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EXISTENCE AND NON-EXISTENCE OF SOLUTIONS FOR A SINGULAR PROBLEM WITH VARIABLE POTENTIALS

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ABSTRACT. The purpose of this article is to prove some existence and nonexistence theorems for the inhomogeneous singular Dirichlet problem

$$-\Delta_p u = \frac{\lambda k(x)}{u^\delta} \pm h(x)u^q.$$

For proving our results we use the sub and super solution method, and monotonicity arguments.

1. Introduction

In this paper we are interested in the following quasilinear and singular problem with variable potentials:

$$\begin{gathered} -\Delta_p u = \lambda k(x)u^{-\delta} \pm h(x)u^q \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0 \quad \text{in } \Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$, $(N \geq 2)$ is a bounded domain with smooth boundary, λ is a positive parameter, $1 < p < \infty$, $p-1 < q \leq p^*-1$, and $0 < \delta < 1$. As usual, $p^* = \frac{Np}{N-p}$ if $1 < p < N$, $p^* \in (p,\infty)$ is arbitrarily large if $p = N$, and $p^* = \infty$ if $p > N$, and the variable weight functions h, $k \in L^\infty(\Omega)$ satisfy

$$\operatorname{ess\,inf}_{x\in\Omega} k(x) > 0 \quad \text{and} \quad \operatorname{ess\,inf}_{x\in\Omega} h(x) > 0. \tag{1.2}$$

Associated with problem (1.1) we have the singular functional $E_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$E_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\lambda}{1-\delta}\int_\Omega k(x)u^{1-\delta}\,\mathrm{d}x \pm \frac{1}{q+1}\int_\Omega h(x)u^{q+1}\,\mathrm{d}x \tag{1.3}$$

in the Sobolev space $W_0^{1,p}(\Omega)$.

Definition 1.1. $u \in W_0^{1,p}(\Omega)$ is called a *weak* solution (or solution, for short) of problem (1.1), that is, for functions $u \in W_0^{1,p}(\Omega)$ satisfying $\operatorname{ess\,inf}_K u > 0$ over every compact set $K \subset \Omega$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \,\mathrm{d}x = \lambda \int_\Omega k(x)u^{-\delta}\phi\,\mathrm{d}x \pm \int_\Omega h(x)u^q\phi\,\mathrm{d}x \tag{1.4}$$

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for all $\phi \in C_c^\infty(\Omega)$. As usual, $C_c^\infty(\Omega)$ denotes the space of all C^∞ functions $\phi\colon \Omega \to \mathbb{R}$ with compact support.

Obviously, every critical point of E_λ is a weak solution of the problem (1.1).

$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, where $p > 1$ is a real constant is called the $p-$Laplacian or the p-Laplace operator. The p-Laplacian is an elliptic partial differential equation, which is degenerate if $p > 2$ and singular if $p < 2$. If $p = 2$, then the p-Laplacian reduces to the simpler classical linear Laplace equation $\Delta u := \nabla.\nabla u$ and in the case of one spatial dimension, we have $\Delta_p u = (|u'|^{p-2}u')'$.

The class of problems (1.1) appears in many nonlinear phenomena, for instance, in the theory of quasi-regular and quasi-conformal mappings (for this see [17, 23]), in the generalized reaction-diffusion theory [13], in the turbulent flow of a gas in a porous medium and in the non-Newtonian fluid theory [7]. In the non-Newtonian fluid theory, the quantity p is the characteristic of the medium. If $p < 2$, the fluids are called pseudo-plastics, if $p = 2$, the fluids are called Newtonian, and if $p > 2$, the fluids are called dilatants.

This kind of problems with convex and concave nonlinearities have been extensively studied by many authors. We refer the reader to the celebrate paper of Ambrosetti-Brezis-Cerami [1], Saoudi [19], Santos [22] with their references therein. For $p = 2$, we refer the reader to [18, 3] and references therein. The basic work in our direction is the paper [4] where Coclite-Palmieri have been considered the nonlinear elliptic equation containing singular term

$$\begin{gathered}
-\Delta u = u^p + \lambda u^{-\gamma}, \quad \text{in } \Omega,\\
u > 0, \quad \text{in } \Omega,\\
u = 0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.5}$$

where $\Omega \subset R^N$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$ and λ is a positive parameter. The exponent p of the sublinear satisfies $0 < p < 1$. The exponent γ of the singular term satisfies $0 < \gamma < 1$. In [4] has been shown that problem (1.5) possesses at least one solution for $\lambda > 0$ small enough, and has no solution when λ is large. We mention that in the work [4] the authors have been extended the results of Crandall-Rabinowitz-Tartar [5].

Problem (1.5) have been also studied with different elliptic operators. We refer the reader to [4, 5, 8, 9, 10, 11, 14, 15, 20, 21] and references therein.

The aim of this work is to extend the results obtained in [4] to the more general problems (1.1). Precisely, the main goal of this paper is to prove some existence and non-existence theorems for the non-linear singular elliptic problem (1.1). Firstly, we state the following definitions.

Definition 1.2. A function $\underline{u} \in W_0^{1,p}(\Omega)$ is called a weak sub-solution to $(1.1)_+$ if $\underline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ and

$$\begin{gathered}
-\Delta_p \underline{u} \leq \lambda k(x)\underline{u}^{-\delta} + h(x)\underline{u}^q \quad \text{in } \Omega,\\
\underline{u}|_{\partial\Omega} = 0, \quad \underline{u} > 0 \quad \text{in } \Omega,
\end{gathered}$$

A function $\overline{u} \in W_0^{1,p}(\Omega)$ is called a weak super-solution to $(1.1)_+$ if $\overline{u} \in C^2(\Omega) \cap C(\overline{\Omega})$ and

$$\begin{gathered}
-\Delta_p \overline{u} \leq \lambda k(x)\overline{u}^{-\delta} + h(x)\overline{u}^q \quad \text{in } \Omega,\\
\overline{u}|_{\partial\Omega} = 0, \quad \overline{u} > 0 \quad \text{in } \Omega,
\end{gathered}$$

Definition 1.3. A solution u_λ of problem $(1.1)_+$ is called minimal if $u_\lambda \le v$ almost everywhere in Ω for any further solution v of problem $(1.1)_+$.

We state below the results that we will prove.

Theorem 1.4. *Assume $0 < \delta < 1$, $p-1 < q < p^*-1$. Then there exists a positive number Λ^* such that the following properties hold:*
(1) *For all $\lambda \in (0, \Lambda^*)$ problem $(1.1)_+$ has a minimal solution u_λ.*
(2) *Problem $(1.1)_+$ has a solution if $\lambda = \Lambda^*$;*
(3) *Problem $(1.1)_+$ does not have any solution if $\lambda > \Lambda^*$.*

Theorem 1.5. *Assume $0 < \delta < 1$, $p-1 < q < p^*-1$. Then there exists a positive number Λ_* such that the following properties hold:*
(1) *If $\lambda > \Lambda_*$, then problem $(1.1)_-$ has at least one solution;*
(2) *If $\lambda < \Lambda_*$, then problem $(1.1)_-$ does not have any solution.*

A comparison between our main result (Theorems 1.4 and 1.5) and some of those the previously cited ones, is now in order: in the present paper, we extended the main result of Giacomoni-Schindler-Takáč [11, Theorem 2.1] to a class of perturbed singular functionals, this feature gains a remarkable importance in the applications. Moreover, it is worth noticing that, since parameter $k(x)$ and $h(x)$ in problem $(1.1)_\pm$, is variable, causes that the quasilinear singular problem is investigate in a complete form. On the other hand, the main difference between Theorems 1.4 and 1.5 above and the main result of Rădulescu-Repovš [18, Theorems 1.1 and 1.2] in applications consists in different from two directions: one is the operator considered in this work is more general than in [18], the other is with considering singular term instead of Rădulescu and Repovš in [18].

2. Proof of Theorem 1.4

The proof is organized in several steps.

Step 1: Existence of minimal solution for $0 < \lambda < \Lambda^*$. Let us define

$$\Lambda^* = \sup\{\lambda > 0\colon (1.1)_+ \text{ has a weak solution}\} \tag{2.1}$$

and let $\lambda_1(\Omega, m) \equiv \lambda_1$ be the first (principal) eigenvalue of $-\Delta_p$ and let Φ_m denote an eigenfunction of $-\Delta_p$ associated to λ_1 i.e., Φ_m solves

$$\begin{gathered} -\Delta_p \Phi_m = \lambda_1 m(x) |\Phi_m|^{p-2}\Phi_m \quad \text{in } \Omega \\ \Phi_m > 0 \quad \text{in } \Omega \\ \Phi_m = 0 \quad \text{in } \partial\Omega. \end{gathered}$$

It is well-known that Φ_m belongs to $C^1(\overline{\Omega})$, that Φ_m may be chosen positive in Ω and that $|\nabla\Phi|$ is positive on a neighborhood of $\partial\Omega$.

To show the existence of a solution to the problem $(1.1)_+$, we construct a well ordered pair of sub-solution $\underline{u}_\lambda$, and a super-solution $\overline{u}_\lambda$, such that $\underline{u}_\lambda \le \overline{u}_\lambda$.

To find a sub-solution, we assume that $m(x) = \min\{k(x), h(x)\}$ and $\lambda_1 \le \lambda$. Define $\psi_c = c\Phi_m^{\frac{p}{p-1+\delta}}$. By a straightforward calculation, we have

$$\nabla\psi_c = c\Big(\frac{p}{p-1+\delta}\Big)\Phi_m^{\frac{1-\delta}{p-1+\delta}}\nabla\Phi_m$$

and

$$-\Delta_p(\psi_c)$$

$$
\begin{aligned}
&= -\operatorname{div}(|\nabla\psi_c|^{p-2}\nabla\psi_c)\\
&= \frac{(pc)^{p-1}(\delta-1)(p-1)}{(p-1+\delta)^p}|\nabla\Phi_m|^p\Phi_m^{\frac{-\delta p}{p-1+\delta}} + \lambda_1\Big(\frac{pc}{p-1+\delta}\Big)^{p-1}m(x)\phi_m^p\Phi_m^{\frac{-\delta p}{p-1+\delta}}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&-\Delta_p(\psi_c)\\
&= \frac{(pc)^{p-1}(\delta-1)(p-1)}{(p-1+\delta)^p}|\nabla\Phi_m|^p\Phi_m^{\frac{-\delta p}{p-1+\delta}} + \lambda_1 m(x)\Big(\frac{pc}{p-1+\delta}\Big)^{p-1}\phi_m^p\Phi_m^{\frac{-\delta p}{p-1+\delta}}\\
&\le m(x)\Big(\Big(\frac{p}{(p-1+\delta)^p}\Big)^p\frac{c^{p-1+\delta}(\delta-1)(p-1)}{p}|\nabla\Phi_m|^p\psi_c^{-\delta}\\
&\quad + \lambda_1\Big(\frac{p}{p-1+\delta}\Big)^{p-1}c^{p-1-q}c^q\Phi_m^{\frac{p(p-1)}{p-1+\delta}}\Big)\\
&\le m(x)\Big(\Big(\frac{p}{(p-1+\delta)^p}\Big)^p\frac{c^{p-1+\delta}(\delta-1)(p-1)}{p}|\nabla\Phi_m|^p\psi_c^{-\delta}\\
&\quad + \lambda_1\Big(\frac{p}{p-1+\delta}\Big)^{p-1}c^{p-1-q}c^q\Phi_m^{\frac{pq}{p-1+\delta}}\Big)\\
&\le m(x)\Big(\Big(\frac{p}{(p-1+\delta)^p}\Big)^p\frac{c^{p-1+\delta}(\delta-1)(p-1)}{p}|\nabla\Phi_m|^p\psi_c^{-\delta}\\
&\quad + \lambda_1\Big(\frac{p}{p-1+\delta}\Big)^{p-1}c^{p-1-q}\psi_c^q\Big)
\end{aligned}
$$

Therefore, for $c > 0$ small enough, we have

$$
-\Delta_p(\psi_c) \le m(x)\left(\lambda\psi_c^{-\delta} + \psi_c^q\right) \le \lambda k(x)\psi_c^{-\delta} + h(x)\psi_c^q
$$

This shows that ψ_c is a sub-solution of the problem $(1.1)_+$.

Let us now show that problem $(1.1)_+$ has a super-solution. Now, we put $m(x) = \max\{k(x), h(x)\}$ and $\lambda_1 \ge \lambda$. Define $\psi_M = M\Phi_m^{\frac{p}{p-1+\delta}}$ for $M > c$ large enough. Using similar arguments as above we have

$$
\nabla\psi_M = M\Big(\frac{p}{p-1+\delta}\Big)\Phi_m^{\frac{1-\delta}{p-1+\delta}}\nabla\Phi_m
$$

and

$$
\begin{aligned}
&-\Delta_p(\psi_M)\\
&= -\operatorname{div}(|\nabla\psi_M|^{p-2}\nabla\psi_M)\\
&= \frac{(pM)^{p-1}(\delta-1)(p-1)}{(p-1+\delta)^p}|\nabla\Phi_m|^p\Phi_m^{\frac{-\delta p}{p-1+\delta}} + \lambda_1 m(x)\Big(\frac{pM}{p-1+\delta}\Big)^{p-1}\phi_m^p\Phi_m^{\frac{-\delta p}{p-1+\delta}}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&-\Delta_p(\psi_M)\\
&= \Big(\frac{pM}{p-1+\delta}\Big)^{p-1}\Phi_m^{\frac{-\delta p}{p-1+\delta}}\Big[\frac{(\delta-1)(p-1)}{p-1+\delta}|\nabla\Phi_m|^p + \lambda_1 m(x)\phi_m^p\Big]\\
&= \Big(\frac{pM}{p-1+\delta}\Big)^{p-1}\Phi_m^{\frac{-\delta p}{p-1+\delta}}\Big[\frac{(\delta-1)(p-1)}{p-1+\delta}|\nabla\Phi_m|^p + \frac{\lambda_1 m(x)}{2}\phi_m^p\Big]\\
&\quad + \frac{\lambda_1 m(x)}{2}\Big(\frac{pM}{p-1+\delta}\Big)^{p-1}\Phi_m^{\frac{-\delta p}{p-1+\delta}}\phi_m^p
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(\frac{p}{p-1+\delta}\Big)^{p-1} M^{p-1+\delta}\Big[\frac{(\delta-1)(p-1)}{p-1+\delta}|\nabla\Phi_m|^p + \frac{\lambda_1(m)m(x)}{2}\phi_m^p\Big]\psi_M^{-\delta} \\
&\quad + \frac{\lambda_1 m(x)}{2}\Big(\frac{p}{p-1+\delta}\Big)^{p-1} M^{p-1-q}\Phi_m^{\frac{p(p-1-q)}{p-1+\delta}}\psi_M^q
\end{aligned}
$$

Therefore, for $M>0$ may be chosen arbitrarily large, we have

$$
-\Delta_p(\psi_M) \ge m(x)\left(\lambda\psi_M^{-\delta}+\psi_M^q\right) \ge \lambda k(x)\psi_M^{-\delta} + h(x)\psi_M^q
$$

This shows that ψ_M is a super-solution of the problem $(1.1)_+$. It remains to show that $\psi_c = \underline{u}_\lambda \le \psi_M = \overline{u}_\lambda$. Therefore, for $c>0$ small enough and $M>0$ large enough, we obtain

$$
\begin{aligned}
&-\Delta_p(\underline{u}_\lambda) \\
&= \Big(\frac{pc}{p-1+\delta}\Big)^{p-1}\Phi_m^{\frac{-\delta p}{p-1+\delta}}\Big[\frac{(\delta-1)(p-1)}{p-1+\delta}|\nabla\Phi_m|^p + \lambda_1 m(x)\phi_m^p\Big] \\
&\le \Big(\frac{pM}{p-1+\delta}\Big)^{p-1}\Phi_m^{\frac{-\delta p}{p-1+\delta}}\Big[\frac{(\delta-1)(p-1)}{p-1+\delta}|\nabla\Phi_m|^p + \lambda_1 m(x)\phi_m^p\Big] = -\Delta_p(\overline{u}_\lambda).
\end{aligned}
$$

Consequently, we may apply the weak comparison principle (see in [11, Theorem 2.3]) in order to conclude that $\underline{u}_\lambda \le \overline{u}_\lambda$. Thus, By the classical iteration method (1.1)+ has a solution between the sub-solution and the super-solution.

Let us now prove that u_λ is a minimal weak solution of $(1.1)_+$. We use here the weak comparison principle (see Proposition 2.3 in Cuesta and Takáč [6]) and the following monotone iterative scheme:

$$
\begin{gathered}
-\Delta_p u_n - \lambda k(x)u_n^{-\delta} = h(x)u_{n-1}^q \quad \text{in } \Omega; \\
u_n|_{\partial\Omega} = 0,
\end{gathered} \tag{2.2}
$$

where $u_0 = \underline{u}_\lambda$, according to Giacomoni, Schindler and Takáč [11], is the unique solution to the following purely singular problem

$$
\begin{gathered}
-\Delta_p u = \lambda k(x)u^{-\delta} \quad \text{in } \Omega, \\
u|_{\partial\Omega} = 0, \quad u>0 \quad \text{in } \Omega.
\end{gathered}
$$

Note that u_0 is a weak subsolution to $(1.1)_+$ and $u_0 \le U$ where U is any weak solution to $(1.1)_+$. Then, from the weak comparison principle, we obtain easily that $u_0 \le u_1$ and $\{u_n\}_{n=1}^\infty$ is a nondecreasing sequence. Furthermore, $u_n \le U$ and $\{u_n\}_{n=1}^\infty$ is uniformly bounded in $W_0^{1,p}(\Omega)$. Hence, it is easy to prove that $\{u_n\}$ converges weakly in $W_0^{1,p}(\Omega)$ and pointwise to u_λ, a weak solution to the problem $(1.1)_+$. Let us show that u_λ is the minimal solution to $(1.1)_+$ for any $0<\lambda<\Lambda^*$. Let v_λ a weak solution to $(1.1)_+$ for any $0<\lambda<\Lambda^*$. Then, $u_0 = \underline{u}_\lambda \le v_\lambda$. From the weak comparison principle, $u_n \le v_\lambda$ for any $n\ge 0$. Letting $n\to\infty$, we obtain $u_\lambda \le v_\lambda$. This completes the proof of the Step 1.

Step 2: $(1.1)_+$ has no positive solution for $\lambda > \Lambda^*$. Firstly, from Step 1 we have that $\Lambda^*>0$. Now, let us show that $\Lambda^*<\infty$. We argue by contradiction: suppose there exists a sequence $\lambda_n\to\infty$ such that $(1.1)_+$ admits a solution u_n. Denote

$$
m := \min\{\operatorname{ess\,inf}_{x\in\Omega} k(x), \operatorname{ess\,inf}_{x\in\Omega}\inf h(x)\} > 0.
$$

There exists $\lambda^*>0$ such that

$$
m\left(\lambda t^{-\delta}+t^q\right) \ge (\lambda_1+\epsilon)t^{p-1} \quad \text{for all } t>0,\ \epsilon\in(0,1),\ \lambda>\lambda^*
$$

where λ_1 is the first Dirichlet eigenvalue of $-\Delta_p$ is positive and is given by

$$\lambda_1 = \min_{u\neq 0} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} \tag{2.3}$$

(see Lindqvist [16]). Choose $\lambda_n > \lambda^*$. Clearly u_n is a supersolution of the problem

$$\begin{gathered} -\Delta_p u = (\lambda_1 + \epsilon) u^{p-1} \quad \text{in } \Omega; \\ u > 0, \quad u|_{\partial\Omega} = 0. \end{gathered} \tag{2.4}$$

for all $\epsilon \in (0,1)$. We now use the [11, Lemma 3.1] to choose $\mu < \lambda_1 + \epsilon$ small enough so that $\mu\phi_1(x) < u_n(x)$ and $\mu\phi_1$ is a subsolution to problem (2.4). By a monotone interation procedure we obtain a solution to (2.4) for any $\epsilon \in (0,1)$, contradicting the fact that λ_1 is an isolated point in the spectrum of $-\Delta_p$ in $W_0^{1,p}(\Omega)$ (see Anane [2]). This proves the claim and completes the proof of the step 2.

Step 3: Existence of at least one positive weak solution for $\lambda = \Lambda^*$ to $(1.1)_+$. Let $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $\lambda_k \uparrow \Lambda^*$ as $k \to \infty$. Then, from Step 1, there exists $u_k = u_{\lambda_k} \geq \underline{u}_{\lambda_k}$ to a weak positive solution to $(1.1)_+$ for $\lambda = \lambda_k$. Therefore, for any $\phi \in C_c^\infty(\Omega)$, we have:

$$\int_\Omega |\nabla u_k|^{p-2} \nabla u_k \nabla\phi \,\mathrm{d}x = \lambda_k \int_\Omega k(x) u_k^{-\delta} \phi \,\mathrm{d}x + \int_\Omega h(x) u_k^q \phi \,\mathrm{d}x. \tag{2.5}$$

Since $u_k \in W_0^{1,p}(\Omega)$ and $u_k \geq \underline{u}_{\lambda_k}$ it is easy to see that (2.5) holds also for $\phi \in W_0^{1,p}(\Omega)$. Moreover, from above

$$E_{\lambda_k}(u_k) \leq E_{\lambda_k}(\underline{u}_{\lambda_k}) < \frac{1}{p}\int_\Omega |\nabla \underline{u}_{\lambda_k}|^p \,\mathrm{d}x - \frac{\lambda_k}{1-\delta}\int_\Omega k(x) {\underline{u}_{\lambda_k}}^{1-\delta} \,\mathrm{d}x < 0, \tag{2.6}$$

Thus, by Sobolev imbedding and using the fact that $k, h \in L^\infty(\Omega)$ it follows that

$$\sup_k \|u_k\|_p < \infty. \tag{2.7}$$

Hence, there exists $u_{\Lambda^*} \geq \underline{u}_{\lambda_k}$ such that $u_k \rightharpoonup u_{\Lambda^*}$ in $W_0^{1,p}(\Omega)$ as $k \to \infty$ and

$$u_k \rightharpoonup u_{\Lambda^*} \text{ in } L^q(\Omega) \text{ since } p-1 < q < p^*-1, \text{ and pointwise a.e. as } k \to \infty. \tag{2.8}$$

From (2.5), (2.7) and (2.8), for any $\phi \in W_0^{1,p}(\Omega)$ we obtain

$$\int_\Omega |\nabla u_{\Lambda^*}|^{p-2} \nabla u_{\Lambda^*} \nabla\phi \,\mathrm{d}x = \Lambda^* \int_\Omega k(x) u_{\Lambda^*}^{-\delta} \phi \,\mathrm{d}x + \int_\Omega h(x) u_{\Lambda^*}^q \phi \,\mathrm{d}x \tag{2.9}$$

which completes the proof of the Step 3 and gives the proof of Theorem 1.4.

3. Proof of Theorem 1.5

The study of existence of solutions to problem $(1.1)_-$ is done by looking for critical points of the functional $J_\lambda \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x - \frac{\lambda}{1-\delta}\int_\Omega k(x)|u|^{1-\delta} \,\mathrm{d}x + \frac{1}{q+1}\int_\Omega h(x)|u|^{q+1} \,\mathrm{d}x \tag{3.1}$$

in the Sobolev space $W_0^{1,p}(\Omega)$. In the next we adopt the following notations. The norm in $W_0^{1,p}(\Omega)$ will be denoted by

$$\|u\| = \Big(\int_\Omega |\nabla u|^p \,\mathrm{d}x\Big)^{1/p}.$$

The norm in $L^{q+1}(\Omega)$ will be denoted by

$$\|u\|_{q+1} = \Big(\int_\Omega |u|^{q+1}\,\mathrm{d}x\Big)^{1/q+1}.$$

The proof of the theorem is organized in several steps.

Step 1: The energy functional J_λ has a global minimizer. We first prove that J_λ is coercive. In order to verify this claim, we first observe that by using Hölder's and Sobolev's inequalities, we have for any $u \in W_0^{1,p}(\Omega)$ and all $\lambda > 0$

$$J_\lambda(u) \geq \frac{1}{p}\|u\|^p - C_1\|u\|^{1-\delta} + C_2\|u\|_{q+1}^{q+1} \tag{3.2}$$

where $C_1 = \lambda|\Omega|^{D+E(1-\delta)} S^{\frac{\delta-1}{p}} \frac{\|k\|_{L^\infty}}{(1-\delta)}$ with $D = \frac{q+\delta}{q+1}$, $E = \frac{p^*-q-1}{p^*(q+1)}$ and $S > 0$ is the best Sobolev constant and $C_2 = (q+1)^{-1}\operatorname{ess\,inf}_{x\in\Omega} h(x)$ are positive constants. It follows from (3.2) that

$$J_\lambda(u) \geq \frac{1}{p}\|u\|^p - C_1\|u\|^{1-\delta}. \tag{3.3}$$

and hence $J_\lambda(u) \to +\infty$ as $\|u\| \to \infty$. This completes the proof of our Claim.

Now, let $n \mapsto u_n$ be a minimizing sequence of J_λ in $W_0^{1,p}(\Omega)$. The coercivity of J_λ implies the boundedness of u_n in $W_0^{1,p}(\Omega)$. Since $J_\lambda(u) = J_\lambda(|u|)$, without loss of generality, we may assume that $(u_n)_n$ is non-negative, converges weakly to some u in $W_0^{1,p}(\Omega)$ and converges also pointwise. Moreover, by the weak lower semicontinuity of the norm $\|\cdot\|$ and the boundedness of $(u_n)_n$ in $W_0^{1,p}(\Omega)$ we obtain

$$J_\lambda(u) \leq \lim_{n\to\infty}\inf J_\lambda(u_n).$$

Hence u is a global minimizer of J_λ in $W_0^{1,p}(\Omega)$. Which completes the proof of the Step 1.

Step 2: The weak limit u is a non-negative weak solution of problem $(1.1)_-$ if $\lambda > 0$ is sufficiently large. Firstly, observe that $J_\lambda(0) = 0$. So, to prove that the non-negative solution is non-trivial, it suffices to prove that there exists $\lambda_* > 0$ such that

$$\inf_{u\in W_0^{1,p}(\Omega)} J_\lambda(u) < 0 \quad \text{for all } \lambda > 0. \tag{3.4}$$

For this purpose, take any positive u and consider ϵu. Then, for a fixed $\lambda > 0$, $J_\lambda(\epsilon u) < 0$ if $\epsilon > 0$ is small enough. Therefore the minimum is negative for all $\lambda > 0$.

Now, we consider the variational problem with constraints,

$$\begin{aligned}\lambda_* = \inf\Big\{&\frac{1}{p}\int_\Omega |\nabla w|^p\,\mathrm{d}x + \frac{1}{q+1}\int_\Omega h(x)|w|^{q+1}\,\mathrm{d}x : w \in W_0^{1,p}(\Omega) \text{ and}\\ &\frac{1}{1-\delta}\int_\Omega k(x)|w|^{1-\delta}\,\mathrm{d}x = 1\Big\}.\end{aligned} \tag{3.5}$$

and define

$$\Lambda_* = \inf\{\lambda > 0 \colon (1.1)_- \text{ admits a nontrivial weak solution}\}. \tag{3.6}$$

From above, we have

$$J_\lambda(u) = \lambda_* - \lambda < 0 \quad \text{for any } \lambda > \lambda_*.$$

Therefore, the above remarks show that $\lambda_* \geq \Lambda_*$ and that problem $(1.1)_-$ has a solution for all $\lambda > \lambda_*$.

We now argue that problem $(1.1)_-$ has a solution for all $\lambda > \Lambda_*$. Fixed $\lambda > \Lambda_*$, by the definition of Λ_*, we can take $\mu \in (\Lambda_*, \lambda)$ such that that J_μ has a non-trivial critical point $u_\mu \in W_0^{1,p}(\Omega)$. Since $\mu < \lambda$, u_μ is a sub-solution of the problem $(1.1)_-$. In order to find a super-solution of the problem $(1.1)_-$ which dominates u_μ. For this purpose we consider the constrained minimization problem

$$\inf\{J_\lambda(w) : \ w \in W_0^{1,p}(\Omega) \text{ and } w \geq u_\mu.\} \tag{3.7}$$

Arguments similar to those used to treat (3.5) show that the above minimization problem has a solution $u_\lambda > u_\mu$. Moreover, u_λ is also a weak solution of problem $(1.1)_-$ for all $\lambda > \Lambda_*$. With the arguments developed in [11] we deduce that problem $(1.1)_-$ has a solution if $\lambda = \Lambda_*$.

Thus, one applies [2, Theorem A.1], based on the Moser iteration, shows that $u \in L^\infty_{\rm loc}$. Next, again by a bootstrap regularity due to Giacomoni-Schindler-Takáč [11, Theorem B.1] shows that the weak solution $u \in C^{1,\alpha}(\Omega)$ where $\alpha \in (0,1)$. Finally, the non-negative follows immediately by the strong maximum principle (see [11, Theorem 2.3]) since u is a C^1 non-negative weak solution of the differential inequality

$$-\nabla(|\nabla u|^{p-2}\nabla u) + h(x)u^q \geq 0 \quad \text{in } \Omega.$$

We deduce that u is positive everywhere in Ω. The proof of the step 2 is now complete.

Step 3: Non-existence for $\lambda > 0$ small. The same monotonicity arguments as in Step 2 show that $(1.1)_-$ does not have any solution if $\lambda < \Lambda_*$. Which completes the proof of the Theorem 1.5.

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