L^\infty(Q)} \|f\|_{L^2_{\omega,1/\alpha}(S_\tau;L^2(\Omega))}, \tag{5.41}$$

which yields

$$e^{\omega \int_0^\tau \alpha(s)\,ds} \|\chi^\varepsilon(\cdot,\tau)\|_{L^2(\Omega)} \le C_{10} \|f\|_{L^2_{\omega,1/\alpha}(S_\tau;L^2(\Omega))}, \quad \tau \in S, \tag{5.42}$$

$$\|\chi^\varepsilon\|_{L^2_{\omega,\alpha}(S_\tau;H^1_0(\Omega))} \le C_{11} \|f\|_{L^2_{\omega,1/\alpha}(S_\tau;L^2(\Omega))}, \quad \tau \in S, \tag{5.43}$$

where C_{10}, C_{11} are positive constants which do not depend on ε.

Since $L^2_{\omega,\alpha}(S; H^1_0(\Omega))$ is a Hilbert space, then estimate (5.43) yield the existence of function $\chi \in L^2_{\omega,\alpha}(S; H^1_0(\Omega))$ such that convergence (5.24) holds.

Convergence (5.24), (5.35) imply that we can pass to the limit in (5.40) as $\varepsilon \to 0$ and we obtain that function χ satisfies (5.27), (5.28), so it suffices to prove that function χ satisfies condition (5.29) and convergence (5.25), (5.26).

From (5.40) and the first statement of Lemma 2.1, for arbitrary $\tau_1, \tau_2 \in S$ ($\tau_1 < \tau_2$) we obtain

$$\int_{\tau_1}^{\tau_2} \|\chi^\varepsilon_t\|^2_{H^{-1}(\Omega)}\,dt \le \int_{\tau_1}^{\tau_2}\int_\Omega \Big(\sum_{j=1}^n \big|\sum_{i=1}^n a_{ij}\chi^\varepsilon_{x_i}\big|^2 + |(\widetilde{a}_0+u)\chi^\varepsilon + wy^\varepsilon|^2\Big)dx\,dt. \tag{5.44}$$

By condition (A1), and (5.43), estimate (5.44) implies that

$$\int_{\tau_1}^{\tau_2} \|\chi^\varepsilon_t(\cdot,t)\|^2_{H^{-1}(\Omega)}\,dt \le C_{12}, \tag{5.45}$$

where $\tau_1, \tau_2 \in S$ ($\tau_1 < \tau_2$) are arbitrary, $C_{12} > 0$ is a constant which depends on τ_1 and τ_2, but does not depend on k.

Having estimates (5.43), (5.45), we can conclude (similarly as it was done for (5.9)) that there exists a subsequence of $\{v^\varepsilon, y^\varepsilon\}$ (which is also denoted by $\{v^\varepsilon, y^\varepsilon\}$) such that

$$\chi^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \chi \quad \text{strongly in } L^2_{\rm loc}(S; L^2(\Omega)). \tag{5.46}$$

Let us prove the following convergence:

$$\forall \tau \in S: \quad \chi^\varepsilon(\cdot,\tau) \underset{\varepsilon\to 0}{\longrightarrow} \chi(\cdot,\tau) \quad \text{strongly in} \quad L^2(\Omega). \tag{5.47}$$

For this purpose, we subtract identity (5.37) from identity (5.27). To the resulting identity, we apply Lemma 2.1 with $z = \chi - \chi^\varepsilon$, $g_0 = (\widetilde{a}_0 + u)(\chi - \chi^\varepsilon) - wy^\varepsilon$, $g_i = \sum_{j=1}^n a_{ij}(\chi_{x_j} - \chi^\varepsilon_{x_j})$ $(i = \overline{1,n})$, $\theta(t) = 2(t - \tau + 1)$, $\tau_1 = \tau - 1$, $\tau_2 = \tau$, where $\tau \in S$ is arbitrary. Consequently, we obtain

$$\begin{aligned}
&\int_\Omega |\chi(x,\tau) - \chi^\varepsilon(x,\tau)|^2\,dx - \int_{\tau-1}^{\tau}\int_\Omega |\chi - \chi^\varepsilon|^2\,dx\,dt \\
&+ \int_{\tau-1}^{\tau}\int_\Omega \Big[\sum_{i,j=1}^n a_{ij}(\chi_{x_i} - \chi^\varepsilon_{x_i})(\chi_{x_j} - \chi^\varepsilon_{x_j})\Big]\theta\,dx\,dt \\
&+ \int_{\tau-1}^{\tau}\int_\Omega \big((\widetilde{a}_0 + u)(\chi - \chi^\varepsilon) - wy^\varepsilon\big)(\chi - \chi^\varepsilon)\theta\,dx\,dt = 0.
\end{aligned} \tag{5.48}$$

Taking into account (A1), we obtain

$$\begin{aligned}
&\int_\Omega |\chi(x,\tau) - \chi^\varepsilon(x,\tau)|^2\,dx \\
&\le \int_{\tau-1}^{\tau}\int_\Omega |\chi - \chi^\varepsilon|^2\,dx\,dt + \int_{\tau-1}^{\tau}\int_\Omega |w||y^\varepsilon||\chi - \chi^\varepsilon|\,dx\,dt.
\end{aligned} \tag{5.49}$$

Using Cauchy-Schwarz inequality, the above inequality yields

$$\begin{aligned}
\int_\Omega |\chi(x,\tau) - \chi^\varepsilon(x,\tau)|^2\,dx \le C_{13}\Big[&\int_{\tau-1}^{\tau}\int_\Omega |\chi - \chi^\varepsilon|^2\,dx\,dt \\
&+ \Big(\int_{\tau-1}^{\tau}\int_\Omega |\chi - \chi^\varepsilon|^2\,dx\,dt\Big)^{1/2}\Big(\int_{\tau-1}^{\tau}\int_\Omega |y^\varepsilon|^2\,dx\,dt\Big)^{1/2}\Big],
\end{aligned}$$

where $C_{13} > 0$ is a constant depending on $\|w\|_{L^\infty(Q)}$ only.

By (5.35) and (5.46), we obtain (5.47). Taking into account (5.47), and letting $\varepsilon \to 0$ in (5.42), the resulting inequality, according to condition (3.5), implies (5.29). □

Lemma 5.3. *There exists a unique weak solution of* (4.6)*, and if* $\omega < K$*, then it belongs to* $L^2_{-\omega,1/\alpha}(S; H^1_0(\Omega))$ *and satisfies the following estimates:*

$$e^{-\omega\int_0^\tau \alpha(s)ds}\|p(\cdot,\tau)\|_{L^2(\Omega)}\,dx \le C_{14}\|p(\cdot,0)\|_{L^2(\Omega)}, \quad \tau \in S, \tag{5.50}$$

$$\|p\|_{L^2_{-\omega,1/\alpha}(S;H^1_0(\Omega))} \le C_{14}\|p(\cdot,0)\|_{L^2(\Omega)}, \tag{5.51}$$

where $C_{14} > 0$ *is a constant independent of* p*.*

Proof. The existence of a unique weak solution p of (4.6) is a well-known fact. Lemma 2.1 yield $p_t \in L^2_{\rm loc}(S; H^{-1}(\Omega))$. To conclude, it suffices to prove estimates (5.50) and (5.51).

According to Lemma 2.1 when $\tau_1 = \tau < 0, \tau_2 = 0$, $z = -p$, $g_0 = (\widetilde{a}_0 + u)p$, $g_i = \sum_{j=1}^n a_{ij}p_{x_j}$ $(i = \overline{1,n})$, while $\theta \in C^1(S)$ is arbitrary function, we obtain

$$\begin{aligned}
&\frac12\theta(0)\int_\Omega |p(x,0)|^2\,dx - \frac12\theta(\tau)\int_\Omega |p(x,\tau)|^2\,dx - \frac12\int_\tau^0\int_\Omega |p|^2\theta'\,dx\,dt\\
&- \int_\tau^0\int_\Omega \big[\sum_{i,j=1}^n a_{ij}p_{x_i}p_{x_j} + (\widetilde{a}_0 + u)|p|^2\big]\theta\,dx\,dt = 0.
\end{aligned}$$

Taking $\theta(t) = e^{-2\omega\int_0^t \alpha(s)ds}$, $t \in S$, we obtain

$$\begin{aligned}
&\frac12 e^{-2\omega\int_0^\tau \alpha(s)ds}\int_\Omega |p(x,\tau)|^2\,dx - \omega\int_\tau^0\int_\Omega \alpha(t)e^{-2\omega\int_0^t \alpha(s)ds}|p|^2\,dx\,dt\\
&+ \int_\tau^0\int_\Omega e^{-2\omega\int_0^t \alpha(s)ds}\big[\sum_{i,j=1}^n a_{ij}p_{x_i}p_{x_j} + (\widetilde{a}_0 + u)|p|^2\big]\,dx\,dt\\
&= \frac12\int_\Omega |p(x,0)|^2\,dx.
\end{aligned}$$

From this, using condition (A1) we have

$$\begin{aligned}
&e^{-2\omega\int_0^\tau \alpha(s)\,ds}\int_\Omega |p(x,\tau)|^2 dx - 2\omega\int_\tau^0\int_\Omega \alpha(t)e^{-2\omega\int_0^t \alpha(s)\,ds}|p|^2 dx\,dt\\
&+ 2(\delta + 1 - \delta)\int_\tau^0\int_\Omega \alpha(t)e^{-2\omega\int_0^t \alpha(s)\,ds}|\nabla p|^2\,dx\,dt\\
&\le \int_\Omega |p(x,0)|^2\,dx.
\end{aligned}$$

According to (2.4), for arbitrary $\delta \in (0,1)$ we obtain

$$\begin{aligned}
&e^{-2\omega\int_0^\tau \alpha(s)\,ds}\int_\Omega |p(x,\tau)|^2 dx + 2(\delta K - \omega)\int_\tau^0\int_\Omega \alpha(t)e^{-2\omega\int_0^t \alpha(s)\,ds}|p|^2 dx\,dt\\
&+ 2(1 - \delta)\int_\tau^0\int_\Omega \alpha(t)e^{-2\omega\int_0^t \alpha(s)\,ds}|\nabla p|^2\,dx\,dt\\
&\le \int_\Omega |p(x,0)|^2\,dx.
\end{aligned}$$

Since $\omega < K$ we choose $\delta \in [0,1)$ such that $\delta K - \omega > 0$ and obtain

$$
\begin{aligned}
&e^{-2\omega\int_0^\tau \alpha(s)ds}\int_\Omega |p(x,\tau)|^2\,dx + \int_\tau^0\int_\Omega \alpha(t)e^{-2\omega\int_0^t \alpha(s)ds}|\nabla p|^2\,dx\,dt \\
&\le C_{11}\int_\Omega |p(x,0)|^2\,dx,
\end{aligned}
\tag{5.52}
$$

where $C_{11} > 0$ is a constant depending on ω and K only. From (5.52) according to Lemma 5.1 we easily obtain (5.50) and (5.51). □

Proof of Theorem 4.2. Let u be an optimal control of problem (4.3), $v \in U_\partial$ be an arbitrary, then using the same notations as in the proof of Lemma 5.2, for all $\varepsilon \in (0,1)$ we have

$$
J(v^\varepsilon) - J(u) \ge 0. \tag{5.53}
$$

Multiplying variational inequality (5.53) by $1/\varepsilon$ and denoting $w = v - u$, we obtain

$$
\begin{aligned}
0 &\le \frac{1}{\varepsilon}\big(J(v^\varepsilon) - J(u)\big) \\
&= \frac{1}{\varepsilon}\Big[\int_\Omega \big(y(x,0;v^\varepsilon) - z_0(x)\big)^2 dx - \int_\Omega \big(y(x,0;u) - z_0(x)\big)^2 dx\Big] \\
&= \frac{1}{\varepsilon}\int_\Omega \big(y^2(x,0;v^\varepsilon) - y^2(x,0;u) - 2z_0(x)y^2(x,0;v^\varepsilon) + 2z_0(x)y(x,0;u)\big)\,dx \\
&= \int_\Omega \frac{y(x,0;v^\varepsilon) - y(x,0;u)}{\varepsilon}\Big[y(x,0;v^\varepsilon) + y(x,0;u) - 2z_0(x)\Big]dx.
\end{aligned}
$$

We rewrite the above inequality as

$$
\int_\Omega \Big(\chi^\varepsilon(x,0)\big(y^\varepsilon(x,0) + y(x,0)\big) - 2\chi^\varepsilon(x,0)z_0(x)\Big)dx \ge 0. \tag{5.54}
$$

According to (5.26) and (5.36), we pass to the limit in (5.54) as $\varepsilon \to 0+$. As a result we obtain the following variational inequality

$$
\int_\Omega \chi(x,0)\big(y(x,0) - z_0(x)\big)dx \ge 0. \tag{5.55}
$$

Applying formula 2.10 of Lemma (2.1) for the adjoint problem (4.6) with test function χ, i.e. $q = \chi$, for any $\tau \in S$, we obtain

$$
\begin{aligned}
0 &= \int_\tau^0 \big(-p_t - \sum_{i,j=1}^n (a_{ij}p_{x_i})_{x_j} + (\widetilde{a}_0 + u)p, \chi\big)\,dt \\
&= -\int_\tau^0 (p_t,\chi)\,dt + \int_\tau^0\int_\Omega \Big[\sum_{i,j=1}^n a_{ij}p_{x_i}\chi_{x_j} + (\widetilde{a}_0 + u)p\chi\Big]dx\,dt = 0.
\end{aligned}
$$

Using integration by parts and condition (A1) (the symmetry of coefficients a_{ij}), we obtain

$$
\begin{aligned}
0 &= -\int_\Omega p\chi\,dx\Big|_{t=\tau}^{t=0} + \int_\tau^0 (p,\chi_t)\,dt \\
&\quad + \int_\tau^0 \int_\Omega \Big[\sum_{i,j=1}^n a_{ij}p_{x_i}\chi_{x_j} + (\widetilde{a}_0+u)p\chi\Big]\,dx\,dt \\
&= -\int_\Omega \chi(x,0)p(x,0)\,dx + \int_\Omega \chi(x,\tau)p(x,\tau)\,dx \\
&\quad + \int_\tau^0 (\chi_t,p)\,dt + \int_\tau^0\int_\Omega \Big[\sum_{i,j=1}^n a_{ij}\chi_{x_i}p_{x_j} + (\widetilde{a}_0+u)\chi p\Big]dx\,dt \\
&= -\int_\Omega \chi(x,0)p(x,0)\,dx + \int_\Omega \chi(x,\tau)p(x,\tau)\,dx \\
&\quad + \int_\tau^0 \big(\chi_t - \sum_{i,j=1}^n (a_{ij}\chi_{x_i})_{x_j} + (\widetilde{a}_0+u)\chi, p\big) - \int_\Omega \chi(x,0)p(x,0)\,dx \\
&\quad + \int_\Omega \chi(x,\tau)p(x,\tau)\,dx - \int_\tau^0\int_\Omega wyp\,dx\,dt.
\end{aligned}
\tag{5.56}
$$

By the weak solution formulation for problem (5.27)–(5.29), from (5.56) we obtain

$$
\int_\Omega \big(y(x,0)-z_0(x)\big)\chi(x,0)\,dx = \int_\Omega p(x,\tau)\chi(x,\tau)\,dx - \int_\tau^0\int_\Omega pyw\,dx\,dt. \tag{5.57}
$$

Let us show that we can pass to the limit in (5.57) as $t\to-\infty$. Indeed, according to (5.50) and (5.29), we have

$$
\begin{aligned}
\int_\Omega |p(x,\tau)\chi(x,\tau)|\,dx &\le \|p(\cdot,\tau)|_{L^2(\Omega)}\|\chi(\cdot,\tau)\|_{L^2(\Omega)} \\
&\le e^{\omega\int_0^\tau \alpha(s)ds}\|p(\cdot,0)\|_{L^2(\Omega)}\|\chi(\cdot,\tau)\|_{L^2(\Omega)} \\
&= \|p(\cdot,0)\|_{L^2(\Omega)}\gamma(\tau),
\end{aligned}
\tag{5.58}
$$

where because of condition (5.29), the function $\gamma(t) := e^{\omega\int_0^\tau \alpha(s)ds}\|\chi(\cdot,\tau)\|_{L^2(\Omega)}$, $t\in S$, is such that $\gamma(t)\to 0$ as $t\to-\infty$.

Condition (A2), Theorem 3.3 (estimate (3.7)) and estimate (5.51), by the Cauchy-Schwarz inequality, imply

$$
\begin{gathered}
\int_\tau^0\int_\Omega |pyw|\,dx\,dt \le \Big(\int_\tau^0\int_\Omega [\alpha]^{-1}e^{-2\omega\int_0^t\alpha(s)ds}|p(x,t)|^2dx\,dt\Big)^{1/2}, \\
\Big(\int_\tau^0\int_\Omega \alpha e^{2\omega\int_0^t\alpha(s)ds}|y|^2dx\,dt\Big)^{1/2} \le C_2C_{10}\|p(\cdot,0)\|^2_{L^2(\Omega)}\|f\|_{L^2_{\omega,1/\alpha}(S;L^2(\Omega))},
\end{gathered}
$$

which yields $wpy\in L^1(Q)$.

According to this and (5.58), we pass to the limit in (5.57) as $\tau\to-\infty$, and obtain

$$
\int_\Omega \big(y(x,0)-z_0(x)\big)\chi(x,0)\,dx = -\iint_Q pyw\,dx\,dt. \tag{5.59}
$$

From (5.55) taking into account (5.59) we obtain (4.7). □

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