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TIME-PERIODIC STRONG SOLUTIONS OF THE 3D NAVIER-STOKES EQUATIONS WITH DAMPING

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Abstract. This article concerns the incompressible Navier-Stokes equations with damping and homogeneous Dirichlet boundary conditions in 3D bounded domains. We find conditions on parameters to guarantee that the problem has a strong time-periodic solution and that the weak solutions of the problem converge to a unique time-periodic solution as $t \to \infty$.

1. Introduction

In this article we consider the three-dimensional Navier-Stokes equations with nonlinear damping

$$
\begin{gathered}
u_t + (u\cdot\nabla)u - \nu\Delta u + \alpha|u|^{\beta-1}u = -\nabla p + f, \quad x\in\Omega,\ t>0,\\
\operatorname{div} u = 0, \quad x\in\Omega,\ t>0,\\
u(x,t) = 0, \quad x\in\partial\Omega,\ t>0,\\
u(x,0) = u_0(x), \quad x\in\Omega,
\end{gathered} \tag{1.1}
$$

where $\nu>0, \alpha>0$ and $\beta\ge 1$ are constants and $f(x,t)$ is the external force. $\Omega\subset\mathbb{R}^3$ is an open bounded set with the boundary $\partial\Omega$ smooth enough. The unknown functions $u(x,t)$ and $p(x,t)$ are velocity and pressure of the flow, respectively.

For the case $\alpha=0$, the equations are the three-dimensional Navier-Stokes equations and the regularity and the uniqueness of weak solutions remain completely open in spite of interests of many mathematicians.

For the case $\alpha>0$, the problem describes the flow with the resistance to the motion such as porous media flow and drag or friction effects (see [3] and references therein). From a mathematical viewpoint, (1.1) can be viewed as a modification of the classic Navier-Stokes equations with the regularizing term $\alpha|u|^{\beta-1}u$. So, it is important to find the conditions on parameters to guarantee the regularity properties and uniqueness of the weak solutions of (1.1).

For the case $\alpha>0$, problem (1.1) has been studied in [3, 7, 8, 10, 11]. In [3], it was proved that the Cauchy problem in $\Omega=\mathbb{R}^3$ has week solutions if $\beta\ge 1$ and global strong solutions if $\beta\ge 7/2$ and the strong solution is unique if $5\ge\beta\ge 7/2$.

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In [10], they proved for $\beta > 3$ that problem (1.1) has a global strong solution and the strong solution is unique when $5 \geq \beta > 3$. Later, in [11], they proved that the strong solution exists globally for $\beta = 3$ and $\alpha = \nu = 1$ and the strong solution is unique even among weak solutions in $L^\infty(0, T; (L^2(\Omega))^3)$ for $\beta \geq 1$. In [7, 8], the existence of global attractor for the problem in bounded set $\Omega \subset \mathbb{R}^3$ was proved provided $5 \geq \beta \geq 7/2$. However, the existence of global strong solutions was not proved when $\beta = 3$ and $1 > \alpha\nu > 0$ if $\Omega = \mathbb{R}^3$ and when $7/2 > \beta \geq 3$ if $\Omega \subset \mathbb{R}^3$ is bounded. Also, for the problem (1.1), whether the weak solution, which is not smooth, is unique or not is still open.

On the other hand, the time-periodic solutions of the Navier-Stokes equations and variants of it have been studied by several authors (see e.g. [1, 2, 4, 5, 6]). They proved the existence of the time-periodic solutions of the Navier-Stokes equations in an unbounded domain in [1] and [6], of the Euler-Voigt and Navier-Stokes-Voigt models in [2] and of 2D stochastic Navier-Stokes equations in [4], and they proved the uniqueness of the time-periodic solutions of the Navier-Stokes equations in [5] under the some assumptions. However, it seems that there is no result for the time-periodic solutions of (1.1).

Motivated by the above work, we shall study the incompressible Navier-Stokes Equations with damping and homogeneous Dirichlet boundary conditions in 3D bounded domains. First, we are going to improve conditions on parameters to guarantee global existence of strong solutions of (1.1) as $\beta = 3$, $\alpha\nu > 1/4$. Second, we will prove that the problem has the strong T-periodic solutions provided $5 > \beta > 3$, $\alpha > 0$ or $\beta = 3$, $\alpha\nu > 1/4$ and $f \in W^{1,2}_{\rm loc}(0, \infty; H)$ is T-periodic. Finally, we find the conditions on parameters to guarantee that the weak solutions of the problem converge to the unique T-periodic strong solution as $t \to \infty$.

We define the usual space

$$V_\infty := \{v \in (C_0^\infty(\Omega))^3, \operatorname{div} u = 0\}.$$

Let H and V denote the closure of V_∞ in the space $(L^2(\Omega))^3$ and $(H_0^1(\Omega))^3$, respectively. H and V are endowed, respectively, with the inner products

$$(u, v) = \int_\Omega u \cdot v \, dx, \quad \forall u, v \in H,$$
$$((u, v)) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad \forall u, v \in V.$$

Denote the dual space of V by V' and the norms in $(L^q(\Omega))^3$ by $\|\cdot\|_q (1 \leq q \leq \infty)$ and $\|\cdot\| = \|\cdot\|_2$.

A function $u \in L^\infty_{\rm loc}(0, \infty; H) \cap L^2_{\rm loc}(0, \infty; V) \cap L^{\beta+1}_{\rm loc}(0, \infty; (L^{\beta+1}(\Omega))^3)$ is said to be a weak solution of (1.1) if it satisfies

$$\begin{gathered} \frac{d}{dt}(u, v) + \nu((u, v)) + b(u, u, v) + \alpha(|u|^{\beta-1}u, v) = (f, v), \\ \forall v \in V \cap (L^{\beta+1}(\Omega))^3, \quad t > 0 \\ u(0) = u_0, \end{gathered} \tag{1.2}$$

where

$$b(u, v, w) = \Sigma_{i,j=1}^3 \int_\Omega u_i \partial_i v_j w_j \, dx, \quad \partial_i := \partial/\partial x_i.$$

The weak formulation (1.2) is equivalent to the abstract equation

$$\frac{du}{dt} + \nu Au + B(u) + G(u) = Pf, \quad t > 0$$
$$u(0) = u_0,$$

where $Au = -P\Delta u$ and P is the orthogonal projection of $(L^2(\Omega))^3$ onto H, $G(u) = \alpha P|u|^{\beta-1}u$ and $\langle B(u), w\rangle = b(u,u,w)$(see [8]). The existence of global weak solutions of (1.1) was proved when $\beta \geq 1, \alpha > 0, u_0 \in H$, and $f \in L^2_{\rm loc}(0,\infty;(L^2(\Omega))^3)$, by Galerkin method(see [7, Theorem 1], [8, Theorem 2.1] and [3, Theorem 2.1]).

The weak solution obtained as the limit of Galerkin approximations is said to be a G-weak solution.

We say that $(u(x,t), p(x,t))$ is a strong solution of (1.1), if $u(x,t)$ is a weak solution and $u \in L^\infty_{\rm loc}(0,\infty;V \cap (L^{\beta+1}(\Omega))^3) \cap L^2_{\rm loc}(0,\infty;D(A))$. Now we state our main results as follows.

Theorem 1.1. *Suppose $u_0 \in V$ and $f \in W^{1,2}_{\rm loc}(0,\infty;(L^2(\Omega))^3)$. Under the assumptions*

$$5 > \beta > 3,\ \alpha > 0 \quad \text{or} \quad \beta = 3,\ \alpha\nu > \frac{1}{4},$$

the G-weak solutions of (1.1) are strong solutions.

Theorem 1.2. *If*

$$\beta > 3,\ \alpha > 0 \quad \text{or} \quad \beta = 3,\ \alpha\nu \geq \frac{1}{4},$$

then there is a constant δ (not necessarily positive) such that

$$\|u(t) - v(t)\|^2 \leq \|u_0 - v_0\|^2 \exp(-\delta t) \tag{1.3}$$

for all $t \geq 0$ and for any $u_0, v_0 \in H$, where u and v are the G-weak solutions with initial data u_0, v_0, respectively. Further, if $\beta \geq 3, \alpha > 0$ and

$$\lambda_1^{\beta-3}\alpha^2(2\nu)^{2\beta-4} > \frac{(2\beta-4)^{2\beta-4}}{(\beta-1)^{2\beta-2}}, \tag{1.4}$$

then there is a $\delta > 0$, which satisfies (1.3), where λ_1 is the first eigenvalue of the Stokes operator A.

Remark 1.3. We conclude that the G-weak solution of (1.1) is unique under the first condition of Theorem 1.2.

Note that Theorem 1.2 is independent of Theorem 1.1, but we have the following combination of these two theorems.

Theorem 1.4. *Suppose $f \in W^{1,2}_{\rm loc}(0,\infty;(L^2(\Omega))^3)$ and $f(\cdot,t) = f(\cdot,t+T)(\forall t \geq 0)$. If*

$$5 > \beta > 3,\ \alpha > 0 \quad \text{or} \quad \beta = 3,\ \alpha\nu > \frac{1}{4},$$

then there exists a T-periodic strong solution of (1.1). Further, if (1.4) is fulfilled, the periodic solution of (1.1) is unique and the weak solutions of (1.1) converge exponentially to the periodic solution as $t \to \infty$.

2. Proof of the Theorem 1.1

The proof is based on some a priori estimations which we will obtain by standard energy and Sobolev estimates. When $\Omega = \mathbb{R}^3$, $-\Delta u$ can be used as a test function to get some regularity of weak solutions. But, it is not an allowed test function for the Dirichlet boundary value problem. So, the relation

$$-(|u|^{\beta-1}u, \Delta u) = \||u|^{(\beta-1)/2}\nabla u\|^2 + \frac{\beta-1}{4}\||u|^{(\beta-3)/2}\nabla |u|^2\|^2$$

can not be used to obtain a regularity property, while it has been used well for the Cauchy problem(see [3, 11]). It is a difficulty for our case.

Suppose $u_0 \in V$, $f \in W^{1,2}_{\rm loc}(0,\infty;(L^2(\Omega))^3)$ and $5 > \beta > 3, \alpha > 0$ or $\beta = 3, \alpha\nu > 1/4$. Let u be a G-weak solution of (1.1) and fix $T > 0$.

Assuming that u is smooth, in (1.2), we replace v by Au, we have

$$(u_t, Au) + \nu((u, Au)) + b(u, u, Au) = -\alpha(|u|^{\beta-1}u, Au) + (f(t), Au).$$

Since

$$(u_t, Au) = \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2, ((u, Au)) = \|Au\|^2,$$

this relation can be written

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \nu\|Au\|^2 + b(u, u, Au) = -\alpha(|u|^{\beta-1}u, Au) + (f(t), Au). \tag{2.1}$$

Using

$$|b(u, u, Au)| \le C\|\nabla u\|^{3/2}\|Au\|^{3/2}$$

(see [9, (2.32)]) and Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \nu\|Au\|^2 \\
&\le C\|\nabla u\|^{3/2}\|Au\|^{3/2} + \alpha\|u\|^{\beta}_{2\beta}\|Au\| + \|f(t)\|\|Au\| \\
&\le \frac{\nu}{2}\|Au\|^2 + C\|\nabla u\|^6 + C'\|u\|^{2\beta}_{2\beta} + \frac{1}{\nu}\|f(t)\|^2.
\end{aligned} \tag{2.2}$$

For the estimate of $\|u\|^{2\beta}_{2\beta}$ recall that $5 > \beta \ge 3$ and $\|u\|^2_\infty \le C\|\nabla u\|\|Au\|$ (see [9]). Thus we get that

$$\begin{aligned}
\|u(t)\|^{2\beta}_{2\beta} &= \int_\Omega |u(x,t)|^{2\beta-6}|u(x,t)|^6\,dx \\
&\le C + \|u(t)\|^{2\beta-6}_\infty\|u(t)\|^6_6 \\
&\le C + C\|u(t)\|^{2\beta-6}_\infty\|\nabla u(t)\|^6 \\
&\le C + C\|\nabla u(t)\|^{\beta+3}\|Au(t)\|^{\beta-3} \\
&\le C + C\|\nabla u(t)\|^{2(\beta+3)/(5-\beta)} + \frac{\nu}{4C'}\|Au(t)\|^2.
\end{aligned} \tag{2.3}$$

Substituting (2.3)) into (2.2) and putting $q := (\beta+3)/(5-\beta)(\ge 3)$, we obtain

$$\frac{d}{dt}\|\nabla u\|^2 + \frac{\nu}{2}\|Au\|^2 \le C + C\|\nabla u\|^{2q} + \frac{2}{\nu}\|f(t)\|^2. \tag{2.4}$$

Momentarily, dropping the term $\nu/2\|Au\|^2$, we have a differential inequality,

$$\frac{d}{dt}y(t) \le C(C_1 + y^q(t)), y(t) := \|\nabla u(t)\|^2, \quad C_1 := C + \frac{2}{\nu}\sup_{0\le t\le T}\|f(t)\|^2.$$

Here, we used $f \in C([0,T];(L^2(\Omega))^3)$, because $f \in W^{1,2}_{\rm loc}(0,\infty;(L^2(\Omega))^3)$. The differential inequality

$$\frac{d}{dt}z(t) \le Cz^q(t), \quad z = 1+y, \quad z(0) = z_0 := 1 + \|\nabla u_0\|^2$$

has a solution defined on $[0,T']$, where

$$T' = \frac{1}{2Cqz_0^{q-1}} > 0. \tag{2.5}$$

And by comparison, we have

$$\begin{gathered} z(t) \le \frac{z_0}{(1 - CqT'z_0^{q-1})^{1/(q-1)}} = 2^{1/(q-1)}z_0\,, \\ \|\nabla u(t)\|^2 \le 2^{1/(q-1)}(1+\|\nabla u_0\|^2), \quad \forall t \in [0,T']. \end{gathered} \tag{2.6}$$

Therefore, by (2.4) and (2.6), we have

$$\sup_{0\le t\le T'} \|\nabla u(t)\|^2 + \int_0^{T'} \|Au(t)\|^2 dt \le C_2(\|\nabla u_0\|^{2q}). \tag{2.7}$$

Thus, we proved the existence of local strong solutions to (1.1).

Multiplying the first equation of (1.1) by u_t, integrating the resulting equation on Ω, and using Young's inequality and [9, (2.32)], we have

$$\begin{aligned} &\|u_t\|^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla u\|^2 + \frac{\alpha}{\beta+1}\frac{d}{dt}\|u\|_{\beta+1}^{\beta+1} \\ &= -((u\cdot\nabla)u, u_t) + (f(t), u_t) \\ &\le \frac{1}{4}\|u_t\|^2 + C\|\nabla u\|^{3/2}\|Au\|^{1/2}\|u_t\| + \|f(t)\|^2 \\ &\le \frac{1}{2}\|u_t\|^2 + C(\|\nabla u\|^6 + \|Au\|^2 + \|f(t)\|^2). \end{aligned} \tag{2.8}$$

Integrating over $[0,T']$ and considering (2.7), it follows that

$$\int_0^{T'} \|u_t(t)\|^2 dt \le C_3(\|\nabla u_0\|^{2q}). \tag{2.9}$$

Therefore, there is a constant $0 < t_1 < T'$ such that

$$\|u_t(t_1)\|^2 \le \frac{C_3}{T'} = 2Cq(1+\|\nabla u_0\|^2)^{q-1}C_3(\|\nabla u_0\|^{2q}) \tag{2.10}$$

by (2.5) and (2.9).

Differentiating the first equation of (1.1) with respect to t and taking the inner product with u_t in H, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \nu\|\nabla u_t\|^2 + \alpha((|u|^{\beta-1}u)_t, u_t) = -((u\cdot\nabla u)_t, u_t) + (f_t(t), u_t). \tag{2.11}$$

Since

$$\begin{aligned} ((|u|^{\beta-1}u)_t, u_t) &= (|u|^{\beta-1}u_t, u_t) + \frac{(\beta-1)}{4}\int_\Omega |u|^{\beta-3}|\frac{\partial}{\partial t}|u|^2|^2\,dx \\ &\ge (|u|^{\beta-1}u_t, u_t), \end{aligned}$$

$$\begin{aligned} -((u\cdot\nabla u)_t,u_t) &= -((u_t\cdot\nabla)u,u_t)-(u\cdot\nabla u_t,u_t)\\ &= -((u_t\cdot\nabla)u,u_t)=((u_t\cdot\nabla)u_t,u)\\ &\le \epsilon_1\nu\|\nabla u_t\|^2+\frac{1}{4\epsilon_1\nu}\||u||u_t|\|^2, \end{aligned}$$

$$(f_t,u_t)\le \frac14\|u_t\|^2+\|f_t\|^2,$$

for any $\epsilon_1>0$, It follows that (2.11) implies

$$\begin{aligned} &\frac12\frac{d}{dt}\|u_t\|^2+\nu(1-\epsilon_1)\|\nabla u_t\|^2+\alpha\||u|^{(\beta-1)/2}|u_t|\|^2\\ &\le \frac{1}{4\epsilon_1\nu}\||u||u_t|\|^2+\frac14\|u_t\|^2+\|f_t(t)\|^2. \end{aligned} \tag{2.12}$$

On the other hand, in (2.8), we use

$$-((u\cdot\nabla)u,u_t)=((u\cdot\nabla)u_t,u)\le \nu\epsilon_2\|\nabla u_t\|^2+\frac{1}{4\nu\epsilon_2}\|u\|_4^4$$

to have

$$\begin{aligned} &\frac12\|u_t\|^2+\frac{\nu}{2}\frac{d}{dt}\|\nabla u\|^2+\frac{\alpha}{\beta+1}\frac{d}{dt}\|u\|_{\beta+1}^{\beta+1}\\ &\le \nu\epsilon_2\|\nabla u_t\|^2+\frac{1}{4\nu\epsilon_2}\|u\|_4^4+\|f(t)\|^2, \end{aligned} \tag{2.13}$$

for any $\epsilon_2>0$. Adding (2.12) and (2.13), we have

$$\begin{aligned} &\frac{d}{dt}(\frac12\|u_t\|^2+\frac{\nu}{2}\|\nabla u\|^2+\frac{\alpha}{\beta+1}\|u\|_{\beta+1}^{\beta+1})+\frac14\|u_t\|^2\\ &+\nu(1-\epsilon_1-\epsilon_2)\|\nabla u_t\|^2+\alpha\||u|^{(\beta-1)/2}|u_t|\|^2-\frac{1}{4\epsilon_1\nu}\||u||u_t|\|^2\\ &\le \frac{1}{4\epsilon_2\nu}\|u\|_4^4+\|f(t)\|^2+\|f_t(t)\|^2\\ &\le C(1+\|u\|_{\beta+1}^{\beta+1})+\|f(t)\|^2+\|f_t(t)\|^2. \end{aligned} \tag{2.14}$$

Suppose $\beta=3$ and $\alpha\nu>1/4$. Then we choose constants $\epsilon_1>0$ and $\epsilon_2>0$ which satisfy

$$1>\epsilon_1>0,\quad \alpha-\frac{1}{4\epsilon_1\nu}>0,\quad 1-\epsilon_1-\epsilon_2>0.$$

If $\beta>3$ and $\alpha>0$, there are constants $\epsilon_1>0$, $\epsilon_2>0$ and $\gamma>0$ which satisfy $1>\epsilon_1>0$, $1-\epsilon_1-\epsilon_2>0$ and

$$\frac{\alpha}{2}\||u|^{(\beta-1)/2}|u_t|\|^2-\gamma\|u_t\|^2\le \alpha\||u|^{(\beta-1)/2}|u_t|\|^2-\frac{1}{4\epsilon_1\nu}\||u||u_t|\|^2$$

As above, substituting $\epsilon_1>0$, $\epsilon_2>0$ and $\gamma>0$ in (2.14), we obtain

$$\begin{aligned} &\frac{d}{dt}(\frac12\|u_t\|^2+\frac{\nu}{2}\|\nabla u\|^2+\frac{\alpha}{\beta+1}\|u\|_{\beta+1}^{\beta+1})\\ &\le C(1+\|u\|_{\beta+1}^{\beta+1})+\gamma\|u_t\|^2+\|f(t)\|^2+\|f_t(t)\|^2, \end{aligned} \tag{2.15}$$

for any $t_1\le t\le T$. Applying Gronwall's lemma into (2.15) and dropping unnecessary terms, we have

$$\|\nabla u(t)\|^2\le C(T)(\|u_t(t_1)\|^2+\|\nabla u(t_1)\|^2+\|u(t_1)\|_{\beta+1}^{\beta+1}$$

$$+\int_0^T (\|f(t)\|^2 + \|f_t(t)\|^2)dt),$$

for any $t_1 \le t \le T$. Thus using (2.6), (2.10) and $\|u\|_{\beta+1} \le C\|\nabla u\|$, we can show that

$$\sup_{T' \le t \le T} \|\nabla u(t)\|^2 \le C(T)(1 + \|\nabla u_0\|^{4q(q-1)} + \int_0^T (\|f(t)\|^2 + \|f_t(t)\|^2)dt). \quad (2.16)$$

From this and (2.6), we deduce that $\|\nabla u(t)\|$ is bounded on $[0,T]$. Therefore, integrating (2.4) over $[0,T]$, we have $u \in L^\infty(0,T;V) \cap L^2(0,T;D(A))$, and thus we have $u \in L^\infty_{\rm loc}(0,\infty;V \cap (L^{\beta+1}(\Omega))^3) \cap L^2_{\rm loc}(0,\infty;D(A))$. This completes the proof.

3. Proof of the Theorem 1.2

First, we introduce the elementary inequality which will be used later.

Lemma 3.1. *For any $x, y \in \mathbb{R}^N$ and $\beta \ge 1$, the inequality*

$$(|x|^{\beta-1}x - |y|^{\beta-1}y)\cdot(x-y) \ge \frac{1}{2}(|x|^{\beta-1} + |y|^{\beta-1})|x-y|^2 \quad (3.1)$$

holds and the coefficient $1/2$ is the best.

Proof. Let $x, y \in \mathbb{R}^N$ and $\beta \ge 1$. Then it follows that

$$0 \le (|x|^{\beta-1} - |y|^{\beta-1})(|x|^2 - |y|^2) = |x|^{\beta+1} + |y|^{\beta+1} - |x|^{\beta-1}|y|^2 - |y|^{\beta-1}|x|^2,$$
$$|x|^{\beta+1} + |y|^{\beta+1} \ge |x|^{\beta-1}|y|^2 + |y|^{\beta-1}|x|^2.$$

Adding this to $|x|^{\beta+1} + |y|^{\beta+1} - 2(|x|^{\beta-1} + |y|^{\beta-1})x\cdot y$, we obtain

$$\begin{aligned}
&2(|x|^{\beta-1}x - |y|^{\beta-1}y)\cdot(x-y) = 2|x|^{\beta+1} + 2|y|^{\beta+1} - 2(|x|^{\beta-1} + |y|^{\beta-1})x\cdot y \\
&\ge |x|^{\beta+1} + |y|^{\beta+1} - 2(|x|^{\beta-1} + |y|^{\beta-1})x\cdot y + |x|^{\beta-1}|y|^2 + |y|^{\beta-1}|x|^2 \\
&= (|x|^{\beta-1} + |y|^{\beta-1})|x-y|^2.
\end{aligned}$$

Therefore, we have (3.1). The coefficient $1/2$ is the best since the equality is fulfilled in (3.1) if $x = -y$ or $x = y$. Then proof is complete. □

Lemma 3.2. *Let λ_1, ν, α be positive and $\beta \ge 3$. If*

$$\lambda_1^{\beta-3}\alpha^2(2\nu)^{2\beta-4} > \frac{(2\beta-4)^{2\beta-4}}{(\beta-1)^{2\beta-2}}, \quad (3.2)$$

then there are $\delta > 0$ and $1 > \epsilon > 0$ such that

$$2\nu\lambda_1(1-\epsilon) + \alpha(|u|^{\beta-1} + |v|^{\beta-1}) - \frac{1}{4\nu\epsilon}(|u|^2 + |v|^2) \ge \delta,$$

for any $u, v \in \mathbb{R}^N$.

Proof. Let $\alpha > 0, \nu > 0$ and $\epsilon > 0$. Suppose $\beta > 3$ and consider the function

$$g(x,y) := 2\nu\lambda_1(1-\epsilon) + \alpha(x^{(\beta-1)/2} + y^{(\beta-1)/2}) - \frac{1}{4\nu\epsilon}(x+y).$$

The minimum of $g(x,y)$ in $\{(x,y) \in R^2 : x \ge 0, y \ge 0\}$ attains at

$$(x_0, y_0) = ((2\nu\epsilon\alpha(\beta-1))^{-2/(\beta-3)}, (2\nu\epsilon\alpha(\beta-1))^{-2/(\beta-3)}).$$

Further, it is satisfied that

$$g(x_0, y_0) = 2\nu\lambda_1(1-\epsilon) - (\beta-3)\alpha^{-2/(\beta-3)}(2\nu\epsilon(\beta-1))^{-(\beta-1)/(\beta-3)}.$$

Find $1 \geq \epsilon > 0$ which fulfills
$$2\nu\lambda_1(1-\epsilon) > (\beta-3)\alpha^{-2/(\beta-3)}(2\nu\epsilon(\beta-1))^{-(\beta-1)/(\beta-3)}.$$
This is equivalent to
$$\lambda_1^{\beta-3}\alpha^2(2\nu)^{2\beta-4}(1-\epsilon)^{\beta-3}\epsilon^{\beta-1} > \frac{(\beta-3)^{\beta-3}}{(\beta-1)^{\beta-1}}. \tag{3.3}$$
The maximum of $(1-\epsilon)^{\beta-3}\epsilon^{\beta-1}$ in $0 \leq \epsilon \leq 1$ is
$$\frac{(\beta-3)^{\beta-3}(\beta-1)^{\beta-1}}{(2\beta-4)^{2\beta-4}}.$$
Substituting this into (3.3), we have (3.2). If $\beta = 3$, (3.2) is equivalent to $\alpha\nu > 1/4$. The proof is complete. □

Proof of Theorem 1.2. Suppose $\beta > 3$ and $\alpha > 0$, or $\beta = 3$ and $\alpha\nu \geq 1/4$. Denote u and v be the G-weak solutions with initial data $u_0, v_0 \in H$, respectively. Fix $T > 0$. Let $u_m(m \geq 1)$ and $v_n(n \geq 1)$ be subsequences of Galerkin approximations of u and v of (1.1), respectively, which satisfy the conditions
$$u_m \to u \quad \text{weakly in } L^2(0,T;V) \text{ and strongly in } L^2(0,T;H), \text{ as } m \to \infty$$
$$v_n \to v \text{ weakly in } L^2(0,T;V) \text{ and strongly in } L^2(0,T;H) \text{ as } n \to \infty,$$
(see [3, 7]). Denote $w_{m,n} = u_m - v_n$. Using the inequality (3.1) and
$$((u_m\cdot\nabla)u_m, w_{m,n}) - ((v_n\cdot\nabla)v_n, w_{m,n}) = -((w_{m,n}\cdot\nabla)w_{m,n}, u_m),$$
we get
$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_{m,n}\|^2 + \nu\|\nabla w_{m,n}\|^2 \\
&= -\alpha(|u_m|^{\beta-1}u_m - |v_n|^{\beta-1}v_n, u_m - v_n) - ((u_m\cdot\nabla)u_m, w_{m,n}) \\
&\quad + ((v_n\cdot\nabla)v_n, w_{m,n}) \\
&\leq -\frac{\alpha}{2}\int_\Omega (|u_m|^{\beta-1} + |v_n|^{\beta-1})|w_{m,n}|^2\,dx + \int_\Omega |u_m||w_{m,n}||\nabla w_{m,n}|\,dx \\
&\leq -\frac{\alpha}{2}\int_\Omega (|u_m|^{\beta-1} + |v_n|^{\beta-1})|w_{m,n}|^2\,dx + \nu\epsilon\|\nabla w_{m,n}\|^2 \\
&\quad + \frac{1}{4\nu\epsilon}\int_\Omega |u_m|^2|w_{m,n}|^2\,dx,
\end{aligned} \tag{3.4}$$
for $t > 0$, where $0 < \epsilon \leq 1$. Changing the order of u_m and v_n, we have
$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_{m,n}\|^2 + \nu\|\nabla w_{m,n}\|^2 \\
&\leq -\frac{\alpha}{2}\int_\Omega (|u_m|^{\beta-1} + |v_n|^{\beta-1})|w_{m,n}|^2\,dx + \nu\epsilon\|\nabla w_{m,n}\|^2 \\
&\quad + \frac{1}{4\nu\epsilon}\int_\Omega |v_n|^2|w_{m,n}|^2\,dx,
\end{aligned} \tag{3.5}$$
Adding (3.4) and (3.5) and using $\lambda_1\|w_{m,n}\|^2 \leq \|\nabla w_{m,n}\|^2$, it follows that
$$\begin{aligned}
&\frac{d}{dt}\|w_{m,n}\|^2 + 2\lambda_1\nu(1-\epsilon)\|w_{m,n}\|^2 \\
&\leq \int_\Omega \{-\alpha(|u_m|^{\beta-1} + |v_n|^{\beta-1}) + \frac{1}{4\nu\epsilon}(|u_m|^2 + |v_n|^2)\}|w_{m,n}|^2\,dx.
\end{aligned} \tag{3.6}$$

Suppose $\beta > 3$ and $\alpha > 0$, or $\beta = 3$ and $\alpha\nu \geq \frac{1}{4}$. Then there are constants δ (not necessarily positive) and $\epsilon \in (0,1]$ such that

$$2\lambda_1\nu(1-\epsilon) + \alpha(|u_m|^{\beta-1} + |v_n|^{\beta-1}) - \frac{1}{4\nu\epsilon}(|u_m|^2 + |v_n|^2) \geq \delta. \tag{3.7}$$

Therefore, from (3.6), (3.7) and Gronwall's lemma we have

$$\|w_{m,n}(t)\|^2 \leq \|w_{m,n}(0)\|^2 e^{-\delta t}, \quad \forall t \in [0,T]. \tag{3.8}$$

Passing to $m, n \to \infty$ in (3.8) implies

$$\|u(t) - v(t)\|^2 \leq \|u_0 - v_0\|^2 e^{-\delta t}, t > 0, \tag{3.9}$$

for almost all $t \in [0,T]$. If we redefine $u(t)$ on a set of measure zero so that they continuous in $[0,T]$, (29) is fulfilled for all $t \in [0,T]$ and for all $t \geq 0$ from arbitrariness of $T > 0$. The proof is complete. $\square$

Remark 3.3. Since the weak solution u of (1.1) is in $L^\infty(0,T;H) \cap L^2(0,T;V) \cap L^{\beta+1}(0,T;(L^{\beta+1}(\Omega))^3)$, $|Bu|_{V'} \in L^4(0,T)$ and $|Gu|_{V'} \in L^{(\beta+1)/\beta}(0,T)$, du/dt belongs to only $L^{(\beta+1)/\beta}(0,T;V')$. Thus, we can not know whether the formula

$$\langle \frac{du(t)}{dt}, u(t)\rangle_{V'V} = \frac{1}{2}\frac{d}{dt}\|u(t)\|^2$$

is true or false, while the formula is true if $du/dt \in L^2(0,T;V')$ and $u \in L^2(0,T;V)$. So, for general weak solutions, we can not know whether the formula (29) is true or false.

4. Proof of Theorem 1.4

Suppose the assumptions of Theorem 1.4 are fulfilled. Denote

$$F := \|f\|^2_{W^{1,2}(0,T;(L^2(\Omega))^3)}.$$

Let S be the mapping defined by

$$(Su_0)(x) := u(x,T;u_0),$$

where $u(x,t;u_0)$ is the unique G-weak solution of (1.1) with initial data $u_0 \in H$ (see Remark 1.3). Denote

$$X := \{u_0 \in H : \|u_0\|^2 \leq C_4 := \frac{F}{\lambda_1\nu(1-e^{-\lambda_1\nu T})}\}.$$

Let $u_0 \in X$. Multiplying the first equation of (1.1) by u, integrating the resulting equation on Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu\|\nabla u\|^2 + \alpha\|u\|^{\beta+1}_{\beta+1} \leq \frac{\lambda_1\nu}{2}\|u\|^2 + \frac{1}{2\lambda_1\nu}\|f(t)\|^2. \tag{4.1}$$

Dropping the term $\alpha\|u\|^{\beta+1}_{\beta+1}$, considering $\lambda_1\|u\|^2 \leq \|\nabla u\|^2$ and applying Gronwall's lemma, we have

$$\|u(t)\|^2 \leq e^{-\lambda_1\nu t}\|u_0\|^2 + \frac{F}{\lambda_1\nu}, \quad \forall t \in [0,T]. \tag{4.2}$$

Therefore, $\|u(T;u_0)\|^2 \leq C_4$ if $\|u_0\|^2 \leq C_4$. That is, $SX \subset X$.

Assuming $\|u_0\|^2 \leq C_4$, integrating (4.1) over $[0,T]$ and substituting (31), it follows that

$$\int_0^T \|\nabla u(t)\|^2 dt \leq \frac{\lambda_1 C_4 T}{2} + \frac{F}{2\lambda_1\nu^2} := C_5.$$

This implies that there is a time $t_0 \in (0,T)$ satisfying

$$\|\nabla u(t_0)\|^2 dt \le \frac{C_5}{T}. \tag{4.3}$$

The inequality (2.16), with $t = 0$ is replaced by $t = t_0$, gives us

$$\begin{aligned} \|\nabla u(T)\|^2 &\le C(T)(1 + \|\nabla u(t_0)\|^{4q(q-1)} + F) \\ &\le C(T)(1 + (\frac{C_5}{T})^{2q(q-1)} + F). \end{aligned} \tag{4.4}$$

Therefore, SX is precompact in H.

The continuity of $S : X \to X$ follows from Theorem 1.2. It is clear that X is a closed, bounded, and convex set in H. Since $S : X \to X$ is a compact and continuous mapping, from Schauder's fixed point theorem we know that S has a fixed point in X. Thus there is a $u_0 \in X$ such that $u(T; u_0) = u_0$. Further, from Theorem 1.1 and $u_0 = u(T; u_0) \in V$, we know that the periodic G-weak solutions are the periodic strong solutions.

Suppose (1.4) is fulfilled. Since the weak solution u belongs to $C((0,T]; H)$ for any $T > 0$, it follows that there is a constant $\delta > 0$ such that

$$\|u(t) - v(t)\|^2 \le \|u(s) - v(s)\|^2 \exp(-\delta(t - s))$$

for any $t > s > 0$ and the weak solutions u and v by (3.9). Then the last part of the Theorem 1.4 is proved, and proof is complete.

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