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DETERMINATION OF AN UNKNOWN SOURCE TERM TEMPERATURE DISTRIBUTION FOR THE SUB-DIFFUSION EQUATION AT THE INITIAL AND FINAL DATA

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Abstract. We consider a class of problems modeling the process of determining the temperature and density of nonlocal sub-diffusion sources given by initial and finite temperature. Their mathematical statements involve inverse problems for the fractional-time heat equation in which, solving the equation, we have to find the an unknown right-hand side depending only on the space variable. The results on existence and uniqueness of solutions of these problems are presented.

1. Introduction

Many instances are known in which the practical needs lead to the problems of determining the coefficients or the right-hand-side of a differential equation from some available data about the solution. These are called the inverse problems of mathematical physics. Inverse problems arise in various areas of human activity such as seismology, mineral exploration, biology, medicine, quality control of industrial goods, etc. All these circumstances place inverse problems among the important problems of modern mathematics.

The purpose of this paper is to study inverse problems for the nonlocal heat equation with involution of space variable x. We consider the heat equation with variable coefficient

$$t^{-\beta}\mathcal{D}_t^{\alpha}u(x,t) - u_{xx}(x,t) + \varepsilon u_{xx}(a+b-x,t) = f(x), \tag{1.1}$$

for $(x,t) \in \Omega = \{-\infty < a < x < b < \infty,\ 0 < t < T < \infty\}$, $0 < \alpha < 1$, $\beta \geq 0$, where $\mathcal{D}_t^{\alpha}$ is the Caputo derivative (see definition 1.3) and ε is a real number.

Differential equations with modified arguments are equations in which the unknown function and its derivatives are evaluated with modifications of time or space variables; such equations are called, in general, functional differential equations. Among such equations, one can single out, equations with involutions [3].

Definition 1.1 ([1, 21]). A function $\omega(x) \not\equiv x$ maps bijectively a set of real numbers Γ, such that

$$\omega(\omega(x)) = x, \quad \text{or} \quad \omega^{-1}(x) = \omega(x)$$

is called an involution on Γ.

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Equations containing involution are equations with an alternating deviation (at $x^* < x$ being equations with advanced, and at $x^* > x$ being equations with delay, where x^* is a fixed point of the mapping $\omega(x)$).

Furthermore, for the equations containing transformation of the spatial variable in the diffusion term, we can cite Cabada and Tojo [4], where an example that describes a concrete situation in physics is given: Consider a metal wire around a thin sheet of insulating material in a way that some parts overlap some others as shown in Figure 1.

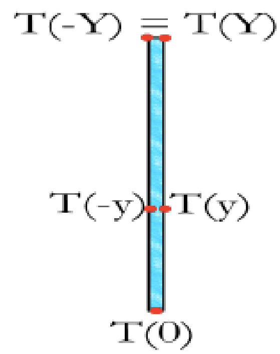

FIGURE 1. An application of heat equation with involution

Assuming that the position $y = 0$ is the lowest of the wire, and the insulation goes up to the left at $-Y$ and to the right up to Y.

For the proximity of two sections of wires they added the third term with modifications on the spatial variable to the right-hand side of the heat equation with respect to the wire:

$$\frac{\partial T}{\partial t}(y,t) = \alpha \frac{\partial^2 T}{\partial y^2}(y,t) + \beta \frac{\partial^2 T}{\partial y^2}(-y,t),$$

where T is the temperature at (y,t). Such equations have also a purely theoretical value.

Concerning the inverse problems for local and nonlocal heat equations, some recent works have been done by Kaliev [5], [6], Kirane [9, 10], Sadybekov [17, 18].

The heat equation also describes the diffusion process. So, the equation of the form (1.1) with fractional derivatives with respect to the time variable is called the sub-diffusion equation. This equation describes the slow diffusion [20]. When $\alpha = \frac{1}{2}$, $\varepsilon = 0$ the equation was interpreted by Nigmatullin [16] within a percolation (pectinate) model. The solution (in an unbounded domain in the space variable) was investigated by Mainardi [12] and others by means of integral transformations.

Now, for the formulation of the problems, we need to define the fractional differentiation operator.

Definition 1.2 ([7]). The Riemann-Liouville fractional integral I^α of order $\alpha > 0$ for an integrable function is defined by

$$I^\alpha[f](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t \in [a,b],$$

where Γ denotes the Euler gamma function.

Definition 1.3 ([7]). The Caputo fractional derivative of order $0 < \alpha < 1$ of a differentiable function is defined by

$$\mathcal{D}_*^\alpha[f](t) = I^{1-\alpha}[\frac{d}{dt}f(t)], \quad t \in [a,b].$$

2. Statement of problems

This article concerns two inverse problems of the time fractional heat equation with involution type in the space variable.

Problem 2.1. Find a couple of functions $(u(x,t), f(x))$ satisfying equation (1.1), under the conditions

$$u(x,0) = \varphi(x), \quad x \in [a,b], \tag{2.1}$$

$$u(x,T) = \psi(x), \quad x \in [a,b], \tag{2.2}$$

and the homogeneous Dirichlet boundary conditions

$$u(a,t) = u(b,t) = 0, \quad t \in [0,T], \tag{2.3}$$

where $\varphi(x)$ and $\psi(x)$ are given sufficiently smooth functions.

Problem 2.2. Find the couple of functions $(u(x,t), f(x))$ in the domain Ω satisfying equation (1.1), conditions (2.1), (2.2) and the homogeneous Neumann boundary conditions

$$u_x(a,t) = u_x(b,t) = 0, \quad t \in [0,T]. \tag{2.4}$$

A regular solution of problems 2.1 and 2.2 is the pair of functions $(u(x,t), f(x))$ where $u \in C_{x,t}^{2,1}(\bar{\Omega})$ (space of two times and one time continuously differentiable functions on $\bar{\Omega}$ according to x and t respectively) and $f \in C([a,b])$.

Note that similar problems for the heat equation and their fractional analogues have been considered in [2, 8, 15, 19].

3. Spectral properties of the Sturm-Liouville problem with involution

Application of the Fourier method for solving problems 2.1 and 2.2 in the form $u(x,t) = \tau(x)u(t)$ leads to the eigenvalue problem defined by the equation

$$\tau''(x) - \varepsilon\tau''(a+b-x) + \lambda\tau(x) = 0, \quad a < x < b, \tag{3.1}$$

and one of the following boundary conditions

$$\tau(a) = 0, \quad \tau(b) = 0, \tag{3.2}$$

$$\tau'(a) = 0, \quad \tau'(b) = 0. \tag{3.3}$$

It is easy to see that the Sturm-Liouville problem for equation (3.1) with one of the boundary conditions (3.2), (3.3) is self-adjoint. It is known that the self-adjoint problem has real eigenvalues and their eigenfunctions form a complete orthonormal basis in $L^2([a,b])$ [14]. To further investigate the problems under consideration, we need to calculate the explicit form of the eigenvalues and eigenfunctions.

For $|\varepsilon| < 1$ problem (3.1), (3.2) has eigenvalues

$$\lambda_{2k} = \frac{(1+\varepsilon)(2k\pi)^2}{(b-a)^2}, \quad k \in \mathbb{N},$$

$$\lambda_{2k+1} = \frac{(1-\varepsilon)((2k+1)\pi)^2}{(b-a)^2}, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup [0]$$

and eigenfunctions

$$\begin{gathered} y_{2k} = \sqrt{\frac{2}{b-a}} \sin \frac{2k\pi}{b-a}(x-a), \quad k \in \mathbb{N}, \\ y_{2k+1} = \sqrt{\frac{2}{b-a}} \sin \frac{(2k+1)\pi}{b-a}(x-a), \quad k \in \mathbb{N}_0. \end{gathered} \tag{3.4}$$

Similarly, problem (3.1), (3.3) has eigenvalues

$$\begin{gathered} \mu_{2k+1} = \frac{(1+\varepsilon)((2k+1)\pi)^2}{(b-a)^2}, \quad k \in \mathbb{N}_0, \\ \mu_{2k} = \frac{(1-\varepsilon)(2k\pi)^2}{(b-a)^2}, \quad k \in \mathbb{N}_0, \end{gathered}$$

and corresponding eigenfunctions

$$\begin{gathered} z_0 = \frac{1}{\sqrt{b-a}}, \\ z_{2k+1} = \sqrt{\frac{2}{b-a}} \cos \frac{((2k+1)\pi)}{b-a}(x-a), \quad k \in \mathbb{N}_0, \\ z_{2k} = \sqrt{\frac{2}{b-a}} \cos \frac{2k\pi}{b-a}(x-a), \quad k \in \mathbb{N}. \end{gathered} \tag{3.5}$$

Lemma 3.1. *The systems of functions* (3.4)*,* (3.5) *are complete and orthonormal in* $L^2([a,b])$*.*

Proof. We prove completeness. System (3.4) is complete in $L^2([a,b])$ if the equalities

$$\begin{gathered} \int_a^b f(x) \sin 2k\pi \frac{x-a}{b-a} dx = 0, \quad k \in \mathbb{N}, \\ \int_a^b f(x) \sin (2k+1)\pi \frac{x-a}{b-a} dx = 0, \quad k \in \mathbb{N}_0, \end{gathered}$$

for $f \in L^2([a,b])$ lead to $f(x) = 0$ in $L^2([a,b])$.

Further, replacing $\pi \frac{x-a}{b-a}$ by ξ, we have:

$$\begin{gathered} \int_0^\pi f\big(\frac{b-a}{\pi}\xi + a\big) \sin 2k\xi \, d\xi = 0, \quad k \in \mathbb{N}, \\ \int_0^\pi f\big(\frac{b-a}{\pi}\xi + a\big) \sin (2k+1)\xi \, d\xi = 0, \quad k \in \mathbb{N}_0. \end{gathered}$$

From the second equation we obtain

$$\begin{aligned} &\int_0^\pi f\big(\frac{b-a}{\pi}\xi + a\big) \sin(2k+1)\xi \, d\xi \\ &= \int_0^{\pi/2} f\big(\frac{b-a}{\pi}\xi + a\big) \sin(2k+1)\xi \, d\xi + \int_{\frac{\pi}{2}}^\pi f\big(\frac{b-a}{\pi}\xi + a\big) \sin(2k+1)\xi \, d\xi \\ &= \int_0^{\pi/2} \Big(f\big(\frac{b-a}{\pi}\xi + a\big) - f\big(b - \frac{b-a}{\pi}\xi\big) \Big) \sin(2k+1)\xi \, d\xi = 0. \end{aligned}$$

Then by the completeness of the system $\{\sin(2k+1)\xi\}_{k\in\mathbb{N}_0}$ in $L^2([0,\frac{\pi}{2}])$ [13], we obtain $f\big(\frac{b-a}{\pi}\xi+a\big)=f(b-\frac{b-a}{\pi}\xi)$, $0<\xi<\frac{\pi}{2}$.

Similarly

$$
\begin{aligned}
&\int_0^{\pi} f\big(\frac{b-a}{\pi}\xi+a\big)\sin 2k\xi\,d\xi\\
&=\int_0^{\pi/2} f\big(\frac{b-a}{\pi}\xi+a\big)\sin 2k\xi\,d\xi+\int_{\frac{\pi}{2}}^{\pi} f\big(\frac{b-a}{\pi}\xi+a\big)\sin 2k\xi\,d\xi\\
&=\int_0^{\pi/2}\big(f\big(\frac{b-a}{\pi}\xi+a\big)+f\big(b-\frac{b-a}{\pi}\xi\big)\big)\sin 2k\xi\,d\xi=0.
\end{aligned}
$$

Then by the completeness of the system $\{\sin 2k\xi\}_{k\in\mathbb{N}}$ in $L^2([0,\frac{\pi}{2}])$ [13], we have

$$
f\big(\frac{b-a}{\pi}\xi+a\big)=-f(b-\frac{b-a}{\pi}\xi),\quad 0<\xi<\frac{\pi}{2}.
$$

Whereupon, $f\big(\frac{b-a}{\pi}\xi+a\big)=0$ in $L^2([0,\frac{\pi}{2}])$, and consequently $f\big(\frac{b-a}{\pi}\xi+a\big)=0$ in $L^2([0,\pi])$. From this it follows that $f(x)=0$ in $L^2([a,b])$.

The completeness of the system (3.5) is proved similarly. □

4. Main results

For problems 2.1 and 2.2 the following theorems hold.

Theorem 4.1. *Let $|\varepsilon|<1$, $\varphi,\psi\in C^3([a,b])$ and $\varphi^{(i)}(a)=\varphi^{(i)}(b)=\psi^{(i)}(a)=\psi^{(i)}(b)=0$, $i=0,1,2$. Then the solution of the problem 2.1 exists, is unique and it can be written in the form*

$$
\begin{aligned}
&u(x,t)\\
&=\varphi(x)+\sum_{k=0}^{\infty}\frac{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1}t^{\alpha}))\sin\sqrt{\frac{\lambda_{2k+1}}{1-\varepsilon}}(x-a)}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1}T^{\alpha}))\frac{\lambda_{2k+1}}{1-\varepsilon}}(\varphi^{(2)}_{2k+1,1}-\psi^{(2)}_{2k+1,1})\\
&\quad+\sum_{k=1}^{\infty}\frac{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k}t^{\alpha}))\sin\sqrt{\frac{\lambda_{2k}}{1+\varepsilon}}(x-a)}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k}T^{\alpha}))\frac{\lambda_{2k}}{1+\varepsilon}}(\varphi^{(2)}_{2k,1}-\psi^{(2)}_{2k,1}),
\end{aligned}
$$

$$
\begin{aligned}
f(x)&=-\varphi''(x)+\varepsilon\varphi''(a+b-x)\\
&\quad+\sum_{k=0}^{\infty}\frac{(1-\varepsilon)(\varphi^{(2)}_{2k+1,1}-\psi^{(2)}_{2k+1,1})}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1}T^{\alpha}))}\sin\sqrt{\frac{\lambda_{2k+1}}{1-\varepsilon}}(x-a)\\
&\quad+\sum_{k=1}^{\infty}\frac{(1+\varepsilon)(\varphi^{(2)}_{2k,1}-\psi^{(2)}_{2k,1})}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k}T^{\alpha}))}\sin\sqrt{\frac{\lambda_{2k}}{1+\varepsilon}}(x-a),
\end{aligned}
$$

where $\varphi^{(2)}_{2k+1,1}=(\varphi''(x),y_{2k+1})$, $\varphi^{(2)}_{2k,1}=(\varphi''(x),y_{2k})$, $\psi^{(2)}_{2k+1,1}=(\psi''(x),y_{2k+1})$, $\psi^{(2)}_{2k,1}=(\psi''(x),y_{2k})$, and $E_{\alpha,l,m}(z)$ is the Mittag-Leffler type function

$$
E_{\alpha,l,m}(z)=\sum_{k=0}^{\infty}\frac{z^k}{C(\alpha,l,m)},\quad C(\alpha,l,m)=\prod_{p=0}^{k}\frac{\Gamma(\alpha p+l)}{\Gamma(\alpha p+m)}.
$$

Theorem 4.2. *Let $\varphi, \psi \in C^3[a,b]$ and $\varphi^{(i)}(a) = \varphi^{(i)}(b) = \psi^{(i)}(a) = \psi^{(i)}(b) = 0, i = 0,1,2$. Then the solution of problem 2.2 exists, is unique and it can be written in the form*

$$\begin{aligned} u(x,t) &= \varphi(x) + \frac{t}{T}(\psi_{0,2} - \varphi_{0,2}) \\ &\quad + \sum_{k=1}^{\infty} \frac{(1 - E_{\alpha+\beta,1,1-\alpha}(-\mu_{2k}t^{\alpha}))\cos\sqrt{\frac{\mu_{2k}}{1-\varepsilon}}(x-a)}{(1 - E_{\alpha+\beta,1,1-\alpha}(-\mu_{2k}T^{\alpha}))\frac{\mu_{2k}}{1-\varepsilon}}(\psi^{(2)}_{2k,2} - \varphi^{(2)}_{2k,2}) \\ &\quad + \sum_{k=0}^{\infty} \frac{(1 - E_{\alpha+\beta,1,1-\alpha}(-\mu_{2k+1}t^{\alpha}))\cos\sqrt{\frac{\mu_{2k+1}}{1+\varepsilon}}(x-a)}{(1 - E_{\alpha+\beta,1,1-\alpha}(-\mu_{2k+1}T^{\alpha}))\frac{\mu_{2k+1}}{1+\varepsilon}}(\psi^{(2)}_{2k+1,2} - \varphi^{(2)}_{2k+1,2}), \end{aligned}$$

$$\begin{aligned} f(x) &= -\varphi''(x) + \varepsilon\varphi''(a+b-x) \\ &\quad + \sum_{k=1}^{\infty} \frac{(1-\varepsilon)(\varphi^{(2)}_{2k,2} - \psi^{(2)}_{2k,2})}{(1 - E_{\alpha+\beta,1,1-\alpha}(-\mu_{2k}T^{\alpha}))} \cos\sqrt{\frac{\mu_{2k}}{1-\varepsilon}}(x-a) \\ &\quad + \sum_{k=0}^{\infty} \frac{(1+\varepsilon)(\varphi^{(2)}_{2k+1,2} - \psi^{(2)}_{2k+1,2})}{(1 - E_{\alpha+\beta,1,1-\alpha}(-\mu_{2k+1}T^{\alpha}))} \cos\sqrt{\frac{\mu_{2k+1}}{1+\varepsilon}}(x-a), \end{aligned}$$

where

$$\begin{gathered} \varphi_{0,2} = (\varphi(x), z_0), \quad \varphi^{(2)}_{2k,2} = (\varphi''(x), z_{2k}), \quad \varphi^{(2)}_{2k+1,2} = (\varphi''(x), z_{2k+1}), \\ \psi_{0,2} = (\psi(x), z_0), \quad \psi^{(2)}_{2k,2} = (\psi''(x), z_{2k}), \psi^{(2)}_{2k+1,2} = (\psi''(x), z_{2k+1}). \end{gathered}$$

5. Proof of existence of the solution for problem 2.1

We give the full proof for problem 2.1. The existence of the solution of problem 2.2 is proved analogously.

As the eigenfunctions for system (3.4) of problem 2.1 form an orthonormal basis in $L^2([a,b])$ (this follows from the self-adjoint problem (3.1), (3.2)), the functions $u(x,t)$ and $f(x)$ can be expanded as follows

$$u(x,t) = \sum_{k=0}^{\infty} u_{2k+1,1}(t) \sin\sqrt{\frac{\lambda_{2k+1}}{1-\varepsilon}}(x-a) + \sum_{k=1}^{\infty} u_{2k,1}(t) \sin\sqrt{\frac{\lambda_{2k}}{1+\varepsilon}}(x-a), \quad (5.1)$$

$$f(x) = \sum_{k=0}^{\infty} f_{2k+1,1} \sin\sqrt{\frac{\lambda_{2k+1}}{1-\varepsilon}}(x-a) + \sum_{k=1}^{\infty} f_{2k,1} \sin\sqrt{\frac{\lambda_{2k}}{1+\varepsilon}}(x-a), \quad (5.2)$$

where $f_{2k+1,1}$, $f_{2k,1}$, $u_{2k+1,1}(t)$, $u_{2k,1}(t)$ are unknown. Substituting (5.1) and (5.2) into (1.1), we obtain the following equation for the functions $u_{2k+1,1}(t)$, $u_{2k,1}(t)$ and the constants $f_{2k+1,1}$, $f_{2k,1}$:

$$\begin{gathered} t^{-\beta}\mathcal{D}^{\alpha} u_{2k+1,1}(t) + \lambda_{2k+1} u_{2k+1,1}(t) = f_{2k+1,1}, \\ t^{-\beta}\mathcal{D}^{\alpha} u_{2k,1}(t) + \lambda_{2k} u_{2k,1}(t) = f_{2k,1}. \end{gathered}$$

Solving these equations [7], we obtain

$$u_{2k+1,1}(t) = \frac{f_{2k+1,1}}{\lambda_{2k+1}} + C_{2k+1} E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1}t^{\alpha}),$$

$$u_{2k,1}(t) = \frac{f_{2k,1}}{\lambda_{2k}} + C_{2k} E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k} t^\alpha),$$

where the constants C_{2k+1}, C_{2k}, $f_{2k+1,1}$, $f_{2k,1}$ are unknown. To find these constants, we use conditions (2.1). Let $\varphi_{2k+1,1}, \varphi_{2k,1}, \psi_{2k+1,1}, \psi_{2k,1}$ be the coefficients of the expansions of $\varphi(x)$ and $\psi(x)$

$$\varphi_{2k+1,1} = \sqrt{\frac{2}{b-a}} \int_a^b \varphi(x) \sin\sqrt{\frac{\lambda_{2k+1}}{1-\varepsilon}}(x-a)dx,$$

$$\varphi_{2k,1} = \sqrt{\frac{2}{b-a}} \int_a^b \varphi(x) \sin\sqrt{\frac{\lambda_{2k}}{1+\varepsilon}}(x-a)dx,$$

$$\psi_{2k+1,1} = \sqrt{\frac{2}{b-a}} \int_a^b \psi(x) \sin\sqrt{\frac{\lambda_{2k+1}}{1-\varepsilon}}(x-a)dx,$$

$$\psi_{2k,1} = \sqrt{\frac{2}{b-a}} \int_a^b \psi(x) \sin\sqrt{\frac{\lambda_{2k}}{1+\varepsilon}}(x-a)dx.$$

We first find C_{2k+1}.

$$u_{2k+1,1}(0) = \frac{f_{2k+1,1}}{\lambda_{2k+1,1}} + C_{2k+1} = \varphi_{2k+1,1},$$

$$u_k(T) = \frac{f_{2k+1,1}}{\lambda_{2k+1}} + C_{2k+1} E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1} T^\alpha) = \psi_{2k+1,1},$$

$$\varphi_{2k+1,1} - C_{2k+1} + C_{2k+1} E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1} T^\alpha) = \psi_{2k+1,1}.$$

Then

$$C_{2k+1} = \frac{\varphi_{2k+1,1} - \psi_{2k+1,1}}{1 - E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1} T^\alpha)}.$$

The constant $f_{2k+1,1}$ is represented as

$$f_{2k+1,1} = \lambda_{2k+1}(\varphi_{2k+1,1} - C_{2k+1}).$$

Now we find C_{2k}.

$$u_{2k,1}(0) = \frac{f_{2k,1}}{\lambda_{2k}} + C_{2k} = \varphi_{2k,1},$$

$$u_{2k,1}(T) = \frac{f_{2k,1}}{\lambda_{2k}} + C_{2k} E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k} T^\alpha) = \psi_{2k,1},$$

$$\varphi_{2k,1} - C_{2k} + C_{2k} E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k} T^\alpha) = \psi_{2k,1}.$$

Then we obtain

$$C_{2k} = \frac{\varphi_{2k,1} - \psi_{2k,1}}{1 - E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k} T^\alpha)}.$$

For the constant $f_{2k,1}$, we found

$$f_{2k,1} = \lambda_{2k}(\varphi_{2k,1} - C_{2k}).$$

Substituting $u_{2k+1,1}(t)$, $u_{2k,1}(t)$, $f_{2k+1,1}$, $f_{2k,1}$ into (5.1) and (5.2), we find

$$u(x,t) = \varphi(x) + \sum_{k=0}^\infty C_{2k+1}(E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1} t^\alpha) - 1) \sin\sqrt{\frac{\lambda_{2k+1}}{1-\varepsilon}}(x-a)$$

$$+ \sum_{k=1}^\infty C_{2k}(E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k} t^\alpha) - 1) \sin\sqrt{\frac{\lambda_{2k}}{1+\varepsilon}}(x-a).$$

Suppose that

$$\varphi^{(i)}(a)=0,\quad \varphi^{(i)}(b)=0,\quad i=0,1,2;$$
$$\psi^{(i)}(a)=0,\quad \psi^{(i)}(b)=0,\ i=0,1,2.$$

we have

$$\begin{aligned} C_{2k+1} &= \frac{\varphi_{2k+1,1}-\psi_{2k+1,1}}{1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1}T^{\alpha})} \\ &= -\frac{\varphi^{(2)}_{2k+1,1}-\psi^{(2)}_{2k+1,1}}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1}T^{\alpha}))\frac{\lambda_{2k+1}}{1-\varepsilon}}. \end{aligned}$$

Similarly, for C_{2k} we obtain

$$C_{2k}=-\frac{\varphi^{(2)}_{2k,1}-\psi^{(2)}_{2k,1}}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k}T^{\alpha}))\frac{\lambda_{2k}}{1+\varepsilon}}.$$

Then

$$\begin{aligned} &u(x,t)\\ &=\varphi(x)+\sum_{k=0}^{\infty}\frac{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1}t^{\alpha}))\sin\sqrt{\frac{\lambda_{2k+1}}{1-\varepsilon}}(x-a)}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1}T^{\alpha}))\frac{\lambda_{2k+1}}{1-\varepsilon}}(\varphi^{(2)}_{2k+1,1}-\psi^{(2)}_{2k+1,1})\\ &\quad+\sum_{k=1}^{\infty}\frac{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k}t^{\alpha}))\sin\sqrt{\frac{\lambda_{2k}}{1+\varepsilon}}(x-a)}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k}T^{\alpha}))\frac{\lambda_{2k}}{1+\varepsilon}}(\varphi^{(2)}_{2k,1}-\psi^{(2)}_{2k,1}). \end{aligned}$$

Similarly,

$$\begin{aligned} f(x)&=-\varphi''(x)+\varepsilon\varphi''(a+b-x)\\ &\quad+\sum_{k=0}^{\infty}\frac{(1-\varepsilon)(\varphi^{(2)}_{2k+1,1}-\psi^{(2)}_{2k+1,1})}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1}T^{\alpha}))}\sin\sqrt{\frac{\lambda_{2k+1}}{1-\varepsilon}}(x-a)\\ &\quad+\sum_{k=1}^{\infty}\frac{(1+\varepsilon)(\varphi^{(2)}_{2k,1}-\psi^{(2)}_{2k,1})}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k}T^{\alpha}))}\sin\sqrt{\frac{\lambda_{2k}}{1+\varepsilon}}(x-a). \end{aligned}$$

Now for the convergence of the series, we have the estimate

$$\begin{aligned} |u(x,t)|&\le C|\varphi(x)|+C\sum_{k=0}^{\infty}\frac{|\varphi^{(2)}_{2k+1,1}|+|\psi^{(2)}_{2k+1,}|}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1}T^{\alpha}))\frac{\lambda_{2k+1}}{1-\varepsilon}}\\ &\quad+C\sum_{k=1}^{\infty}\frac{|\varphi^{(2)}_{2k,1}|+|\psi^{(2)}_{2k,1}|}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k}T^{\alpha}))\frac{\lambda_{2k}}{1+\varepsilon}}, \end{aligned}\tag{5.3}$$

where C is a constant. Similarly for $f(x)$ we obtain the estimate

$$\begin{aligned} |f(x)|&\le C|\varphi(x)|+C|\varphi(a+b-x)|\\ &\quad+C\sum_{k=0}^{\infty}\frac{|\varphi^{(2)}_{2k,1}|+|\psi^{(2)}_{2k,1}|}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1}T^{\alpha}))}\\ &\quad+C\sum_{k=1}^{\infty}\frac{|\varphi^{(2)}_{2k,1}|+|\psi^{(2)}_{2k,1}|}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k}T^{\alpha}))} \end{aligned}\tag{5.4}$$

where C is a constant.

Since by hypotheses of Theorem 4.1, the functions $\varphi^{(2)}$, $\psi^{(2)}$ are continuous on $[0,\pi]$, then by the Bessel inequality for the trigonometric series the following series converge:

$$\sum_{k=0}^{\infty}|\varphi^{(2)}_{2k+1,1}|^2 \le C\|\varphi^{(2)}(x)\|^2_{L_2(a,b)}, \tag{5.5}$$

$$\sum_{k=1}^{\infty}|\varphi^{(2)}_{2k,1}|^2 \le C\|\varphi^{(2)}(x)\|^2_{L_2(a,b)}, \tag{5.6}$$

$$\sum_{k=0}^{\infty}|\psi^{(2)}_{2k+1,1}|^2 \le C\|\psi^{(2)}(x)\|^2_{L_2(a,b)}, \tag{5.7}$$

$$\sum_{k=1}^{\infty}|\psi^{(2)}_{2k,1}|^2 \le C\|\psi^{(2)}(x)\|^2_{L_2(a,b)}, \tag{5.8}$$

which implies the boundedness of the set

$$\{\varphi^{(2)}_{2k+1,1}, \psi^{(2)}_{2k+1,1}, \varphi^{(2)}_{2k,1}, \psi^{(2)}_{2k,1}\}.$$

Therefore, by the Weierstrass M-test (see[11]), series (5.3) and (5.4) converge absolutely and uniformly in the region $\bar{\Omega}$.

Now we show the possibility of termwise differentiation of the series (5.3) twice in the variable x and once in the variable t. For this purpose, we prove that the obtained term by term differentiation of the series converge absolutely and uniformly in the domain $\bar{\Omega}$. Given the estimates (5.5) and (5.7) we have

$$\begin{aligned} |u_{xx}(x,t)| \le C|\varphi''(x)| &+ C\sum_{k=0}^{\infty}\frac{|\varphi^{(2)}_{2k+1,1}| + |\psi^{(2)}_{2k+1,1}|}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1}T^{\alpha}))} \\ &+ C\sum_{k=1}^{\infty}\frac{|\varphi^{(2)}_{2k+1,1}| + |\psi^{(2)}_{2k+1,1}|}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k}T^{\alpha}))} < \infty, \end{aligned}$$

$$\begin{aligned} |\mathcal{D}^{\alpha}_t u(x,t)| \le C &\sum_{k=0}^{\infty}\frac{|\varphi^{(2)}_{2k+1,1}| + |\psi^{(2)}_{2k+1,1}|}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k+1}T^{\alpha}))} \\ &+ \sum_{k=1}^{\infty}\frac{|\varphi^{(2)}_{2k+1,1}| + |\psi^{(2)}_{2k+1,1}|}{(1-E_{\alpha+\beta,1,1-\alpha}(-\lambda_{2k}T^{\alpha}))} < \infty. \end{aligned}$$

Hence the obtained solution satisfies (1.1) point-wise; by construction, it satisfies the conditions (2.1)-(2.3).

6. Proof of uniqueness for the solution of problem 2.2

Suppose that there are two solutions $\{u_1(x,t), f_1(x)\}$ and $\{u_2(x,t), f_2(x)\}$ of problem 2.2. Denote

$$\begin{gathered} u(x,t) = u_1(x,t) - u_2(x,t), \\ f(x) = f_1(x) - f_2(x). \end{gathered}$$

Then the functions $u(x,t)$ and $f(x)$ satisfy (1.1) and the homogeneous conditions (2.1) and (2.3). Let

$$u_{0,2}(t) = \frac{1}{\sqrt{b-a}} \int_a^b u(x,t)dx, \tag{6.1}$$

$$u_{2k+1,2}(t) = \sqrt{\frac{2}{b-a}} \int_a^b u(x,t) \cos \sqrt{\frac{\mu_{2k+1}}{1+\varepsilon}}(x-a)dx, k \in \mathbb{N}, \tag{6.2}$$

$$u_{2k,2}(t) = \sqrt{\frac{2}{b-a}} \int_a^b u(x,t) \cos \sqrt{\frac{\mu_{2k}}{1-\varepsilon}}(x-a)dx, k \in \mathbb{N}, \tag{6.3}$$

$$f_{0,2} = \frac{1}{\sqrt{b-a}} \int_a^b f(x)dx, \tag{6.4}$$

$$f_{2k+1,2} = \sqrt{\frac{2}{b-a}} \int_a^b f(x) \cos \sqrt{\frac{\mu_{2k+1}}{1+\varepsilon}}(x-a)dx, k \in \mathbb{N}, \tag{6.5}$$

$$f_{2k,2} = \sqrt{\frac{2}{b-a}} \int_a^b f(x) \sin \sqrt{\frac{\mu_{2k}}{1-\varepsilon}}(x-a)dx, k \in \mathbb{N}. \tag{6.6}$$

Applying the operator $\mathcal{D}^\alpha$ to the equation (6.1) we have

$$\begin{aligned}
\mathcal{D}^\alpha u_{0,2}(t) &= \frac{1}{\sqrt{b-a}} \int_a^b \mathcal{D}_t^\alpha u(x,t)dx \\
&= \frac{1}{\sqrt{b-a}} \int_a^b (u_{xx}(x,t) - \varepsilon u_{xx}(a+b-x,t))dx + f_{0,2}.
\end{aligned}$$

Integrating by parts and taking into account the homogeneous conditions (2.1) and (2.2), we obtain

$$\mathcal{D}^\alpha u_{0,2}(t) = f_{0,2}, \quad u_{0,2}(0) = 0, \quad u_{0,2}(T) = 0.$$

Consequently, $f_{0,2} \equiv 0, u_{0,2}(t) \equiv 0$.

In a similar way for the functions (6.2), (6.3), (6.4), (6.5), (6.6) one can prove that

$$f_{2k+1,2} = 0, f_{2k,2} = 0, u_{2k+1,2}(t) \equiv 0, u_{2k,2}(t) \equiv 0.$$

Further, by the completeness of the system (3.5) in $L^2([a,b])$ we obtain

$$f(t) \equiv 0, u(x,t) \equiv 0, \quad 0 \le t \le T, \ a \le x \le b.$$

Uniqueness of the solution of the problem 2.2 is proved.

Uniqueness of the solution of problem 2.1 can be proved similarly.

6.1. **Analytical and numerical examples.** As an illustration, we present here a simple example solution for the inverse problem 2.1 with $a = 0, b = \pi$. For this purpose, we consider the following choice of conditions (2.1):

$$u(x,0) = 0, \quad u(x,T) = \sin x, \quad x \in [0,\pi],$$

i.e., we have

$$\varphi(x) = 0 \quad \text{and} \quad \psi(x) = \sin x.$$

Calculating the coefficients of the series solutions as given in Theorem 4.1, we obtain

$$\begin{aligned}
u(x,t) &= \frac{1 - E_{\alpha+\beta,1,1-\alpha}(-(1-\varepsilon)t^\alpha)}{1 - E_{\alpha+\beta,1,1-\alpha}(-(1-\varepsilon)T^\alpha)} \sin x, \\
f(x) &= \frac{1-\varepsilon}{1 - E_{\alpha+\beta,1,1-\alpha}(-(1-\varepsilon)T^\alpha)} \sin x.
\end{aligned}$$

If $\alpha = 1/2$ and $\beta = 0$, then

$$\begin{aligned} &E_{1/2,1,1/2}(-(1-\varepsilon)t^{\alpha}) \\ &= \exp(-(1-\varepsilon)^2 x) - t^{-1/2}\exp(-(1-\varepsilon)^2 t)(-1+\operatorname{erfc}(-(1-\varepsilon)\sqrt{t})) \end{aligned}$$

These solutions are illustrated in Figures 2, 3, 4.

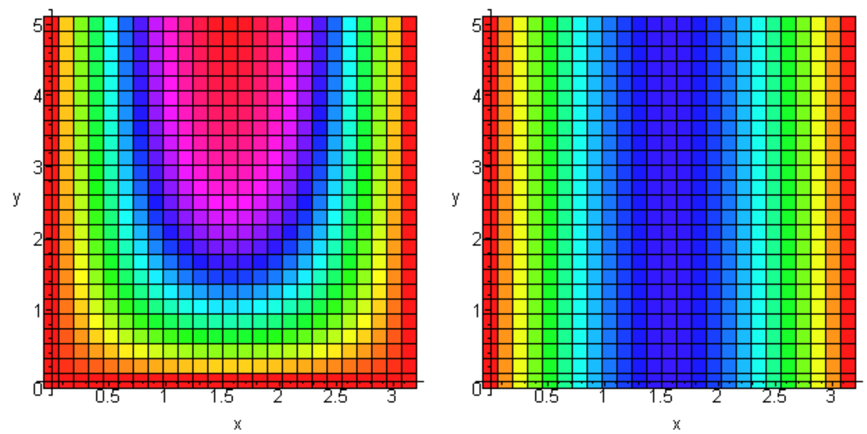

FIGURE 2. Graphs of $u(x,t)$ and $f(x)$ (right) for $\varepsilon = 0.6$ and $T = 5$.

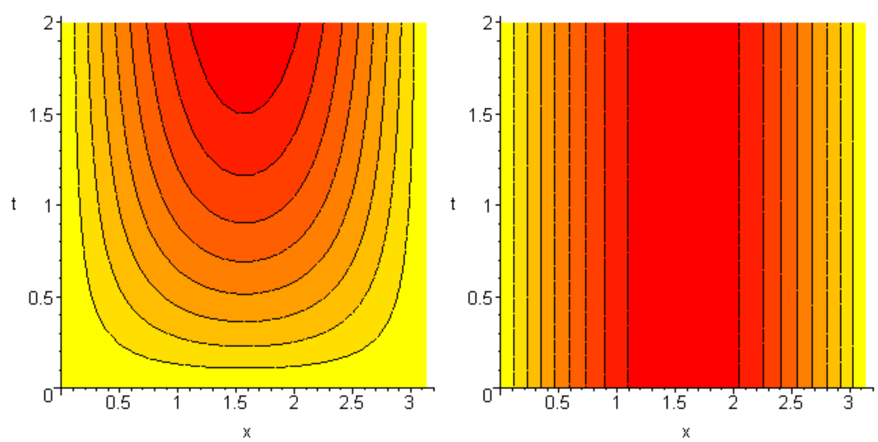

FIGURE 3. Graphs of $u(x,t)$ and $f(x)$ (right) for $\varepsilon = 0.9$ and for $T = 2$.

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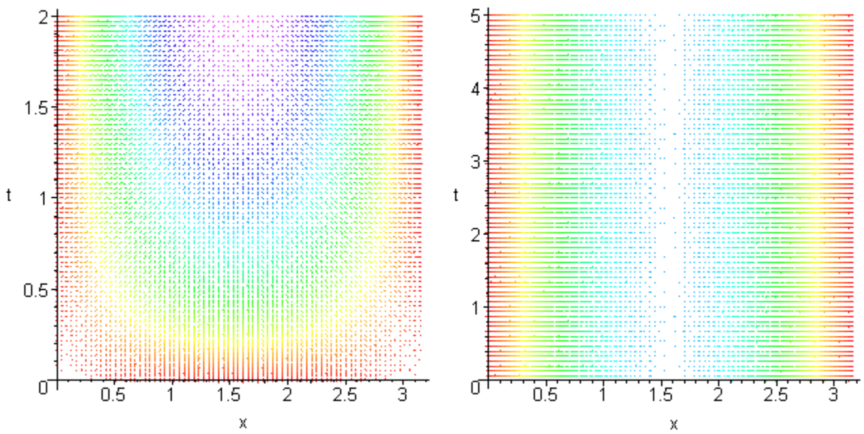

FIGURE 4. Graphs of $u(x,t)$ and $f(x)$ (right) for $\varepsilon = 0.8$ and $T = 2$.

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