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POSITIVE SOLUTIONS FOR p-LAPLACIAN EQUATIONS OF KIRCHHOFF TYPE PROBLEM WITH A PARAMETER

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Abstract. In this article, we consider the existence and non-existence of positive solutions for the Kirchhoff type equation

$$
\begin{gathered}
-\Big(a+\lambda M\Big(\int_\Omega |\nabla u|^p dx\Big)\Big)\Delta_p u = f(u), \quad \text{in } \Omega, \\
u = 0, \quad \text{on } \partial\Omega,
\end{gathered}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$, a is a positive constant, $N \geq 3$, $\lambda \geq 0$, $2 \leq p < N$, M and f are positive continuous functions. Under some weak assumptions on f, we show that the above problem has at least one positive solution when λ is small and has no nonzero solution when λ is large. Our argument is based on iterative technique and variational methods.

1. Introduction and main results

The well-known nonlinear Kiffchhoff type equation has attracted massive attention as it stems from interesting physical problems, see [1, 10, 19]. The pioneer research on Kirchhoff type problem belongs to Pohozaev [25] and Bernstein [3]. But only after the work of Lions [20], in which an abstract functional framework to the equation was set, the equation received extensive attention.

In this article, we are interested in the existence of positive solutions for the nonlinear Kirchhoff equation

$$
\begin{gathered}
-\Big(a+\lambda M\Big(\int_\Omega |\nabla u|^p dx\Big)\Big)\Delta_p u = f(u), \quad \text{in } \Omega, \\
u = 0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.1}
$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 3$, $a > 0$ is a positive constant, $\lambda \geq 0$ is a parameter, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $2 \leq p < N$, and $M : \mathbb{R}_+ \to \mathbb{R}_+$ with $\mathbb{R}_+ = [0, +\infty)$. Moreover, the nonlinearity $f(t)$ satisfies the following basic assumptions:

- (A1) f is Lipschitz continuous and $\lim_{t\to 0^+} \frac{f(t)}{t^{p-1}} = 0$;
- (A2) $\lim_{t\to +\infty} \frac{f(t)}{t^{p^*-1}} = 0$, where $p^* = \frac{pN}{N-p}$;
- (A3) $\lim_{t\to +\infty} \frac{f(t)}{t^{p-1}} = +\infty$.

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Since we are only interested in positive solutions, without loss of generality, we suppose that $f(t) \equiv 0$ for $t < 0$.

Problem (1.1) has been widely researched in recent years, especially on the existence of positive solutions, multiple solutions and sign-changing solutions, see [2, 7, 8, 16, 17, 23]. For example, Ourraoui [23] considered problem (1.1) involving critical Sobolev exponent. They got their results via the variational principle of Ekeland. Correa et al [7] also studied problem (1.1). They established sufficient conditions on M and the nonlinearity f under which (1.1) possesses positive solutions. Later, based on the fountain theorem, Huang et al in [17] proved the existence and multiplicity of solutions of problem (1.1) when the nonlinearity is concave-convex.

The generalization of problem (1.1) to unbounded domain also attracted much attention. For some interesting results, we refer to [4, 5, 9, 13, 24]. Chen, Song and Xiu [4] studied the following general case:

$$\begin{gathered} M\Big(\int_{\mathbb{R}^N}(|\nabla u|^p + V(x)|u|^p)dx\Big)(-\Delta u + V(x)|u|^{p-2}u) \\ = f(x,u) + g(x), \quad \text{in } \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to +\infty. \end{gathered} \tag{1.2}$$

Under different assumptions on the nonlinear term $f(x,u)$, multiple solutions of problem (1.2) was constructed by applying the Mountain Pass Theorem, Ekeland's variational principle and Krasnoselskii's genus theory in [28]. Cheng and Dai [13] also considered a class of generalized form of problem (1.1). They used a cut-off function to get the bounded Palais-Smale sequences and proved the existence of a positive solution. In addition, when $M(t) = t$, Chen and Zhu [5] utilized the Nehari manifold method to study problem (1.1). They obtained that there exists at least a positive ground state solution.

In the special case of $p = 2$, there are much more works than that of general p. For example, Li et al [22] studied the existence of a positive solution to the nonlinear Kirchhoff type problem

$$\Big(a + \lambda\int_{\mathbb{R}^N}|\nabla u|^2dx + \lambda\int_{\mathbb{R}^N}u^2dx\Big)(-\Delta u + bu) = f(u), \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Where $N \geq 3$ and a, b are positive constants. Under the condition $\lim_{t\to+\infty} f(t)/t = +\infty$ and λ is sufficient small, a positive solution of problem (1.3) was obtained by using a cut-off function and monotonicity trick. But they didn't show whether there still exists at least one positive solution when λ is not so small. More recently, Liu, Liao and Tang [21] investigated problem (1.3) further. They considered two cases where the nonlinearity respectively satisfied asymptotically linear and superlinear conditions at the infinity. The most important is, they proved that if λ is large enough the problem (1.3) has no nonzero solution. For more interesting results, we refer to [11, 12, 26, 29, 30, 31] and the references therein.

In this article, motivated by the papers [13, 21, 22], we discuss the existence and non-existence of positive solutions of problem (1.1). We adopt the method in [15], which studied the solution for semilinear elliptic equation. More precisely, first of all, we will use monotonicity tricks introduced in [18, 27] and iterative technique to establish the existence of positive solutions for equation (1.1) whenever λ is sufficient small. Secondly, we will show that if λ is large enough, equation (1.1) has no nonzero solution.

To state our main results clearly, we firstly introduce some Sobolev spaces and norms. Let $W_0^{1,p}(\Omega)$ be the usual Sobolev space. We denote the usual norm of $L^s(\Omega)$ by $\|u\|_s$ for all $p \le s \le p^*$. Define $H := \{u \in W_0^{1,p}(\Omega) : u(x) = u(|x|)\}$, equipped with the norm

$$\|u\| = \Big(\int_\Omega |\nabla u|^p dx\Big)^{1/p}, \quad \forall u \in H.$$

It is clear that the embedding $H \hookrightarrow L^s(\Omega)$ for $p < s < p^*$ is compact and continuous for $p \le s \le p^*$, namely, there exists constants $\gamma_s > 0$ such that $\|u\|_s \le \gamma_s \|u\|$ for $p \le s \le p^*$. Here and in the sequel, C_i denote positive constants, $i = 1, 2, 3, \dots$. The following theorem is the first main result in the paper.

Theorem 1.1. *Assume that Ω is convex and $\lambda \ge 0$ is a parameter. If the conditions* (A1)–(A3) *hold. Then for any positive continuous function M, there exists λ_0 such that for any $\lambda \in [0, \lambda_0)$, problem* (1.1) *has at least one positive solution.*

Remark 1.2. We note that for the special case $p = 2$ and $\lambda = 0$, the above result has been established in [32] and [14] respectively. Besides, Cheng and Dai [13] also obtained the result under the conditions (A1), (A3) and the assumption

(A4) there exist constants $C > 0$ and $q \in (p, p^*)$ such that

$$|f(t)| \le C(|t|^{p-1} + |t|^{q-1}), \quad \forall t \in \mathbb{R}_+.$$

Evidently, the condition (A4) is stronger than our condition (A2). Thus, our result can be regarded as an extension of these papers mentioned above.

Nevertheless, if the parameter $\lambda > 0$ is big enough, and Ω is unbounded, in addition to the following assumptions:

(A5) there exists a $\tau > 0$ such that $M(t) = t^\tau$ with $p(\tau + 1)/\tau < N$;

(A6) $f \in C(\mathbb{R}, \mathbb{R})$ and $\lim_{t\to 0} \frac{f(t)}{t^{p-1}} = 0$;

(A7) $\limsup_{|t|\to+\infty} \frac{|f(t)|}{t^{p^*-1}} < +\infty$;

we can obtain the following results:

Theorem 1.3. *Assume that $\Omega = \mathbb{R}^N$ with $N \ge 3$ and $\lambda > 0$ is a parameter. If* (A5)–(A7) *hold, then there exists $\Theta > 0$ such that for any $\lambda > \Theta$, problem* (1.1) *has no nontrivial solution.*

The main results are proved in the sections below. In Section 2, some preliminary concepts and results are presented. In Section 3, we prove Theorems 1.1 and 1.3.

2. Preliminary results

Throughout this section, we suppose $T, S > 0$ and $\varphi \in H$ with $\|\varphi\| \le S$. For given $\varphi \in H$ and $\theta \in [\frac{1}{p}, 1]$, we study the energy functional $\Phi_{\varphi,\theta} : H \to \mathbb{R}$ define by

$$\Phi_{\varphi,\theta}(u) = \frac{a}{p}\int_\Omega |\nabla u|^p dx + \frac{\lambda}{p} M(\|\varphi\|^p)\int_\Omega |\nabla u|^p dx - \theta\int_\Omega F(u)dx \tag{2.1}$$

for all $u \in H$, where $F(u) = \int_0^u f(t)dt$. Obviously, the functional $\Phi_{\varphi,\theta}$ is well defined and $\Phi_{\varphi,\theta} \in C^1(H, \mathbb{R})$. Further, for any $u, v \in H$, we have

$$\langle \Phi'_{\varphi,\theta}(u), v\rangle = (a + \lambda M(\|\varphi\|^p))\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v\, dx - \theta\int_\Omega f(u)v\, dx. \tag{2.2}$$

In the process of our argument, we will use the following proposition.

Proposition 2.1 ([18, 27])**.** *Let $(X, \|\cdot\|_X)$ be a Banach space and $I \subset \mathbb{R}_+$ an interval. Consider the family of C^1 functionals on X*

$$J_\mu(u) = A(u) - \mu B(u), \quad \mu \in I,$$

with B nonnegative and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_X \to +\infty$ and such that $J_\mu(0) = 0$.

For any $\mu \in I$, we set

$$\Gamma_\mu = \{\gamma \in C([0,1], X) : \gamma(0) = 0, J_\mu(\gamma(1)) < 0\}.$$

If for every $\mu \in I$ the set Γ_μ is nonempty and $c_\mu = \inf_{\gamma\in\Gamma_\mu} \max_{t\in[0,1]} J_\mu(\gamma(t)) > 0$, then for almost every $\mu \in I$ there is a sequence $\{u_n\} \subset X$ such that

1. *$\{u_n\}$ is bounded;*
2. *$J_\mu(u_n) \to c_\mu$;*
3. *$J'_\mu(u_n) \to 0$ in the dual X^{-1} of X.*

To apply Proposition 2.1, in our case, we let

$$A_\varphi(u) = \frac{a}{p}\|u\|^p + \frac{\lambda}{p} M(\|\varphi\|^p)\|u\|^p, \quad B(u) = \int_\Omega F(u)dx.$$

The following Pohožaev equality is crucial to the proof of the boundedness of the Palais-Smale sequence.

Lemma 2.2. *If $u \in H$ is a critical point of $\Phi_{\varphi,\theta}$, namely, u is a week solution of*

$$\begin{gathered} -\left(a + \lambda M(\|\varphi\|^p)\right)\Delta_p u = \theta f(u), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{2.3}$$

then the following Pohožaev type identity holds

$$\begin{aligned} &[a + \lambda M(\|\varphi\|^p)]\Big[\big(\frac{N}{p} - 1\big)\|u\|^p + \big(1 - \frac{1}{p}\big)\int_{\partial\Omega} |\nabla u|^p (x\cdot\nu)d\sigma\Big] \\ &= \theta N \int_\Omega F(u)dx. \end{aligned} \tag{2.4}$$

Proof. Because $u \in H$ is a week solution of (2.3), by the standard regularity results, we get that $u \in W_0^{2,p}(\Omega) \cap W_0^{1,p}(\overline{\Omega})$. Setting

$$g(u) = \frac{\theta f(u)}{a + \lambda M(\|\varphi\|^p)}.$$

Then, it is obvious that $u \in H$ is also a solution of $-\Delta_p u = g(u)$. Applying the Pohožaev identity in [6], we have

$$\big(1 - \frac{1}{p}\big)\int_{\partial\Omega} |\nabla u|^p (x\cdot\nu)d\sigma = \big(1 - \frac{N}{p}\big)\int_\Omega g(u)u dx + N\int_\Omega G(u)dx, \tag{2.5}$$

where $G(t) = \int_0^t g(s)ds$, we obtain the conclusion. $\square$

Lemma 2.3. *If* (A3) *holds, then there exist $\lambda_0 = \lambda_0(T, S) > 0$ and $u_0 \in H$, such that $\Phi_{\varphi,\theta}(u_0) < 0$ for every $\lambda \in [0, \lambda_0)$.*

Proof. For given $T > 0$, there exists a constant $\lambda_0 = \lambda_0(T, S)$, such that

$$\lambda \max_{\tau \in [0, S^p]} M(\tau) \leq T \tag{2.6}$$

whenever $\lambda \in [0, \lambda_0)$. Choose $\phi \in H$ with $\phi \geq 0$ and $\|\phi\| = 1$. In view of (A3), we have that for any $C_1 > 0$ with $C_1 > (a+T)/\int_\Omega |\phi|^p dx$, there exists $C_2 > 0$ such that

$$F(t) \geq C_1 |t|^p - C_2, \quad t \in \mathbb{R}_+. \tag{2.7}$$

Thus, for any $\lambda \in [0, \lambda_0)$, we have

$$\begin{aligned} \Phi_{\varphi,\theta}(t\phi) &= \frac{1}{p}(a + \lambda M(\|\varphi\|^p)) \int_\Omega |\nabla(t\phi)|^p dx - \theta \int_\Omega F(t\phi) dx \\ &\leq \frac{t^p}{p}(a+T) - \theta \int_\Omega F(t\phi) dx \\ &\leq \frac{t^p}{p}\Big(a + T - C_1 \int_\Omega |\phi|^p dx\Big) + \frac{C_2 |\Omega|}{p}. \end{aligned} \tag{2.8}$$

Hence, we can choose $t > 0$ large enough such that $\Phi_{\varphi,\theta}(t\phi) < 0$, the proof is completed. $\square$

Lemma 2.4. *Under assumptions* (A1) *and* (A2)*, there exists positive constants* α, β *such that*

$$\Phi_{\varphi,\theta}(u) \geq \alpha, \quad \forall u \in H, \quad \|u\| \leq \beta. \tag{2.9}$$

Proof. Using (A1) and (A2), for $\varepsilon \in (0, 1/(2\gamma_p^p))$, there exists a constants $C_3(\varepsilon) > 0$ such that

$$F(t) \leq \frac{\varepsilon}{p} t^p + C_3(\varepsilon) t^{p^*}, \quad t \in \mathbb{R}_+. \tag{2.10}$$

Furthermore, for $u \in H$, by the Sobolev embedding,

$$\begin{aligned} \Phi_{\varphi,\theta}(u) &= \frac{1}{p}(a + \lambda M(\|\varphi\|^p))\|u\|^p - \theta \int_\Omega F(u) dx \\ &\geq \frac{a}{p}\|u\|^p - \int_\Omega \big(\frac{\varepsilon}{p}|u|^p + C_3(\varepsilon)|u|^{p^*}\big) dx \\ &\geq \frac{a}{2p}\|u\|^p - C_3(\varepsilon)\gamma_{p^*}^{p^*}\|u\|^{p^*}. \end{aligned} \tag{2.11}$$

Hence, choosing

$$\beta := \|u\| = \Big(\frac{1}{2p^* C_3(\varepsilon)\gamma_{p^*}^{p^*}}\Big)^{\frac{1}{p^*-p}},$$

one has $\Phi_{\varphi,\theta}(u) \geq \alpha$, where $\alpha = \beta^p(\frac{a}{2p} - \frac{1}{2p^*})$, which is independent of φ and θ. $\square$

Lemma 2.5. *If* (A1)–(A3) *hold, then there exist* $\lambda_0 = \lambda_0(T, S) > 0$ *and a sequence* $\{\theta_k\} \subset I$ *satisfying* $\theta_k \to 1$ *(as* $k \to +\infty$*), such that* Φ_{φ,θ_k} *has a nontrivial critical point* u_{φ,θ_k} *for* $\lambda \in [0, \lambda_0)$.

Proof. Set $I = [\frac{1}{p}, 1]$, from Proposition 2.1, there is $\{\theta_k\} \subset I$ with $\theta_k \to 1$ as $k \to +\infty$, and corresponding sequence $\{u_{n,\varphi,\theta_k}\} \subset H$ such that

$$\{u_{n,\varphi,\theta_k}\} \text{ is bounded and } \Phi_{\varphi,\theta_k}(u_{n,\varphi,\theta_k}) \to c_{\varphi,\theta_k};$$
$$\Phi'_{\varphi,\theta_k}(u_{n,\varphi,\theta_k}) \to 0 \quad \text{in } H^{-1};$$

where $c_{\varphi,\theta_k} = \inf_{\gamma\in\Gamma_{\varphi,\theta_k}} \sup_{u\in\gamma([0,1])} \Phi_{\varphi,\theta_k}(u)$ and

$$\Gamma_{\varphi,\theta_k} = \{\gamma \in C([0,1],H) | \gamma(0) = 0, \Phi_{\varphi,\theta_k}(\gamma(1)) < 0\}.$$

Up to a subsequence, we can assume that there exists u_{φ,θ_k} in H such that

$$\begin{gathered} u_{n,\varphi,\theta_k} \rightharpoonup u_{\varphi,\theta_k}, \quad \text{in } H; \\ u_{n,\varphi,\theta_k} \to u_{\varphi,\theta_k}, \quad \text{on } L^s(\Omega),\ \forall s \in (p,p^*); \\ u_{n,\varphi,\theta_k} \to u_{\varphi,\theta_k}, \quad \text{a.e. on } \Omega. \end{gathered} \tag{2.12}$$

From (A1) and (A2), for any $0 < \varepsilon < \frac{1}{p}$, there exists $C_\varepsilon > 0$ such that

$$|f(t)| \le \varepsilon |t|^{p-1} + \varepsilon |t|^{p^*-1} + C_\varepsilon |t|^{k_0-1}, \quad k_0 \in (p,p^*). \tag{2.13}$$

Then, from Hölder's inequality, we have

$$\begin{aligned} &\big| \int_\Omega f(u_{n,\varphi,\theta_k})(u_{n,\varphi,\theta_k} - u_{\varphi,\theta_k}) dx \big| \\ &\le \int_\Omega |f(u_{n,\varphi,\theta_k})| |u_{n,\varphi,\theta_k} - u_{\varphi,\theta_k}| dx \\ &\le \varepsilon \|u_{n,\varphi,\theta_k}\|_p^{p-1} \|u_{n,\varphi,\theta_k} - u_{\varphi,\theta_k}\|_p + \varepsilon \|u_{n,\varphi,\theta_k}\|_{p^*}^{p^*-1} \|u_{n,\varphi,\theta_k} - u_{\varphi,\theta_k}\|_{p^*} \\ &\quad + C_\varepsilon \|u_{n,\varphi,\theta_k}\|_{k_0}^{k_0-1} \|u_{n,\varphi,\theta_k} - u_{\varphi,\theta_k}\|_{k_0} \\ &\le \varepsilon \gamma_p^p \|u_{n,\varphi,\theta_k}\|^{p-1} \|u_{n,\varphi,\theta_k} - u_{\varphi,\theta_k}\| + \varepsilon \gamma_{p^*}^{p^*} \|u_{n,\varphi,\theta_k}\|^{p^*-1} \|u_{n,\varphi,\theta_k} - u_{\varphi,\theta_k}\| \\ &\quad + C_\varepsilon \gamma_{k_0}^{k_0-1} \|u_{n,\varphi,\theta_k}\|^{k_0-1} \|u_{n,\varphi,\theta_k} - u_{\varphi,\theta_k}\|_{k_0}, \end{aligned}$$

which implies that

$$\int_\Omega f(u_{n,\varphi,\theta_k})(u_{n,\varphi,\theta_k} - u_{\varphi,\theta_k}) dx \to 0. \tag{2.14}$$

Similar to the argument above, we can also conclude that

$$\begin{gathered} \langle \Phi'_{\varphi,\theta_k}(u_{\varphi,\theta_k}), u_{n,\varphi,\theta_k} - u_{\varphi,\theta_k} \rangle \to 0, \\ \int_\Omega f(u_{\varphi,\theta_k})(u_{n,\varphi,\theta_k} - u_{\varphi,\theta_k}) dx \to 0, \quad \text{as } n \to +\infty. \end{gathered}$$

From this and (2.14), one has

$$\begin{aligned} &\langle \Phi'_{\varphi,\theta_k}(u_{n,\varphi,\theta_k}) - \Phi'_{\varphi,\theta_k}(u_{\varphi,\theta_k}), u_{n,\varphi,\theta_k} - u_{\varphi,\theta_k} \rangle \\ &= a \int_\Omega (|\nabla u_{n,\varphi,\theta_k}|^{p-2} \nabla u_{n,\varphi,\theta_k} - |\nabla u_{\varphi,\theta_k}|^{p-2} \nabla u_{\varphi,\theta_k}) \cdot \nabla (u_{n,\varphi,\theta_k} - u_{\varphi,\theta_k}) dx \\ &\quad + \lambda M(\|\varphi\|^p) \int_\Omega \Big(|\nabla u_{n,\varphi,\theta_k}|^{p-2} \nabla u_{n,\varphi,\theta_k} - |\nabla u_{\varphi,\theta_k}|^{p-2} \nabla u_{\varphi,\theta_k} \Big) \\ &\quad \cdot \nabla (u_{n,\varphi,\theta_k} - u_{\varphi,\theta_k}) dx \\ &\quad + \theta_k \int_\Omega [f(u_{n,\varphi,\theta_k}) - f(u_{\varphi,\theta_k})](u_{n,\varphi,\theta_k} - u_{\varphi,\theta_k}) dx \to 0. \end{aligned}$$

Combining this with the standard inequality in $\mathbb{R}^N$ given by

$$(|\zeta|^{p-2}\zeta - |\eta|^{p-2}\eta, \zeta - \eta) \ge \begin{cases} C_p |\zeta - \eta|^p, & p \in [2,+\infty), \\ C_p |\zeta - \eta|^2 (|\zeta| + |\eta|)^{p-2}, & 1 < p < 2. \end{cases} \tag{2.15}$$

We have that $\|u_{n,\varphi,\theta_k} - u_{\varphi,\theta_k}\| \to 0$, that is, $u_{n,\varphi,\theta_k} \to u_{\varphi,\theta_k}$ in H.

It follows from the above discussion that there exist $\lambda_0 = \lambda_0(T,S) > 0$ and a sequences $\{\theta_k\}$ with $\theta_k \to 1$ such that

$$\Phi_{\varphi,\theta_k}(u_{\varphi,\theta_k}) = c_{\varphi,\theta_k} \quad \text{and} \quad \langle \Phi'_{\varphi,\theta_k}(u_{\varphi,\theta_k}), u_{\varphi,\theta_k}\rangle = 0,$$

if $\lambda \in [0,\lambda_0)$. The proof is complete. $\square$

Lemma 2.6. *Let u_{φ,θ_k} be a critical point of Φ_{φ,θ_k} at level c_{φ,θ_k}. Then for $S > 0$ sufficiently large, there exists $\lambda_0 = \lambda_0(T,S)$ such that for any $\lambda \in [0,\lambda_0)$, subject to a subsequence, $\|u_{\varphi,\theta_k}\| \le S$ for all $k \in \mathbb{N}$.*

Proof. On the one hand, since u_{φ,θ_k} be a critical point of Φ_{φ,θ_k}, then from (2.4), u_{φ,θ_k} satisfies the following Pohožaev identity

$$\begin{aligned} &[a+\lambda M(\|\varphi\|^p)]\Big[\big(\frac{N}{p}-1\big)\|u_{\varphi,\theta_k}\|^p + \big(1-\frac{1}{p}\big)\int_{\partial\Omega}|\nabla u_{\varphi,\theta_k}|^p(x\cdot\nu)d\sigma\Big] \\ &= \theta_k N\int_\Omega F(u_{\varphi,\theta_k})dx. \end{aligned} \tag{2.16}$$

We assume μ_1 is an eigenvalue of the operator $-\Delta_p$, and let $\phi_1 > 0$, $x \in \Omega$ be an eigenfunction corresponding to μ_1, in view of (A3), we have that for any $\kappa > 0$ with $\kappa > 2\mu_1(a+T)$, there exists $C_4(\kappa) > 0$ such that

$$\begin{aligned} \mu_1(a+\lambda M(\|\varphi\|^p))\int_\Omega u^{p-1}_{\varphi,\theta_k}\phi_1 dx &= \theta_k\int_\Omega f(u_{\varphi,\theta_k})\phi_1 dx \\ &\ge \kappa\int_\Omega u^{p-1}_{\varphi,\theta_k}\phi_1 dx - C_4(\kappa) \end{aligned} \tag{2.17}$$

and $\int_\Omega u^{p-1}_{\varphi,\theta_k}\phi_1 dx \le C_5(T)$ for a constant $C_5(T) > 0$. Combining this with the results in [14], there is a constant $C_6(T) > 0$ such that $|\nabla u_{\varphi,\theta_k}|^p \le C_6(T)$, $x \in \partial\Omega$. Thus, by (2.16), there exists a constant $C_7(T) > 0$ such that

$$\begin{aligned} &\big(\frac{N}{p}-1\big)\|u_{\varphi,\theta_k}\|^p - \frac{\theta_k N}{a+\lambda M(\|\varphi\|^p)}\int_\Omega F(u_{\varphi,\theta_k})dx \\ &= -\big(1-\frac{1}{p}\big)\int_{\partial\Omega}|\nabla u_{\varphi,\theta_k}|^p(x\cdot\nu)d\sigma \\ &\ge -C_7(T). \end{aligned} \tag{2.18}$$

On the other hand, from Lemmas 2.3 and 2.5, there is a constant $C_8(T) > 0$ such that

$$\begin{aligned} c_{\varphi,\theta_k} &= \Phi_{\varphi,\theta_k}(u_{\varphi,\theta_k}) \\ &\le \max_{t\ge 0}\Phi_{\varphi,\theta_k}(t\phi) \\ &\le \max_{t\ge 0}\Big\{\frac{t^p}{p}(a+T) - \frac{1}{p}\int_\Omega F(t\phi)dx\Big\} \le C_8(T). \end{aligned} \tag{2.19}$$

So, one has

$$\frac{1}{p}\|u_{\varphi,\theta_k}\|^p - \frac{\theta_k}{a+\lambda M(\|\varphi\|^p)}\int_\Omega F(u_{\varphi,\theta_k})dx = \frac{c_{\varphi,\theta_k}}{a+\lambda M(\|\varphi\|^p)} \le C_8(T). \tag{2.20}$$

It follows from (2.18) and (2.20) that $\|u_{\varphi,\theta_k}\|^p \le NC_8(T) + C_7(T)$. Consequently, for given $T > 0$, if we take $S = (NC_8(T) + C_7(T))^{1/p}$, then $\|u_{\varphi,\theta_k}\| \le S$. $\square$

From Lemma 2.6, for any k, if $\varphi = \varphi_0 \equiv 0$, then we know that $\Phi_{\varphi_0,\theta_k}$ has a nontrivial critical point and we denote it by $u_{1,k}$ with $\|u_{1,k}\| \le S$. Let $\varphi = u_{1,k}$, then $\Phi_{u_{1,k},\theta_k}$ has a nontrivial critical point $u_{2,k}$ with $\|u_{2,k}\| \le S$. Therefore, by induction, we can obtain a sequence $u_{m,k}$ with $\|u_{m,k}\| \le S$, $m = 1, 2, \dots$.

3. Proof of the main results

Proof of Theorem 1.1. In view of $u_{n,k}$ with $\|u_{n,k}\| \le S$, for all $n, k \in \mathbb{N}$. For fixed k, up to a subsequence, we assume that $u_{n,k} \rightharpoonup u_k$ in H, $u_{n,k} \to u_k$ on $L^s(\Omega)$ for all $s \in (p, p^*)$ and $u_{n,k}(x) \to u_k(x)$ a.e. in Ω, we also have $\|u_k\| \le S$. Then, one has

$$\begin{aligned} &\langle \Phi'_{u_{n-1,k},\theta_k}(u_k), u_{n,k} - u_k\rangle \\ &= (a + \lambda M(\|u_{n-1,k}\|^p)) \int_\Omega |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla(u_{n,k} - u_k)dx \\ &\quad - \theta_k \int_\Omega f(u_k)(u_{n,k} - u_k)dx \to 0, \quad \text{as } n \to +\infty, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\langle \Phi'_{u_{n-1,k},\theta_k}(u_{n,k}) - \Phi'_{u_{n-1,k},\theta_k}(u_k), u_{n,k} - u_k\rangle \\ &= (a + \lambda M(\|u_{n-1,k}\|^p)) \int_\Omega \Big(|\nabla u_{n,k}|^{p-2}\nabla u_{n,k} - |\nabla u_k|^{p-2}\nabla u_k\Big) \\ &\quad \cdot \nabla(u_{n,k} - u_k)dx - \theta_k \int_\Omega [f(u_{n,k}) - f(u_k)](u_{n,k} - u_k)dx \\ &\to 0, \quad \text{as } n \to +\infty. \end{aligned} \tag{3.2}$$

It follows from (3.2) and (2.15) that $u_{n,k} \to u_k$ as $n \to +\infty$. And for every $v \in H$, one has

$$\begin{aligned} 0 &= \lim_{n\to+\infty} \langle \Phi'_{u_{n-1,k},\theta_k}(u_{n,k}), v\rangle \\ &= \lim_{n\to+\infty} \Big[(a + \lambda M(\|u_{n-1,k}\|^p)) \int_\Omega |\nabla u_{n,k}|^{p-2}\nabla u_{n,k} \cdot \nabla v dx \\ &\quad - \theta_k \int_\Omega f(u_{n,k}) v dx\Big] \\ &= (a + \lambda M(\|u_k\|^p)) \int_\Omega |\nabla u_k|^{p-2}\nabla u_k \cdot \nabla v dx - \theta_k \int_\Omega f(u_k) v dx \\ &= \langle \Phi'_{u_k,\theta_k}(u_k), v\rangle \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \Phi_{u_k,\theta_k}(u_k) &= \frac{1}{p}(a + \lambda M(\|u_k\|^p))\|u_k\|^p - \theta_k \int_\Omega F(u_k)dx \\ &= \lim_{n\to+\infty} \Big[\frac{1}{p}(a + \lambda M(\|u_{n-1,k}\|^p))\|u_{n,k}\|^p - \theta_k \int_\Omega F(u_{n,k})dx\Big] \\ &= \lim_{n\to+\infty} \Phi_{u_{n-1,k},\theta_k}(u_{n,k}). \end{aligned} \tag{3.4}$$

From Lemma 2.4, we have $\Phi_{u_{n-1,k},\theta_k}(u_{n,k}) = c_{u_{n-1,k},\theta_k} \ge \alpha$. Hence $\Phi'_{u_k,\theta_k}(u_k) = 0$, and $\Phi_{u_k,\theta_k}(u_k) \ge \alpha$ follows directly from (3.3) and (3.4).

On the other side, because of $\|u_k\| \le S$, $k \in \mathbb{N}$, without lose of generality, we may suppose that $u_k \rightharpoonup u$ in H, $u_k \to u$ on $L^s(\Omega)$ for all $s \in (p, p^*)$ and $u_k(x) \to u(x)$

a.e. in Ω. Together with the boundedness of $\lambda M(\|u_k\|^p)$, we get

$$
\begin{aligned}
&\langle \Phi'_{u_k,\theta_k}(u), u_k - u\rangle \\
&= (a+\lambda M(\|u_k\|^p))\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(u_k-u) - \theta_k\int_\Omega f(u)(u_k-u)dx \\
&\to 0, \quad \text{as } n\to+\infty,
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
&\langle \Phi'_{u_k,\theta_k}(u_k) - \Phi'_{u_k,\theta_k}(u), u_k - u\rangle \\
&= (a+\lambda M(\|u_k\|^p))\int_\Omega (|\nabla u_k|^{p-2}\nabla u_k - |\nabla u|^{p-2}\nabla u)\cdot\nabla(u_k-u)dx \\
&\quad - \theta_k\int_\Omega [f(u_k)-f(u)](u_k-u)dx.
\end{aligned}
\tag{3.6}
$$

From (2.15), we also get that $u_k \to u$ in H as $k\to+\infty$. Therefore, $\forall\ \omega\in H$, one has

$$
\begin{aligned}
0 &= \lim_{n\to+\infty}\langle \Phi'_{u_k,\theta_k}(u_k), \omega\rangle \\
&= \lim_{n\to+\infty}\Big[(a+\lambda M(\|u_k\|^p))\int_\Omega |\nabla u_k|^{p-2}\nabla u_k\cdot\nabla\omega dx - \theta_k\int_\Omega f(u_k)\omega dx\Big] \\
&= (a+\lambda M(\|u\|^p))\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\omega dx - \int_\Omega f(u)\omega dx,
\end{aligned}
\tag{3.7}
$$

which shows that u is a solution of (1.1). Furthermore,

$$
\begin{aligned}
&\frac{1}{p}(a+\lambda M(\|u\|^p))\|u\|^p - \int_\Omega F(u)dx \\
&= \lim_{k\to+\infty}\Big[\frac{1}{p}(a+\lambda M(\|u_k\|^p))\|u_k\|^p - \theta_k\int_\Omega F(u_k)dx\Big] \\
&= \lim_{k\to+\infty}\Phi_{u_k,\theta_k}(u_k).
\end{aligned}
\tag{3.8}
$$

Combining this with $\Phi_{u_k,\theta_k}(u_k)\geq\alpha>0$, we know that u is nontrivial. By the strong maximum principle, we further obtain that u is positive in Ω. Hence, the proof is complete. □

Proof of Theorem 1.3. From (A6) and (A7), for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$
|f(t)| \leq \varepsilon|t|^{p-1} + C_\varepsilon|t|^{p^*-1}.
$$

Arguing by contradiction, assume that the problem (1.1) has a nontrivial solution $u\in H$, one has

$$
(a+\lambda M(\|u\|^p))\|u\|^p = a\|u\|^p + \lambda\|u\|^{p(\tau+1)} \leq \frac{\varepsilon}{p}\|u\|^p + C_\varepsilon\|u\|^{p^*}. \tag{3.9}
$$

By $p(\tau+1)/\tau < N$, we deduce that $p^* < p(\tau+1)$. Choosing $\varepsilon = a/2$, it follows from (3.9) and the Young inequality that

$$
\begin{aligned}
&\frac{a}{p}\|u\|^p + \lambda\|u\|^{p(\tau+1)} \\
&\le C_\varepsilon\|u\|^{p^*} \\
&= \Big(\frac{a\tau}{p(\tau+1)-p^*}\Big)^{\frac{p(\tau+1)-p^*}{p\tau}} \|u\|^{\frac{p(\tau+1)-p^*}{\tau}} \Big(\frac{p(\tau+1)-p^*}{a\tau}\Big)^{\frac{p(\tau+1)-p^*}{p\tau}} \\
&\quad \times C_\varepsilon\|u\|^{\frac{(p^*-p)(\tau+1)}{\tau}} \\
&\le \frac{a}{p}\|u\|^p + \frac{p^*-p}{p\tau}\Big(\frac{p(\tau+1)-p^*}{a\tau}\Big)^{\frac{p(\tau+1)-p^*}{p^*-p}} C_\varepsilon^{\frac{p\tau}{p^*-p}}\|u\|^{p(\tau+1)}.
\end{aligned} \tag{3.10}
$$

Define

$$
\Theta := \frac{p^*-p}{p\tau}\Big(\frac{p(\tau+1)-p^*}{a\tau}\Big)^{\frac{p(\tau+1)-p^*}{p^*-p}} C_\varepsilon^{\frac{p\tau}{p^*-p}}.
$$

Consequently, for any $\lambda > \Theta$, the problem (1.1) has no nontrivial solution. $\square$

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References

[1] A. Arosio, S. Panizzi; *On the well-posedness of the Kirchhoff string*, Trans. Amer. Math. Soc., **348** (1996), 305-330.

[2] G. Autuori, F. Colasuonno, P. Pucci; *On the existence of stationary solutions for higher-order p-Kirchhoff problems*, Commun. Contemp. Math., **16** (2014), 1450002, pp. 43.

[3] S. Bernstein; *Sur une classe déquations fonctionelles aux dérivées partielles*, Bull. Acad. Sci. URSS. Sér., **4** (1940), 17-26.

[4] C. S. Chen, H. X. Song, Z. H. Xiu; *Multiple solutions for p-Kirchhoff equations in $\mathbb{R}^N$*, Nonlinear Anal., **86** (2013), 146-156.

[5] C. S. Chen, Q. Zhu; *Existence of positive solutions to p-Kirchhoff-type problem without compactness conditions*, Appl. Math. Lett., **28** (2014), 82-87.

[6] D. M. Cao, S. J. Peng, Q. F. Wang; *Pohozaev identities and their applications to nonlinear elliptic equations*, Sci. Sin. Math., **46** (2016), 1649-1674 (in Chinese), doi:10.1360/N012016-00113.

[7] F. J. S. A. Corrêa, G. M. Figueiredo; *On an elliptic equation of p-Kirchhoff type via variational methods*, Bull. Aust. Math. Soc., **74** (2006), 263-277.

[8] F. Colasuonno, P. Pucci; *Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations*, Nonlinear Anal., **74** (2011), 5962-5974.

[9] J. H. Chen, X. H. Tang, Z. Gao; *Existence of ground state sign-changing solutions for p-Laplacian equations of Kirchhoff type*, Math. Meth. Appl. Sci., **40** (2017), 5056-5067.

[10] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano; *Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation*, Adv. Differential Equations, **6** (2001), 701-730.

[11] S. T. Chen, X. H. Tang; *Infinitely many solution for super-quadratic Kirchhoff-type equations with sign-changing potential*, Appl. Math. Lett., **67** (2017), 40-45.

[12] S. T. Chen, X. H. Tang; *Ground state solutions for asymptotically periodic Kirchhoff-type equations with asymptotically cubic or super-cubic nonlinearities*, Mediterr. J. Math., **14** (2017), 209.

[13] X. Y. Cheng, G. W. Dai; *Positive solutions for p-Kirchhoff type problems on $\mathbb{R}^N$*, Math. Meth. Appl. Sci., **38** (2015), 2650-2662.

[14] D. G. De Figueiredo, P. L. Lions, R. D. Nussbaum; *A priori esitimates and existence of positive solutions of semilinear elliptic equations*, J. Math. Pures Appl., **61** (1982), 41-63.
[15] D. G. de Figueiredo, M. Girardim, M. Matzeu; *Semilinear elliptic equations with dependence on the gradient via mountain-pass techniques*, Differential Integral Equations, **17** (2004), 119-126.
[16] A. Hamydy, M. Massar, N. Tsouli; *Existence of solutions for p-Kirchhoff type problems with critical exponent*, Electron. J. Differential Equations, **2011**, No. 105, (2011), 1-8.
[17] J. C. Huang, C. S. Chen, Z. H. Xiu; *Existence and multiplicity results for a p-Kirchhoff equation with a concave-convex term*, Appl. Math. Lett., **26** (2013), 1070-1075.
[18] L. Jeanjean; *Local conditiond insuring bifurcation from the continuous spectrum*, Math. Z., **232** (1999), 651-664.
[19] G. Kirchhoff; *Mechanik*, Teubner, Leipzig, 1883.
[20] J. L. Lions; *On some questions in boundary value problems of mathematical physics, Contemporary Development in Continuum Mechanics and Partial Differential Equations*, in: North-Holland Math. Stud., **30**, North-Holland, Amsterdam, New York, 1978, pp. 284-346.
[21] J. Liu, J. F. Liao, C. L. Tang; *Positive solution for the Kirchhoff-type equations involving general subcritical growth*, Commun. Pure Appl. Anal., **15** (2016), 445-455.
[22] Y. H. Li, F. Y. Li, J. P. Shi; *Existence of a positive solution to Kirchhoff type problems without compactness conditions*, J. Differential Equations, **253** (2012), 2285-2294.
[23] A. Ourraoui; *On a p-Kirchhoff problem involving a critical nonlinearity*, C. R. Math. Acad. Sci. Paris, **352** (2014), 295-298.
[24] P. Pucci, R. Servadei; *Existence, non-existence and regularity of radial ground states for p-Laplacian equations with singular weights*, Ann. Inst. H. Poincaré Anal. Non Linéaire, **25** (2008), 505-537.
[25] S. I. Pohožaev; *On a class of quasilinear hyperbolic equations*. Mat. Sb. (N.S.), **96** (1975), 152-166 (in Russian).
[26] D. D. Qin, Y. B. He, X. H. Tang; *Ground state solutions for Kirchhoff type equations with asymptotically 4-linear nonlinearity*, Comput. Math. Appl., **71** (2016), 1524-1536.
[27] M. Struwe; *Variational methods: applications to nonlinear partial differential equations and Hamiltonian systems*, Ergeb. Math. Grenzgeb. (3), **34** (1996), 1344-1360.
[28] M. Struwe; *Variational Methods, Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, 3rd ed., Springer-Verlag, NewYork, 2000.
[29] X. H. Tang, B. T. Cheng; *Ground state sign-changing solution for Kirchhoff type problems in bounded domains*, J. Differential Equations, **261** (2016), 2384-2402.
[30] X. H. Tang, S. T. Chen; *Ground state solutions of Nehari-Pohozaev type for Kirchhoff-type problems with general potentials*, Calc. Var. Partial Differential Equations, **56** (2017), 110.
[31] J. Zhang, X. H. Tang, W. Zhang; *Existence of multiple solutions of Kirchhoff type equation with sign-changing potential*, Appl. Math. Comput., **242** (2014), 491-499.
[32] Q. G. Zhang, H. R. Sun, J. J. Nieto; *Positive solution for a superlinear Kirchhoff type problem with a parameter*, Nonlinear Anal., **95** (2014), 333-338.

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