OSCILLATION OF FUNCTIONAL DIFFERENTIAL EQUATIONS
OF N-TH ORDER WITH DISTRIBUTED DEVIATING
ARGUMENTS

SESHADEV PADHI, JULIO G. DIX

Abstract. We establish conditions for the oscillation and asymptotic behavior of non-oscillatory solutions of the following functional differential equation with distributed deviating arguments

$$y^{(n)}(t) + p(t)y^{(n-1)}(t) + \int_a^b q(t, \xi)f(y(t), y(t - \tau(t, \xi))) d\sigma(\xi) = 0,$$

We find explicit sufficient conditions for the oscillation as lower bounds for moments of the integral kernel $q$.

1. Introduction

This article concerns the oscillation of solutions to the $n$-th order delay differential equation

$$y^{(n)}(t) + p(t)y^{(n-1)}(t) + \int_a^b q(t, \xi)f(y(t), y(t - \tau(t, \xi))) d\sigma(\xi) = 0, \quad (1.1)$$

where $n \geq 2$, $f$, $p$, $q$, $\tau$ are given continuous functions, and for the Stieltjes integral $\sigma$ is nondecreasing.

Oscillation for $n$-order delay differential equations has been the subject of many publications; see for example the books [1, 6, 9, 12, 14] and references cited therein. The equation $y^{(n)}(t) + q(t)y(t - a) = 0$ has been the starting point for the study of linear and nonlinear delay equations of various types; see for example the articles [2, 4, 7, 10, 13, 15]. However, very few articles have been published on the oscillation for nonlinear equations of the form (1.1). Wang and Ge [15] obtained criteria for the oscillation of solutions to (1.1) with retarded argument $g(t, \xi)$ instead of $t - \tau(t, \xi)$. They assume that $n$ is even, hypothesis (A1) below, and that $f$ has partial derivatives bounded below by positive constants.

Our objective is obtaining verifiable conditions for the oscillation of solutions under hypotheses weaker than those in [15], and valid for $n$ odd and for $n$ even. The basic idea is to find estimates for a non-oscillatory solution $y$ in terms of its $\ell$-th derivative. Then estimate this function in terms of the $n$-derivative. Then use (1.1) to estimate the $n$-derivative in terms of $y$. This completes a cycle of estimates and provides our criteria for oscillation and non-oscillation stated in terms of lower bounds on the moments of the integral kernel $q$.

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We consider only nonconstant solutions, which we assume exist on some interval \([t_0, \infty)\). So that we can define the following terms:

A solution is called \textit{oscillatory} if it has zeros of arbitrarily large value; otherwise the solution is called non-oscillatory.

A function is called \textit{eventually positive} if there exist a value \(t_0\) such that \(x(t) > 0\) for all \(t \geq t_0\). Similarly, we define eventually negative, eventually non-negative, and eventually zero.

In this article we assume the following:

(A1) The integral kernel \(q\) is non-negative and not identically zero on any set \([t_0, +\infty) \times [a, b]\). The coefficient \(p\) is continuous, non-negative, and satisfies condition (2.1) below.

(A2) The delay \(\tau\) is a continuous function such that \(\tau_0(t) \leq \tau(t, \xi) \leq \tau_1(t)\) for all \(\xi\) in \([a, b]\);

(A3) The (nonlinear) function \(f\) is continuous in both variables having \(f(0, 0) = 0\), and a positive real number \(f_1\) such that

\[
\begin{align*}
    f(x, y) &\geq f_1 y \quad \text{when } x \text{ and } y \text{ are positive} \\
    f(x, y) &\leq f_1 y \quad \text{when } x \text{ and } y \text{ are negative}.
\end{align*}
\]

Note that (A3) is weaker than the hypothesis \(\partial u_i f \geq c_i > 0\) stated in [15, (H6)]. In fact, the function \(f(x, y) = x/(1 + x^2) + y\) satisfies (A3) but not [15, (H6)]. On the other hand by setting \(f_1 = \min\{c_1, c_2\}\), assumption (A3) is satisfied.

For short notation we define the function

\[
Q(t) = \int_{a}^{b} q(t, \xi) d\sigma(\xi).
\]

2. \textbf{Main Results}

First, we state some lemmas to be used in this article.

\textbf{Lemma 2.1.} If \(y\) is a solution of (1.1) which is positive on some interval \([t_0, +\infty)\) and

\[
\int_{t_0}^{\infty} \exp \left( - \int_{t_0}^{s} p(s) \, ds \right) \, ds = \infty, \tag{2.1}
\]

then \(y^{(n-1)}\) is eventually positive. Similarly if \(y\) is negative, \(y^{(n-1)}\) is eventually negative.

\textbf{Proof.} A lemma similar to this is proved in Kartsatos [11]; however, we include a proof for completeness and to use some of its arguments in other proofs. We show only the case when \(y(t) > 0\); the other case is similar. Select a large \(t_0\) so that \(y(t) > 0, y(t - \tau_1) > 0, Q(t) > 0\) for all \(t \geq t_0\). Then multiply (1.1) by the integrating factor \(r(t) = \exp(\int_{t_0}^{t} p(s) \, ds)\) which is positive. Using (A3), so that the integral in (1.1) is positive, we have

\[
(r(t)y^{(n-1)}(t))' < 0.
\]

Since \(ry^{(n-1)}\) is decreasing, there are only two possibilities: When \(ry^{(n-1)}\) is positive for all large \(t\), \(y^{(n-1)}\) is eventually positive and the proof is complete. When \(ry^{(n-1)}\) is negative for all large \(t\), we argue as follows. Since \(ry^{(n-1)}\) is decreasing and \(r\) is positive,

\[
y^{(n-1)}(t) \leq r(t_0)y^{(n-1)}(t_0) \frac{1}{r(t)}.
\]
Then integrating,
\[ y^{(n-2)}(t) \leq y^{(n-2)}(t_0) + r(t_0)g^{(n-1)}(t_0) \int_{t_0}^t \frac{1}{r(s)} \, ds. \]

Due to condition (2.1), the integral approaches \( \infty \), and \( y^{(n-2)} \) is eventually negative. Once \( y^{(n-1)} < 0 \) and \( y^{(n-2)} < 0 \) the claim below leads to \( y^{(n-3)} < 0 \). Then repeating this argument, we arrive to \( y < 0 \) which is a contradiction that completes the proof.

Claim: If \( v'(t) < 0 \), \( v''(t) \leq 0 \) for all \( t \) large, then there exist a \( t_1 \) such that \( v(t) < 0 \) for all \( t \geq t_1 \); i.e., \( v \) is eventually negative. To show this claim use the Taylor expansion at a large value \( t_0 \), with remainder of order 2. Then write the inequality without the reminder and take the limit as \( t \to \infty \).

**Lemma 2.2.** Let \( y(t) \) be an eventually positive function which is \( n \) times continuous differentiable, and \( y^{(n)} \) non-positive and not eventually zero. Then each derivative of \( y \) is eventually positive or eventually negative. Also there exists a time \( t_1 \) and an integer \( \ell \), \( 0 \leq \ell \leq n-1 \), such that \( n+\ell \) is odd and for all \( t \geq t_1 \),
\[ y^{(i)}(t) > 0 \quad \text{for } i = 0, 1, 2, \ldots, \ell, \]
\[ (-1)^{i+1}y^{(i)}(t) > 0 \quad \text{for } i = \ell + 1, \ldots, n. \]  

**Sketch of the proof.** The first statement follows from the fact that if \( v' \) is eventually positive or eventually negative, then \( v \) can not have more than one zero; therefore, \( v \) is eventually positive of eventually negative. As stated in [9, Lemma 1.1], the second result follows from the fact that if \( v'' \) and \( v' \) are eventually positive, then \( v \) is eventually positive. See also [12, Lemma 5.2.1]

For \( y \) eventually negative and \( y^{(n)} \) non-negative, similar results hold with the inequality reversed in (2.2).

As a consequence of the above lemma, the set \( N \) of non-oscillatory solutions to (1.1) can be decomposed as the union of disjoint sets \( N_\ell \) consisting of non-oscillatory solutions satisfying (2.1). This is,
\[ N = \begin{cases} 
N_1 \cup N_2 \cup \cdots \cup N_{n-1}, & \text{if } n \text{ is even}, \\
N_0 \cup N_2 \cup \cdots \cup N_{n-1}, & \text{if } n \text{ is odd}.
\end{cases} \]

The following lemma is part of [9, Lemma 1.3] and its results are obtained using repeated integration by parts.

**Lemma 2.3.** Under the conditions of Lemma 2.2, for \( t \geq t_1 \), the following inequalities hold:
\[ \int_{t_0}^\infty s^{n-1-\ell}|y^{(n)}(s)| \, ds < \infty, \]  
\[ y(t) \geq y(t_1) + \frac{1}{(\ell-1)!} \int_{t_1}^t (t-s)^{\ell-1}y^{(\ell)}(s) \, ds, \]  
\[ y^{(\ell)}(t) \geq \frac{1}{(n-\ell-1)!} \int_t^\infty (s-t)^{n-1-\ell} (y^{(n)}(s)) \, ds. \]

We remark that inequalities similar the ones above can be stated for eventually negative functions.
Lemma 2.4. Assume (A1)-(A3) and that there exist a constant $0 < \beta \leq 1$ such that $\beta \leq (t - \tau_1(t))^{-1}$. Also assume that for each $l$ in $\{1, 2, 3, \ldots, n - 1\}$ with $n + l$ odd,

\[
\limsup_{t \to \infty} (t - t_1)^l \int_{t^{1/\beta}}^{\infty} (s - t)^{n-l-1} Q(s) \, ds > \frac{l!(n-l-1)!}{f_1} \tag{2.6}
\]

If there exists a non-oscillatory solution of (1.1), then it belongs to $N_0$; i.e. $\ell = 0$ in (2.2).

**Proof.** Let $y(t)$ be a non-oscillatory solution of (1.1). This proof is done only for $y$ eventually positive, since the negative case is similar. If necessary, increase the reminder of order $\ell$ for all $t \geq t_1$. Now, we claim that $0 < \ell \leq n - 1$. Since $y^{(\ell+1)}$ is negative, $y^{(l)}$ is decreasing and by (2.4),

\[
y(t) \geq y^{(l)}(t) \frac{1}{(l-1)!} \int_{t_1}^{t} (t - s)^{l-1} \, ds = \frac{y^{(l)}(t)(t - t_1)^l}{l!} \tag{2.7}
\]

Note that this inequality also follows from the Taylor expansion of $y$ about $t_1$ with reminder of order $\ell$. Using (2.5),

\[
y(t) \geq (t - t_1)^l \frac{1}{(l-1)!} \int_{t_1}^{\infty} (s - t)^{n-l-1} (-y^{(n)}(s)) \, ds. \tag{2.8}
\]

From (1.1), (A2), (A3), and that $y$ is increasing, we obtain

\[
-y^{(n)}(s) \geq \int_{a}^{b} q(s, \xi) f(y(s), y(s - \tau(s, \xi))) \, d\xi \\
\geq \int_{a}^{b} q(s, \xi) f_1 y(s - \tau(s, \xi)) \, d\xi \\
\geq f_1 y(s - \tau_1(s)) Q(s). \tag{2.9}
\]

Recall that $(s - \tau_1(s))' \geq \beta > 0$ and that $t - \tau_1(t) > 0$. For $s \geq t + t/\beta$, we have

\[
t \leq (s - t)\beta = \int_{t}^{s} \beta \, dw \\
\leq \int_{t}^{s} (w - \tau_1(w))' \, dw \\
\leq t - \tau_1(t) + \int_{t}^{s} (w - \tau_1(w))' \, dw = s - \tau_1(s).
\]

Since $y$ is increasing, $y(t) \leq y(s - \tau_1(s))$. Then the inequality $\int_{t}^{s} \leq \int_{t+t/\beta}$, (2.8) and (2.9) imply

\[
y(t) \geq (t - t_1)^l f_1 y(t) \frac{1}{l!(n-l-1)!} \int_{t^{1/\beta}}^{\infty} (s - t)^{n-l-1} Q(s) \, ds
\]

which contradicts (2.6) and completes the proof. $\square$

At this point, we have the result obtained in [15]: By lemma 2.3 the existence of a non-oscillatory solution requires $\ell = 0$. However, when $n$ is even, $\ell$ can not be zero because $n + \ell$ is odd. Therefore, all solutions of (1.1) with $n$ even are oscillatory.
**Remark 2.5.** The conditions of Lemma 2.4 are satisfied if for example
\[
Q(s) \geq (s - t)^{-n} \frac{l((n - 1 - t)!2l^2)}{f_1 \beta^l}
\]
for \( s \geq t + t/\beta \). This is, \( Q(s) \) does not decay faster than \( s^{-n} \). Note that for a fixed \( t \) and a function \( G(s) \geq (s - t)^{-n} \), we have \((s - t)^{n-1-l}G(s) \geq (s - t)^{-1-l}\) and
\[
\int_{t+1/\beta}^{\infty} (s - t)^{n-1-l}G(s) \, ds \geq \int_{t+1/\beta}^{\infty} (s - t)^{-1-l} \, ds = \frac{\beta^l}{l!}.
\]
Multiplying \( G(s) \) by \( l((n - 1 - t)!2l^2) \) and using that \( \lim_{t \to \infty} (t - t_1)/t = 1 \), we obtain an inequality that implies condition (2.6).

**Lemma 2.6.** Assume \((A2)-(A3)\) and that \( \tau_0(t) \) in \((A2)\) is non-decreasing and positive for \( t \) large. Also assume that there exists a positive constant \( b \) such that
\[
\limsup_{t \to \infty} \int_{t}^{t+\tau_0(t)+b} Q(s) \, ds > \frac{(n - 1)!3}{2f_1}.
\]
Then non-oscillatory solutions of \((1.1)\) cannot belong to \( N_0 \); i.e., \( \ell > 0 \) in \((2.2)\).

**Proof.** Let \( y(t) \) be a non-oscillatory solution of \((1.1)\). Without loss of generality, we assume that \( y \) is eventually positive; the negative case is proved similarly. If necessary, increase the value of \( t_1 \) in Lemma 2.2 so that \( y(t) > 0 \), \( y^{(n)}(t) < 0 \), \( \tau_0(t) > 0 \), and \( \tau_0 \) is increasing, for all \( t \geq t_1 \). Now, we claim that \( 0 < \ell \leq n - 1 \) in \((2.2)\). On the contrary assume that \( \ell = 0 \). Then \( y^{(1)} \) is non-decreasing, we have
\[
-y^{(n)}(s) \geq \int_{a}^{b} q(s, \xi) f(y(s), y(s - \tau(s, \xi))) \, d\xi
\]
\[
\geq \int_{a}^{b} q(s, \xi) f_1 y(s - \tau(s, \xi)) \, d\xi
\]
\[
\geq f_1 y(s - \tau_0(s)) Q(s).
\]
From \((2.5)\) and the inequality \( \int_{t}^{\infty} \geq f_1^{t+\tau_0(t)+b} \),
\[
y(t) \geq \frac{1}{(n - 1)!} \int_{t}^{\infty} (s - t)^{n-1} \left( - y^{(n)}(s) \right) \, ds
\]
\[
\geq \frac{f_1}{(n - 1)!} \int_{t}^{t+\tau_0(t)+b} (s - t)^{n-1} y(s - \tau_0(s)) Q(s) \, ds
\]
\[
\geq \frac{f_1}{(n - 1)!} y(t + b) \int_{t}^{t+\tau_0(t)+b} (s - t)^{n-1} Q(s) \, ds
\]
Therefore,
\[
\frac{y(t)}{y(t + b)} \geq \frac{f_1}{(n - 1)!} \int_{t}^{t+\tau_0(t)+b} Q(s) \, ds
\]
Since \( y \) is positive and decreasing, the \( \lim_{t \to \infty} y(t) \) exists. If this limit is non-zero, then \( \lim_{t \to \infty} y(t)/y(t + b) = 1 \). On the other hand if this limit is zero, we apply L’Hopital’s Rule, that \( y \) is decreasing while \( y’ \) is increasing to obtain
\[
1 \leq \lim_{t \to \infty} \frac{y(t)}{y(t + b)} = \lim_{t \to \infty} \frac{y’(t)}{y’(t + b)} \leq 1.
\]
Thus there exists a $t_2$ such that $\frac{1}{2} < \frac{y(t)}{y(t+b)} < \frac{3}{2}$ and

$$\frac{3f_1}{(n-1)!2} \geq \int_t^{t+\tau_0(t)+b} (s-t)^{n-1}Q(s)\,ds$$

for $t \geq t_2$. This contradicts (2.10) and completes the proof. \hfill \Box

We remark that if there exist a positive constant $b$ such that

$$Q(s) \geq \frac{3n!}{\tau_0(t+b)2f_1}$$

for all $s \geq t$, then condition (2.10) is satisfied. This is, $Q$ does not decay faster than $1/\tau_0$. Certainly this condition is stronger than the condition in Remark 2.5.

We are ready to state the main result of this paper.

**Theorem 2.7.** Under the assumptions of Lemmas 2.3 and 2.4, all solutions of (1.1) are oscillatory.

**Proof.** By Lemma 2.3 the existence of a non-oscillatory solution requires $\ell = 0$. However, by Lemma 2.4, $\ell$ can not be zero. Therefore, all solutions of (1.1) are oscillatory. \hfill \Box

**Theorem 2.8.** Assume (A1)-(A3) and that $\tau_0(t)$ in (A2) is non decreasing and positive for $t$ large. Also assume that for some value $t_0$,

$$\int_{t_0}^{\infty} (s-t_0)^{n-1}Q(s)\,ds = \infty. \quad (2.12)$$

If $y$ is a non-oscillatory solution of (1.1) belonging to $N_0$, then $\lim_{t \to \infty} y(t) = 0$.

**Proof.** Certainly the limit exits because $y$ is positive and decreasing. Now if $\lim_{t \to \infty} y(t) = y_1 \neq 0$, then there exists a value $t_2$ such that $y_1/2 < y(t) < 3y_1/2$ for all $t \geq t_2$. From (2.5), (2.11), and this inequality, we have

$$\frac{3}{2}y_1 \geq y(t) \geq \frac{f_1}{(n-1)!2} \int_t^{\infty} (s-t)^{n-1}y(s-\tau_0(s))Q(s)\,ds \geq \frac{f_1y_1}{(n-1)!2} \int_t^{\infty} (s-t)^{n-1}Q(s)$$

Note that the integral in (2.12) is an increasing function of $t_0$ (its derivative is negative). Then the above inequality contradicts (2.12), and hence $\lim_{t \to \infty} y(t) = 0$. \hfill \Box

### 3. Same oscillation results under other conditions

In this section, we find alternate conditions for Lemma 2.4, while using Lemma 2.6 to obtain the results of Theorem 2.7. First we present an auxiliary lemma.

**Lemma 3.1.** Under the conditions of Lemma 2.2, we have

$$y^{(\ell)}(t) \geq \frac{1}{(n-1-\ell)!} \prod_{i=1}^{n-1-\ell} (2^i-1)_{\ell} (n-1)_{\ell}^{(n-1-\ell)}(2^{n-1-\ell}-t), \quad (3.1)$$

where the product with no factors, $\prod_{i=1}^{0}$, is understood as 1.
Proof. If \( \ell = n - 1 \) the inequality (3.1) becomes equality and is true. Note that \( \ell \) can not be \( n - 2 \), because by Lemma 2.2, \( n + \ell \) is odd. Therefore, the proof is need only for \( \ell \leq n - 3 \). The following steps are a refinement of the steps taken in the proof of [12, (5.2.3)]. Integrate \( y^{(n-1)} \) which is positive and decreasing, to obtain

\[
-y^{(n-2)}(t) = -y^{(n-2)}(2t) + \int_t^{2t} y^{(n-1)}(s) \, ds
\]

\[
\geq \int_t^{2t} y^{(n-1)}(s) \, ds \geq (2^1 - 1)t y^{(n-1)}(2t).
\]

Integrating \( y^{(n-2)} \) and using that \( y^{(n-1)} \) is decreasing, the above inequality yields

\[
y^{(n-3)}(t) = y^{(n-3)}(2t) - \int_t^{2t} y^{(n-2)}(s) \, ds
\]

\[
\geq \int_t^{2t} sy^{(n-1)}(2s) \, ds
\]

\[
\geq y^{(n-1)}(2^2t) \int_t^{2t} s \, ds = \frac{1}{2!}(2^2 - 1)t^2 y^{(n-1)}(2^2t)
\]

Similarly,

\[
-y^{(n-4)}(t) \geq \frac{1}{3!}(2^3 - 1)(2^2 - 1)(2^2 - 1)t^3 y^{(n-1)}(2^3t),
\]

\[
y^{(n-5)}(t) \geq \frac{1}{4!}(2^4 - 1)(2^3 - 1)(2^3 - 1)t^4 y^{(n-1)}(2^4t).
\]

After \( n - 2 - \ell \) steps we have expression (3.1).

Lemma 3.2. Assume (A1)-(A3) and that there exist a constant \( 0 < \beta \leq 1 \) such that \( \beta \leq (t - \tau_1(t))^{\ell} \). Also assume that for each \( \ell \) in \( \{1, 2, 3, \ldots, n - 1\} \) with \( n + \ell \) odd,

\[
\limsup_{t \to \infty} \int_{t_1}^{t} (t - s)^{l-1}s^{n-1-l} \int_{2^{n-1-l}((s+s)/\beta)}^{\infty} Q(u, du, dt > \frac{(l-1)!(n-1-l)!}{\prod_{i=1}^{n-1-l}(2^i - 1)} Q_1.
\]

(3.2)

If there exists non-oscillatory solution of (1.1), then it belongs to \( N_0 \); i.e. \( \ell = 0 \).

Proof. Let \( y(t) \) be a non-oscillatory solution of (1.1). Without loss of generality, we assume that \( y \) is eventually positive. The negative case is proved similarly. If necessary, increase the value of \( t_1 \) in Lemma 2.2 so that \( y(t) > 0, y^{(n-1)}(t) > 0, t - \tau_1(t) > 0, y(t - \tau_1) > 0 \), for all \( t \geq t_1 \). Now, we claim that \( \ell = 0 \) in (2.2). On the contrary assume that \( 0 < \ell \leq n - 1 \). Integrating \( y^{(n-1)} \) from \( t \) to \( \infty \) and using that \( y^{(n-1)} > 0 \),

\[
y^{(n-1)}(t) = \int_t^{\infty} (-y^{(n)}(s)) \, ds.
\]

Then by (3.1),

\[
y^{(\ell)}(t) \geq \frac{1}{(n-1-\ell)!} \prod_{i=1}^{n-1-\ell} (2^i - 1) t^{n-1-\ell} \int_{2^{n-1-\ell}t}^{\infty} \left( -y^{(n)}(s) \right) \, ds.
\]

By (2.9) and using that \( y^{(2n-1-\ell)} \geq y(t) \), because \( y \) is increasing, we have

\[
y^{(\ell)}(t) \geq \frac{1}{(n-1-\ell)!} \prod_{i=1}^{n-1-\ell} (2^i - 1) t^{n-1-\ell} y(t) \int_{2^{n-1-\ell}(t+t/\beta)}^{\infty} Q(s) \, ds.
\]
Using (2.4),
\[1 \geq \frac{f_1 \prod_{\ell=1}^{n-1} (2\ell - 1)}{(n - 1 - \ell)!}(\ell - 1)! \int_{t_1}^{t} (t - s)^{\ell - 1} s^{n - 1 - \ell} \int_{2\ell - 1 \cdot (s + s/\beta)}^{\infty} Q(u) \, du \, dt\]
which contradicts (3.2) and completes the proof. \(\square\)

The condition of Lemma 3.2 is met if for example \(Q(s) \geq \text{constant}/s^n\). To show this statement, we replace the lower limit of the inner integral by \(t\). Then estimate the outer integral by a quadrature using the critical point of the integrand and the two limits of integration.

**Lemma 3.3.** Let \(r\) be a continuous non-negative function and \(w\) be a positive, twice continuous differentiable function that satisfies
\[w''(t) + r(t)w(t) \leq 0 \quad \text{for } t \geq t_1.\]
Then the differential equation
\[x''(t) + r(t)x(t) = 0\]
does not have oscillatory solutions.

The above lemma is stated as [5, Lemma 3], also it is used in [3, p. 240]. One way to prove this lemma is by defining a sequence of functions: \(x_1(t) = w(t_1)\), and
\[x_{i+1}(t) \geq w(t_1) + \int_{t_1}^{t} \int_{s}^{\infty} r(\xi) x_i(\xi) \, d\xi \, ds.\]
that converges to a non-oscillatory solution of (3.4). Then by the Sturm separation theorem none of the solutions to (3.4) is oscillatory.

**Lemma 3.4.** Assume (A1)-(A3) and that there exist a constant \(0 < \beta \leq 1\) such that \(\beta \leq (t - \tau(t))^\ell\). Also assume that for all \(\ell \in \{1, 2, \ldots, n - 1\}\) the differential equation
\[x''(t) + \frac{f_1 t^{\ell - 1}}{(n - 2 - \ell)!}(\ell - 1)! \int_{t + 2\ell/\beta}^{\infty} (s - t)^{n - 2 - \ell} Q(s) \, ds \, x(t) = 0\]
has oscillatory solutions. If there exists a non-oscillatory solution of (1.1), then it belongs to \(N_0\); i.e. \(\ell = 0\).

**Proof.** Let \(y(t)\) be a non-oscillatory solution of (1.1). The proof is done only for \(y\) is eventually positive, since the negative case is similar. We will find the following estimates: \(y^{(\ell+1)}\) in terms of \(y^{(n)}\), \(y^{(n)}\) in terms of \(y\), and \(y\) in terms of \(y^{(\ell-1)}\). If necessary, increase the value of \(t_1\) in Lemma 2.2 so that \(y(t) > 0\), \(y^{(n-1)}(t) > 0\), \(t - \tau(t) > 0\), \(y(t - \tau) > 0\), for all \(t \geq t_1\). Now, we claim that \(\ell = 0\) in (2.2). On the contrary assume that \(0 < \ell \leq n - 1\). The Taylor expansion of \(y^{(\ell+1)}(t)\) about a point \(s > t\), with reminder of order \(n\), gives
\[y^{(\ell+1)}(t) \leq \frac{1}{(n - 2 - \ell)!} \int_{s}^{t} (t - s)^{n - 2 - \ell} y^{(n)}(s) \, ds = \frac{1}{(n - 2 - \ell)!} \int_{s}^{t} (s - t)^{n - 2 - \ell} y^{(n)}(s) \, ds,\]
where we used that \( n - 2 - \ell \) is odd. In the limit as \( s \to \infty \), and using (2.9),

\[
-y^{(\ell+1)}(t) \geq \frac{1}{(n - 2 - \ell)!} \int_t^\infty (s-t)^{n-2-\ell} y^{(n)}(s) \, ds \\
\geq \frac{1}{(n - 2 - \ell)!} \int_t^\infty (s-t)^{n-2-\ell} f_1 y(s-\tau_1(s))Q(s) \, ds.
\]

From the condition \( \beta \leq (s-\tau_1(s))' \), it follows that \( t + 2t/\beta \leq s \) implies \( 2t \leq s-\tau_1(s) \).

Using the inequality \( \int_t^\infty \leq \int_t^{t+2t/\beta} \), and that \( y \) is increasing, we have

\[
-y^{(\ell+1)}(t) \geq \frac{1}{(n - 2 - \ell)!} f_1 y(2t) \int_{t+2t/\beta}^\infty (s-t)^{n-2-\ell} Q(s) \, ds.
\]

The Taylor expansion of \( y \) about the point \( t \), with (positive) reminder of order \( \ell \), gives

\[
y(2t) \geq y^{(\ell-1)}(t)t^{\ell-1} \frac{1}{(\ell-1)!}.
\]

Combining the two inequalities above,

\[
-y^{(\ell+1)}(t) \geq \frac{f_1 t^{\ell-1}}{(n - 2 - \ell)![(\ell-1)!]} \int_{t+2t/\beta}^\infty (s-t)^{n-2-\ell} Q(s) \, ds y^{(\ell-1)}(t).
\]

Setting \( w(t) = y^{(\ell-1)}(t) \), we have

\[
w''(t) + \frac{f_1 t^{\ell-1}}{(n - 2 - \ell)![(\ell-1)!]} \int_{t+2t/\beta}^\infty (s-t)^{n-2-\ell} Q(s) \, ds w(t) \leq 0.
\]

Since \( w > 0 \), by lemma 3.4, equation (3.5) can not have oscillatory solutions, which contradicts the assumption in this lemma. Therefore, \( \ell = 0 \) and proof is complete.

The condition of Lemma 3.4 is met if for example the coefficient \( r(t) \) of \( x(t) \) satisfies \( \lim_{t \to \infty} t^2 r(t) > 1/4 \); see [14, Chapter 2]. This is achieved if \( Q(s) \geq \text{constant}/(s-t)^n \); i.e., \( Q \) does not decay faster than \( 1/t^n \). Note that \( Q \) has the same order of magnitude as in the remarks after Lemmas 2.4 and 3.2.

We conclude this article by re-stating Theorem 2.7 with lemmas in this section.

**Theorem 3.5.** Under the assumptions of Lemma 2.6 and either Lemma 3.2 or Lemma 3.4, all solutions of (1.1) are oscillatory.

**References**


Seshadev Padhi  
Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762, USA.  
Permanent Address: Department of Applied Mathematics, Birla Institute of Technology, Ranchi-835 215, India  
E-mail address: ses_2312@yahoo.co.in

Julio G. Dix  
Department of Mathematics, Texas State University, San Marcos, TX 78666, USA  
E-mail address: jd01@txstate.edu