

BOUNDING THE ORDER OF A GROUP WITH A
LARGE CONJUGACY CLASS

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LARGE CONJUGACY CLASS

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1. INTRODUCTION

This thesis considers a problem in a branch of abstract algebra known as finite group theory. Finite group theory is concerned with the structure of finite sets of objects related to one another by a binary operator such as multiplication or addition. Often, these structures represent various forms of symmetry. In this thesis, we study a problem inspired by a series of publications.

In 2008, Noah Snyder [14] published a result on a group parameter e . He classified groups based on fixed values of e and bounded the order of a group in terms of e . His definition was this:

Definition Let G be a group of order n and V an irreducible representation of G over \mathbb{C} of dimension d . Define e to be the non-negative integer satisfying $n = d(d + e)$.

The parameter e must be an integer since d divides $|G|$ [7, p.96]. It has the interesting property that if e is small relative to d , then G has a character of large degree.

Before Snyder, Yakov Berkovich classified groups where $e = 1$ and $e = 2$ [1]. Snyder added to this classification by solving the case where $e = 3$. His most noteworthy contribution to this problem, though, is the following theorem:

Theorem 1.1 (Snyder [14]). *Let G be a group of order n and d be the degree of some irreducible character of G and $d(d+e) = n$. If $e > 1$, $n \leq ((2e)!)^2$.*

This sparked a series of publications aiming to improve upon the bound. The first article along this line of inquiry was written by Isaacs [9]. The relevant theorem from his article is as follows:

Theorem 1.2 (Isaacs [9]). *Let G be a group of order n and d be the degree of some irreducible character of G . If $e > 1$, then $n \leq Be^6$ for some universal constant B .*

This result required the simple group classification theorem for full generality, but succeeded in giving a polynomial bound. Interestingly, it spawned another paper [11] whose results were necessary to complete certain cases of Isaacs' proof.

Two students of Isaacs, Christina Durfee and Sara Jensen, were the next to make headway on the problem. They removed the universal constant from Isaacs' bound. Their main result was the following:

Theorem 1.3 (Durfee-Jensen [6]). *For $e > 1$, we have the following bounds on $|G|$ in terms of e :*

1. *If e is divisible by two distinct primes, then $|G| < e^4 + e^3$.*
2. *If e is a prime power then $|G| < e^6 - e^4$.*
3. *If e is a prime, then $|G| < e^4 + e^3$.*

Since e is an integer, this theorem has as an immediate corollary that $|G| < e^6 - e^4$ for all groups G . It is still a polynomial of degree 6, but is significantly smaller than Isaacs' bound.

Mark Lewis made the most recent advance. He gave general conditions [12] for a group to satisfy the stronger bound $|G| \leq e^4 - e^3$.

Theorem 1.4 (Lewis [12]). *Let G be a group with a nontrivial, abelian normal subgroup. Let d be the degree of some irreducible character of G and $|G| = d(d+e)$. If $e > 1$, then $d \leq e^2 - e$ and $|G| \leq e^4 - e^3$. This bound is best possible.*

Notably, all solvable groups satisfy the conditions of this theorem. Since there exist solvable groups where $|G| = e^4 - e^3$ (a result due to Isaacs [9]), this problem is completely solved for solvable groups.

A related problem was the subject of Durfee's dissertation [5]. She studied a parameter that is essentially Snyder's e relative to a fixed normal subgroup.

Definition Let N be a normal subgroup of a finite group G . Let χ and Θ be irreducible characters of G and N , respectively, such that Θ is fixed by the conjugation action of G and χ restricts to a multiple of Θ on N . Let $d = \chi(1)/\Theta(1)$. Define e by $|G/N| = d(d+e)$.

Durfee's parameter, like Snyder's, is always a non-negative integer. In her dissertation, she studies the case where $e = 1$ and $e = 2$ and gives consideration to supersolvable groups and nilpotent groups. Her main result, though, was the following:

Theorem 1.5 (Durfee [5]). *Let N be a normal subgroup of a finite group G where G/N is solvable and let Θ be an irreducible character of N that is G -invariant. Let χ be an irreducible character of G that is a multiple of Θ and let $d = \chi(1)/\Theta(1)$. Write $|G : N| = d(d + e)$ for some non-negative integer e . If $e \geq 1$ and $d > e^5 - e$, then we can find groups X and Y such that:*

1. $N \subseteq X \triangleleft Y \subseteq G$
2. $|Y/X| = (d/e)(d/e + 1)$
3. Y/X is either the group of order 2 or is a 2-transitive Frobenius group.

In light of the aforementioned research, a group theoretic analog is studied in this thesis. There is a close connection between characters and conjugacy classes that often motivates a problem for one based off a result of the other. Sometimes, the theorems produced from this line of research are very similiar. A survey of such results can be found in [4].

Our approach is to preserve the following identity:

$$|G| = \sum_{i=1}^n d_i^2 \tag{1.1}$$

where d_i is the degree of an irreducible character and i indexes the n ordinary irreducible characters of G . This is a well-known identity that holds for all finite groups [7]. We thus want to replace the d_i 's with analogous values for conjugacy classes. Since the number of irreducible characters is the same as the number of conjugacy classes [7, p.96], this is a reasonable goal. Furthermore, the sizes of

conjugacy classes satisfy a similar identity:

$$|G| = \sum_{i=1}^n |x_i^G|$$

where there are n distinct conjugacy classes and the $x_i \in G$ are representatives of each of these conjugacy classes. The primary difference between the two identities is the presence of squared terms in the former. The simplest way to accommodate this is to consider the square root of conjugacy class sizes instead of just conjugacy class sizes:

$$|G| = \sum_{i=1}^n \left(\sqrt{|x_i^G|} \right)^2.$$

This maintains the form of (1.1) even though our terms are no longer rational. The main benefit of this change is that it makes the definition of an analogous e resemble the original very closely.

Taking d to be the square root of a conjugacy class size, we can define an analog e with Snyder's equation: $|G| = d(d + e)$. We still have that $d^2 + de$ is a sum of non-negative integers but, unfortunately, that is the extent of the parallel. This analog can take on irrational values. It does, however, have the property that small values of e relative to d correspond to large conjugacy class sizes (see Chapter 5). For non-trivial groups, this indicates that our e is an adequate analog. In this thesis, we bound the group order in terms of this analogous e and then classify groups attaining the bound.

We also consider a relative parameter in the spirit of Durfee's research. Let N be a fixed normal subgroup of G . Let $x \in G$ and $d = \sqrt{|x^G|/|x^N|}$. Define e via $|G/N| = d(d + e)$. Here we have the same issue as above: e can be irrational. In spite of this, there are still some worthwhile results for the analog. We will provide results for fixed values of e and discuss the groups associated with them.

The rest of this paper proceeds with chapters of background material and then

two chapters for our main results, one for Snyder's analog and the other for Durfee's analog. A solid understanding of group theory is assumed in the following although, where possible, we try to present all supporting results. When it is not feasible to provide a proof, a reference is given.

Our notation will follow closely that used in Isaacs' *Finite Group Theory* [10] and other notation will be introduced as needed.

2. DIHEDRAL GROUPS

Dihedral groups will play an important role in later results. While some familiarity with their structure is assumed, we will review their relevant properties.

We begin with our definition:

Definition A *dihedral group* is a group D having a cyclic subgroup C of index 2 such that $D - C$ contains only elements of order 2.

Remark Both the cyclic subgroup of order 2 and the Klein 4-group are dihedral. For the former, take $C = 1$. In the latter, all non-identity elements are involutions and any subgroup of order 2 will suffice for C .

Since $|D : C| = 2$, we know that $C \triangleleft D$ and D has even order. Dihedral groups also have two generators: There exists a $c \in C$ such that $\langle c \rangle = C$. We also have that $D - C$ is a coset of C . If $t \in D - C$, then $Ct = D - C$ and $\langle c, t \rangle = D$. Another important property of dihedral groups is that, for $c \in C$ and $t \in D - C$, conjugation of c by t yields the inverse of c , that is $c^t = c^{-1}$. Showing this is slightly more involved but still follows quickly from our definition. We know that $ct \in D - C$, so ct must be an involution. Then $(ct)^2 = ctct = 1$. Left multiplication by c^{-1} now gives the desired relation: $tct = c^t = c^{-1}$. This last property is often used as a

defining characteristic in the presentation of dihedral groups:

$$D = \langle c, t : c^n = 1, t^2 = 1, t^{-1}ct = c^{-1} \rangle$$

where $|D| = 2n$ for some $n \in \mathbb{Z}$. That presentation is the same as what we have defined above since $\langle c \rangle$ is a cyclic group and t cannot be in $\langle c \rangle$. Furthermore, $\langle c \rangle t$ contains only involutions because $(c^k t) \cdot (c^k t) = c^k \cdot (t^{-1} c^k t) = c^k c^{-k} = 1$.

Another way to view dihedral groups is as the semi-direct product of $C_n \rtimes_{\varphi} C_2$ where C_n denotes the cyclic group of order n and $\varphi : C_2 \rightarrow \text{Aut}(C_n)$ is the map that sends 1_{C_2} to the identity automorphism and C_2 's other element to the automorphism $c \mapsto c^{-1}$. From this view, we have some element $(c, 1) \in C_n \rtimes C_2$ that generates the subgroup identified with C_n and another element $(1, t) \in C_n \rtimes C_2$ that generates the subgroup identified with C_2 . Furthermore, $(1, t)$ has the desired conjugation action on $(c, 1)$, i.e. $(c, 1)^{(1, t)} = (c^{-1}, 1)$. Finally, we note that $(c, 1)$ and $(1, t)$ comprise a complete set of generators for $C_n \rtimes C_2$. Then

$$\begin{aligned} C_n \rtimes C_2 &= \langle (c, 1), (1, t) : (c, 1)^n = (1, 1), \\ &\quad (1, t)^2 = (1, 1), \\ &\quad (1, t^{-1})(c, 1)(1, t) = (c^{-1}, 1) \rangle \end{aligned}$$

for which there is an obvious isomorphism to $\langle c, t : c^n = 1, t^2 = 1, t^{-1}ct = c^{-1} \rangle$.

While not strictly necessary for our results, we will frequently make use of the fact that, up to isomorphism, there is exactly one dihedral group of a given order.

Proposition 2.1. *All dihedral groups of order $2n$ are isomorphic.*

Proof. Let D_0 and D be dihedral groups of order $2n$. We seek an isomorphism $\varphi : D_0 \rightarrow D$. Let $D_0 = \langle c_0, t_0 \rangle$ and $D = \langle c, t \rangle$ such that $|c_0| = |c| = n$ and

$|t| = |t_0| = 2$. Define $\varphi(c_0^k t_0^m) = c^k t^m$ for $0 \leq k < n$ and $0 \leq m < 2$. Then $\varphi(c_0^k) = c^k$ and $\varphi(t_0) = t$. We now show φ to be a homomorphism. There are 4 cases:

1. $\varphi(c_0^{k_1} c_0^{k_2}) = c^{k_1+k_2} = c^{k_1} c^{k_2} = \varphi(c_0^{k_1}) \varphi(c_0^{k_2})$.
2. $\varphi(c_0^{k_1} (c_0^{k_2} t_0)) = c^{k_1+k_2} t = c^{k_1} (c^{k_2} t) = \varphi(c_0^{k_1}) \varphi(c_0^{k_2} t_0)$.
3. $\varphi((c_0^{k_1} t_0) c_0^{k_2}) = c^{k_1-k_2} t = (c^{k_1} t) c^{k_2} = \varphi(c_0^{k_1} t_0) \varphi(c_0^{k_2})$.
4. $\varphi((c_0^{k_1} t_0) (c_0^{k_2} t_0)) = c^{k_1-k_2} = (c^{k_1} t) (c^{k_2} t) = \varphi(c_0^{k_1} t_0) \varphi(c_0^{k_2} t_0)$.

The above gives that φ respects the multiplication of D_0 and is, hence, a homomorphism. Because every $d \in D$ can be written $d = c^k t^m$ with $0 \leq k < n$, $0 \leq m < 2$ and that $\varphi(c_0^k t_0^m) = c^k t^m$, φ is surjective. To show injectivity, suppose $\varphi(c_0^k t_0^m) = \varphi(c_0^l t_0^r)$ for $0 \leq k, l < n$ and $0 \leq m, r < 2$. Then $c^k t^m = c^l t^r$ which implies that $l \equiv k \pmod{n}$ and $r \equiv m \pmod{2}$. By the bounds imposed on k , m , l , and r , we must have $k = l$ and $m = r$. Then $c_0^k t_0^m = c_0^l t_0^r$. We conclude that φ is an isomorphism and the proof is complete. \square

2.1 Conjugacy Classes

We now discuss the conjugacy class structure of dihedral groups. The number and sizes of conjugacy classes are completely determined by the order of a dihedral group. In the following propositions, let D be a dihedral group and $C \triangleleft D$ be cyclic of index 2.

Proposition 2.2. *If $c \in C$, then $c^D = \{c, c^{-1}\}$.*

Proof. Since C is abelian, $c^C = \{c\}$. Then we must only consider the action of the involutions in $D - C$. Let z be a generator of C and t be an involution in $D - C$.

Then every element of $D - C$ is of the form $z^k t$ for some non-negative integer k . For any $c \in C$:

$$tz^{-k}cz^k t = tct = c^{-1}.$$

Thus every involution of $D - C$ sends an element of C to its inverse via conjugation. It follows that $c^D = \{c, c^{-1}\}$ for all $c \in C$. \square

The remaining conjugacy classes have a different structure depending on the parity of the $|C|$. Note also that if $|C|$ is even, it contains an involution which must have a conjugacy class consisting only of itself. The following lemma is easy, but necessary for the next proposition. It is presented here for completeness.

Lemma 2.3. *Let n be an even integer and k be any integer. Then k and $k \bmod n$ have the same parity.*

Proof. Suppose $k \geq 0$ and let $k \bmod n = r$. Then $r = k - n \cdot m$ for some integer m . Immediately, r and k have the same parity since $n \cdot m$ must be even. If $k < 0$, then $r = -k - n \cdot m$ and the same logic gives that k and $k \bmod n$ have the same parity. \square

Proposition 2.4. *Let D have order $2n$, n even. Then D has 3 distinct conjugacy classes of involutions: two of size $n/2$ and one of size 1.*

Proof. Since C has even order n , it contains a unique involution which lies in its own conjugacy class by the Proposition 2.2. The remaining involutions are of the form $z^k t$ ($k \in \mathbb{Z}$) where $\langle z \rangle = C$ and $t \in D - C$. Consider the following:

$$\begin{aligned} z^{-m}(z^k t)z^m &= z^{-m}(z^k t)z^m t t = z^{k-2m} t \\ tz^{-m}(z^k t)z^m t &= tz^{k-m} t z^m t = z^{2m-k} t, \end{aligned}$$

with m an arbitrary integer. Note that k , $k - 2m$, $2m - k$ are all of the same parity. By Lemma 2.3, we must have that $k \bmod n$, $k - 2m \bmod n$, and $2m - k \bmod n$ share

the same parity as well. Then the conjugacy classes of involutions in $D - C$ are $\{z^{kt} : k \text{ is even}\}$ and $\{z^{kt} : k \text{ is odd}\}$. Each class must contain exactly half of the involutions in $D - C$ so they are both of size $n/2$. \square

Now for the odd case, we have:

Proposition 2.5. *Let D have order $2n$, n odd. Then D has exactly one class of involutions of size n .*

Proof. Each involution of D generates a Sylow 2-subgroup of order 2. All of these are distinct and must be conjugate. Then all involutions of D are in the same conjugacy class. \square

2.2 Factor Groups

We will show that dihedral groups have the property that every factor group is dihedral. Before we get to this, let us discuss the normal subgroups of dihedral groups.

Proposition 2.6. *If $N \triangleleft D$ is not a subgroup of C , then $N = D$ or $|D : N| = 2$.*

Proof. Let D have order $2n$, $\langle z \rangle = C$, $t \in D - C$, and $N \triangleleft D$. Suppose $t \in N$. By normality, $t^D \subseteq N$. If n is odd, then Proposition 2.5 gives us that $zt \in N$. So $zt \cdot t = z \in N$ and thus $N = D$. If n is even, then $z^2t \in N$ by Proposition 2.4. Then $\langle z^2 \rangle < N$ and $\langle z^2 \rangle t \subset N$. We infer that $|N| \geq n$. But then, $|D : N| \leq 2$ and the proposition holds. \square

Proposition 2.7. *If D is dihedral and $N \triangleleft D$, then D/N is dihedral.*

Proof. There are two cases, either N contains an involution from $D - C$ or $N \leq C$. In the former case, we know by the previous proposition that either $N = D$ or $|D/N| = 2$. Since both the trivial group and the cyclic group of order 2 are dihedral, we must only show the remaining case.

Let $N \leq C$. Let a “bar” denote the image under the canonical mapping of $D \mapsto D/N$, e.g. $\bar{D} = D/N$ (the “bar convention”). We know that \bar{C} is cyclic and that $|\bar{D} : \bar{C}| = |D : C| = 2$. Furthermore, for all $t \in D - C$, $\bar{t} \in \bar{D} - \bar{C}$ and $\bar{t}^2 = \bar{t}\bar{t} = 1$. Then \bar{D} is dihedral.

□

2.3 Generalized Dihedral Groups

It is possible to extend the notion of a dihedral group by relaxing the cyclic condition on the subgroup of index 2.

Definition A *generalized dihedral group* is a group G with an abelian subgroup B of index 2 such that $G - B$ contains only involutions.

As in the case of dihedral groups, we may view G under a semi-direct product construction where $G = B \rtimes_{\varphi} C_2$ where B is an abelian group, C_2 is the cyclic group of order 2, and φ sends the non-identity element of C_2 to the inverse map ($b \mapsto b^{-1}$).

It is not surprising then that this class of groups shares a similar conjugacy class structure with the usual dihedral groups in certain cases.

Proposition 2.8. *If G is generalized dihedral of order $2n$ with odd n , then G has a single conjugacy class of involutions.*

Proof. This follows immediately from the Sylow conjugacy theorem. □

We cannot, however, say much about the conjugacy classes of involutions if n is even. Consider the direct product of 3 cyclic groups of order 2: $C_2 \times C_2 \times C_2$. This group satisfies the conditions of generalized dihedral groups but is abelian and every element is an involution! Thus, every conjugacy class consists of only a single element.

3. GROUP ACTIONS

We will need a few properties of group actions in later results. Here we present the required background.

Definition Let G be a group and Ω be a set. A *group action* of G on Ω is a mapping from $\Omega \times G$ to Ω that satisfies:

$$\begin{aligned}\alpha \cdot 1 &= \alpha \\ (\alpha \cdot g) \cdot h &= \alpha \cdot (gh)\end{aligned}$$

for $\alpha \in \Omega$ and $g, h \in G$.

For a simple example, let $\Omega = G$ and define $\alpha \cdot g = \alpha g$. Then $\alpha \cdot 1 = \alpha 1 = \alpha$ and $(\alpha \cdot g) \cdot h = \alpha gh = \alpha \cdot (gh)$ and so G acts on itself by right multiplication.

Conjugation gives a more interesting action. Again take $\Omega = G$ but let $\alpha \cdot g = \alpha^g$. Now, $\alpha \cdot 1 = \alpha^1 = \alpha$ and $(\alpha \cdot g) \cdot h = (\alpha^g)^h = \alpha^{gh} = \alpha \cdot (gh)$.

To extend this idea, let us introduce the concept of an orbit.

Definition The *orbit* of $\alpha \in \Omega$ is the set:

$$\mathcal{O}_\alpha = \{\beta \in \Omega : \alpha \cdot g = \beta \text{ for some } g \in G\}.$$

Orbits partition Ω into distinct classes. This is shown by analyzing the implied relation: $\alpha \sim \beta$ if and only if there exists $g \in G$ such that $\alpha \cdot g = \beta$. A few quick calculations show it to be an equivalence relation:

$$\alpha \cdot 1 = \alpha \quad (\text{reflexive})$$

$$\text{if } \alpha \cdot g = \beta, \text{ then } \beta \cdot g^{-1} = \alpha \quad (\text{symmetric})$$

$$\text{if } \alpha \cdot g = \beta \text{ and } \beta \cdot h = \gamma, \text{ then } \alpha \cdot gh = \gamma \quad (\text{transitive})$$

where $\alpha, \beta, \gamma \in \Omega$ and $g, h \in G$.

For the conjugation action discussed above, orbits are conjugacy classes. That is:

$$\mathcal{O}_g = g^G = \{g^h : h \in G\}.$$

Using properties of orbits, we can easily determine the size of conjugacy classes. This calculation involves the notion of a *stabilizer*.

Definition The *stabilizer* of $\alpha \in \Omega$ is the group

$$G_\alpha = \{g \in G : \alpha \cdot g = \alpha\}.$$

It should be clear that $G_\alpha \leq G$: if $\alpha \cdot g = \alpha$ and $\alpha \cdot h = \alpha$ then $\alpha \cdot gh = (\alpha \cdot g) \cdot h = \alpha$. For the conjugation action mentioned above, the stabilizer of $g \in G$ is exactly $C_G(g)$, the centralizer of g . To see this, let $h \in G_g$. Then $g \cdot h = h^{-1}gh = g$ if and only if $gh = hg$ and so $h \in C_G(g)$. The same argument suffices to prove $h \in C_G(g)$ implies $h \in G_g$. Then $G_g = C_G(g)$ for the conjugation action of G on itself.

We now determine the size of orbits and conjugacy classes.

Theorem 3.1 ([10], p.5). *Let $H = G_\alpha$ and $\Lambda = \{Hx : x \in G\}$. There exists a bijection $\theta : \Lambda \rightarrow \mathcal{O}_\alpha$ such that $\theta(Hx) = \alpha \cdot x$. In particular, $|\mathcal{O}_\alpha| = |G : H|$.*

Proof. We first show that if $Hx = Hy$, then $\alpha \cdot x = \alpha \cdot y$. Since $y \in Hx$, $y = hx$ for some $h \in H$. Then:

$$\alpha \cdot y = \alpha \cdot hx = (\alpha \cdot h) \cdot x = \alpha \cdot x$$

where the final equality holds because H is the stabilizer of α . This shows that each coset of H sends α to exactly one element of \mathcal{O}_α .

Then we can define the map $\theta : \Lambda \rightarrow \mathcal{O}_\alpha$ by $\theta(Hx) = \alpha \cdot x$. Since every element of G lies in some coset of H , it follows from the definition of an orbit that θ is surjective. This leaves injectivity. Suppose $\theta(Hx) = \theta(Hy)$. Then:

$$\alpha = (\alpha \cdot x) \cdot x^{-1} = (\alpha \cdot y) \cdot x^{-1} = \alpha \cdot yx^{-1}.$$

Then $yx^{-1} \in H$ and so $y \in Hx$. It follows that θ is a bijection. Thus, the size of \mathcal{O}_α is equal to the number of cosets of H , i.e. $|\mathcal{O}_\alpha| = |G : H|$.

□

Immediately, we have:

Corollary 3.2. *The size of a conjugacy class g^G is $|G : C_G(g)|$.*

3.1 Transitive Actions

Definition Let G be a group acting on the set Ω . We call this a *transitive action* if and only if it has exactly one orbit.

Equivalently, transitive actions are those that, for all pairs $\alpha, \beta \in \Omega$, there exists a $g \in G$ such that $\alpha \cdot g = \beta$. This view is perhaps more suggestive but, in our results, the former definition will prove more immediately useful.

An easy transitive action is to let a group G act on itself by right multiplication. Take $g \in G$. Then for any $h_1 \in G$ there is an $h_2 \in g$ such that $gh_2 = h_1$. Then all elements of G are in \mathcal{O}_g and the right multiplication action is transitive.

Another example, to build on previous material, is to take G to be a generalized dihedral group such that $|G| = 2n$ with n odd. Then G acts transitively on its involutions.

Later, we will be concerned with groups that permit a transitive action on some finite set. The following proposition may be illustrative.

Theorem 3.3 ([7], p.34). *Let G be a finite group that acts transitively on the finite set Ω . This action is equivalent to one on the right cosets of a subgroup of G .*

Proof. Let $\alpha_0 \in \Omega$ and let H be the stabilizer of α_0 . We first show that there is a one-to-one correspondence between the cosets of H and the set Ω .

Suppose $\alpha_0 \cdot g_\beta = \beta$ for $\beta \in \Omega$ and $g_\beta \in G$. Then $\alpha_0 \cdot hg_\beta = \beta$ for all $h \in H$. All elements of the coset Hg_β then send α_0 to β . Now, take $g \in G$ such that $\alpha_0 \cdot g = \beta$. Then $\alpha_0 \cdot gg_\beta^{-1} = \alpha_0$ and $g \in Hg_\beta$. We infer that Hg_β contains all elements of G sending α_0 to β .

Since each element of G must act on α_0 and send it to some $\beta \in \Omega$, the set $\Omega' = \{Hg_\beta : \beta \in \Omega\}$ is a complete set of cosets.

Now, we show these actions to be equivalent via the bijection $\Theta : \Omega \rightarrow \Omega'$ where $\Theta(\beta) = Hg_\beta$ and $\alpha_0 \cdot g_\beta = \beta$. Consider $Hg_\beta g = Hg_\gamma$ for $\beta, \gamma \in \Omega$. Then

$$\alpha_0 \cdot g_\beta g = \beta \cdot g = \gamma = \alpha_0 g_\gamma.$$

Applying Θ to the middle equality, $\beta \cdot g = \gamma$, gives $(Hg_\beta)g = Hg_\gamma$. This shows that the action of G on Ω is preserved under the bijection Θ and is, hence, isomorphic to the action of G on Ω' . This completes the proof. \square

4. FROBENIUS GROUPS

Frobenius groups will play an important role in the results of this thesis.

Definition Let G be a group and N be a normal subgroup of G with complement A . The group G is a *Frobenius group* if and only if $n^a \neq n$ for non-identity elements $n \in N$ and $a \in A$.

An important example, for our purposes, will be the “odd-order” dihedral groups, that is, dihedral groups of order $2n$ for odd n . In this class of groups, we take N to be the cyclic subgroup of order n and A to be any subgroup generated by an involution (refer to the discussion in Chapter 2). Then A acts on N via conjugation such that $n^a = n^{-1}$ for non-identity elements $a \in A$ and $n \in N$. Moreover, since N has odd order, $n = n^{-1}$ if and only if $n = 1$. Thus, the action of A on N satisfies the definition above and we may conclude that “odd-order” dihedral groups are Frobenius groups.

Similarly, generalized dihedral groups $G = B \rtimes C_2$ are Frobenius if $|B|$ is odd.

4.1 Kernels and Complements

From here on, the normal subgroup N from the definition of Frobenius groups will be referred to as a *Frobenius kernel* and its complement, A , will be a *Frobenius*

complement. These subgroups have many nice properties and relationships of which we will now mention a few.

Lemma 4.1 ([10], p. 177). *Let G be a Frobenius group with kernel N and complement A . Then $|N| \equiv 1 \pmod{|A|}$ and, in particular, $|A|$ and $|N|$ are coprime.*

Proof. Consider the A -orbits of N . By the Orbit-Stabilizer theorem, $|\mathcal{O}_n| = |A : C_A(n)|$ for all $n \in N$. But, $n^a \neq n$ for non-identity elements $n \in N$ and so $C_A(n) = 1$ unless $n = 1$. Exactly one orbit of N is then of size 1 and all others are of size $|A|$. We conclude that $|N| = 1 + k|A|$ for some positive integer k and thus $|N| \equiv 1 \pmod{|A|}$. □

Lemma 4.2 ([10], p. 183). *If G is a Frobenius group with kernel N having complement A , then $A \cap A^g = 1$ for $g \in G - A$.*

Proof. Suppose $A \cap A^x > 1$ for some $x \in G$. Since $G = AN$, we may write $x = an$ for some $a \in A$ and $n \in N$. Then $A \cap A^{an} = A \cap A^n > 1$ and there exists some non-identity $b^n \in A \cap A^n$. This gives us that $[b, n] = b^{-1}b^n \in A$. Since N is normal, $[b, n] \in N$ for $b \in A$ and $n \in N$. But then $[b, n] \in A \cap N = 1$ and so b must centralize n . Since G is Frobenius we are forced to conclude $n = 1$ and that $x \in A$. Thus, $A \cap A^g = 1$ for $g \in G - A$. □

Other properties can be found in [10], [7], [8] for the interested reader.

4.2 Frobenius' Theorem

Frobenius groups can be viewed as those with a transitive action on a set satisfying the properties that:

1. Its stabilizers are non-trivial.

2. Only the identity fixes more than one letter.

To see this, let G be a Frobenius group with complement A and consider right multiplication action of G on the cosets of A . We know that this action is transitive, so all that remains to be shown is that it satisfies the aforementioned properties.

The subgroup A^g of G fixes the coset Ag of A , giving the former property. For the latter, the bulk of the work has already been done in Lemma 4.2. If $x \in G$ fixes A and Ag , then $x \in A \cap A^g = 1$.

Now, consider the elements not fixing any cosets of A :

$$X = G - \bigcup_{g \in G} A^g.$$

It turns out that $X \cup 1$ is exactly the Frobenius kernel of G and is, hence, normal.

This is the content of the next theorem:

Theorem 4.3 (Frobenius, [7], p. 140). *Let G be a Frobenius group acting on a set Ω . Let A be the subgroup fixing $\alpha_0 \in \Omega$. Then the set:*

$$N = (G - \bigcup_{g \in G} A^g) \cup \{1\}$$

is a normal subgroup of order $|G : A|$.

Unfortunately, no character free proof of this result has been discovered. The proof has been omitted here as the background material would significantly lengthen this thesis. For the motivated reader, one may consult Isaacs' *Algebra: A Graduate Course* [8, Ch. 15] for a chapter devoted to reaching this result as quickly as possible.

A useful corollary is this:

Corollary 4.4. *Let G be a Frobenius group with kernel N and complement A . If $g \in G - N$, then g is contained in some conjugate of A .*

Moving on, we use these results to prove some other relevant characterizations.

4.3 Characterizations

Lemma 4.5 ([10], p. 183). *Let G be a finite group and N be a normal subgroup complemented by A . The following are equivalent:*

1. G is Frobenius.
2. $C_G(a) \leq A$ for all non-identity $a \in A$.
3. $C_G(n) \leq N$ for all non-identity $n \in N$.

Proof. We will show that (1) is equivalent to (2) and then that (3) is equivalent to (1).

Suppose that G is Frobenius. Let $a \in A$ be a non-identity element and take $x \in C_G(a)$. Then $a \in A \cap A^x$. Since the intersection is non-trivial, $x \in A$ and (2) holds.

Now, assume (2) and we show (1). Let $a \in A$ be a non-identity element. Then:

$$C_N(a) = N \cap C_G(a) \subseteq N \cap A = 1$$

where the subset relation holds by (2) and the final equality holds because A is a complement of N . So (1) follows.

Again, suppose G is Frobenius. Let $n \in N$ be a non-identity element. Assume that $C_G(n) \not\leq N$. By Corollary 4.4, $a^g \in C_G(n)$ for some non-identity $a \in A$ and $g \in G$. If $m = n^{g^{-1}}$, then $a \in C_G(m)$. Since N is normal, $m \in N$ and, because G is Frobenius, we must have $m = 1$. But then $n = 1$ — a contradiction. Therefore, $C_G(n) \leq N$.

Finally, we prove (1) from (3). Let $n \in N$ be a non-identity element. Then

$$C_A(n) = A \cap C_G(n) \subseteq A \cap N = 1$$

where the subset relation holds by (3) and the latter equality because A complements N . Thus, G is Frobenius. □

Lemma 4.6 ([7], p. 39). *Let A be a non-trivial subgroup of G . Then G is Frobenius with complement A if and only if A intersects its conjugates trivially and is its own normalizer.*

Proof. Let G be Frobenius with complement A . By Lemma 4.2, we know that $A \cap A^g = 1$ for all $g \in G - A$.

Suppose $A \cap A^g = 1$ for all $g \in G - A$. Then $C_G(a) \leq A$ for all non-identity $a \in A$. According to Lemma 4.5, this is equivalent to the statement that G is Frobenius.

All that remains to be shown is that A is self-normalizing. Let $x \in N_G(A)$. Then $A \cap A^x = A$. Lemma 4.2 gives that $x \in A$. It follows that $N_G(A) = A$ and this completes the proof. □

5. AN ANALOG TO SNYDER'S PARAMETER

Here begin the main results of this thesis. The following chapter is devoted to a group theoretic analog to Snyder's parameter. We replace his use of irreducible character degrees with the square roots of conjugacy class sizes. Both of these satisfy the identity:

$$|G| = \sum_{i=1}^n d_i^2$$

where d_i is either character degrees or square roots of conjugacy class sizes and i indexes the total set of characters or conjugacy classes (the number of objects indexed is the same in both cases [7, p.96]). After initial definitions and examples, we will analyze the properties of and range of values e can take. Then, we will bound the order of a group by our parameter and characterize groups attaining this bound.

Without further ado:

Definition Let G be a finite group. For $x \in G$ let $c(x) = |C_G(x)|$ and $k(x) = |G : C_G(x)|$. We define the parameter e as follows:

$$e = \min\{(c(x) - 1) \cdot \sqrt{k(x)} : x \in G\}.$$

To see that this really corresponds with Snyder's e , let $x \in G$ be such that

$e = (c(x) - 1) \cdot \sqrt{k(x)}$ and consider the following:

$$\begin{aligned} e &= (c(x) - 1) \cdot \sqrt{k(x)} \\ &= c(x) \frac{k(x)}{\sqrt{k(x)}} - \sqrt{k(x)} \\ &= \frac{|G|}{\sqrt{k(x)}} - \sqrt{k(x)}. \end{aligned}$$

If we let $d = \sqrt{k(x)}$, we arrive at the equation $|G| = d(d + e)$. Since $k(x)$ is precisely the size of the conjugacy class containing x , for a fixed G , when the conjugacy class of x is very large, e is very small. This behavior is similar to Snyder's parameter with character degrees.

For a more detailed analysis of this relation, consider the function $f(c) = (c - 1)\sqrt{n/c}$ where n is the order of a group and c is the order of a centralizer. Then e is the minimum of f over the domain of possible centralizer orders. Since $f'(c) = \sqrt{n/c} \cdot (c + 1)/2c$ and $c \geq 1$, the function f must be monotonically increasing. Thus, small values of e are correlated with small centralizers or, equivalently, large conjugacy class sizes. In particular, the smallest possible values of e are attained only by groups with a conjugacy class that contains exactly half of the group's elements, the largest non-trivial conjugacy class possible relative to a group's order (see Theorem 5.6).

For abelian groups, calculating e is nearly trivial. Since all conjugacy classes are singletons in such groups, we have $e = n - 1$. Another easy example is the symmetric groups. The largest conjugacy class in S_n is that of the $(n - 1)$ -cycles and it has size $n!/(n - 1)$ (see Appendix). The value of e for S_n is then:

$$e = (n - 2) \cdot \sqrt{\frac{n!}{n - 1}}.$$

We now analyze the smallest possible values our parameter e can take. Note that the groups here are very small and that e necessarily increases for larger groups (a closer look at the function f mentioned above reveals that f increases as n does). In fact, given the order of a group, we can specify the range of values e can take. We need only look at the largest possible conjugacy class and the smallest possible conjugacy class. If G is a non-trivial group, its largest conjugacy class can contain up to half of the group's elements. This case corresponds to a centralizer of order 2. For the other extreme, a conjugacy class can contain only a single element. This happens in abelian groups and the entire group is the centralizer for every element. Then, for a group G of order n :

$$\sqrt{\frac{n}{2}} \leq e \leq n - 1. \quad (5.1)$$

This fact makes it easy to classify groups with small values of e .

Theorem 5.1. *Let G be a finite group such that $e \leq 2$. Then one of the following holds:*

1. $e = 0$ if and only if G is trivial.
2. $e = 1$ if and only if G is C_2 .
3. $e = \sqrt{3}$ if and only if G is S_3 .
4. $e = 2$ if and only if G is C_3 .

Proof. We will begin by showing that statements (1) - (4) hold in isolation and then prove that there are no other possibilities if $e \leq 2$. For the following, let $x \in G$ be an element such that $e = (c(x) - 1) \cdot \sqrt{k(x)}$.

Suppose $e = 0$. Then $c(x) = 1$. Only the trivial group has a trivial centralizer. This is sufficient for (1).

Let $e = 1$. Then $c(x) = 2$ and $k(x) = 1$. It must be the case that $|G| = 2$ and so G is C_2 . Reversing this argument proves (2).

Take $e = \sqrt{3}$. We must have that $c(x) = 2$ and $k(x) = 3$. This implies that G has order 6. There are only two groups of this order: C_6 and S_3 [2]. The former is abelian and thus all its conjugacy classes are singletons. The latter has already been shown to have $e = \sqrt{3}$.

Now, let $e = 2$. There are two cases. If $c(x) = 3$ and $k(x) = 1$, then G must be C_3 . Since C_3 must have $e = 2$, we need only consider the remaining case. Assume $c(x) = 2$ and $k(x) = 4$. This implies that $|G| = 8$. Since $c(x) < 8$, G cannot be abelian. The only two groups of order 8 that are not abelian are D_8 and Q_8 [2], the dihedral group of order 8 and quaternion group respectively. Neither of these groups have a self-centralizing involution and thus G can be neither. This completes (4).

We will now show that there are no other values of $e \leq 2$. By equation (5.1), if a group is non-trivial, $e \geq 1$. So consider groups G such that $1 < e < 2$. The largest possible order of G is 7 since $\sqrt{7/2} < 2$ but $\sqrt{8/2} \geq 2$. There are no non-abelian groups of order 3, 4, 5, or 7 [2] so any group where $1 < e < 2$ must have order 6. As we showed above, the only possibility is S_3 .

□

Next, we extend the work above and classify groups with prime values of e . This result is the beginning of a complete collection of understood e values. Even though e is not always an integer, it is based on integral values and, hence, does not range over all real numbers (as we saw above, e takes exactly one value between 1 and 2). Furthermore, this theorem illustrates that e has a strong relationship with the structure of a group. Before we get to this classification, we present a required lemma.

Lemma 5.2. *Let P be a group of order p^2 for some prime p . Then P is abelian.*

Proof. Let P act on $Z(P)$, the group's center, by conjugation. We know that every element of $Z(P)$ has an orbit of size one under this action and every other orbit must have p -power size. Then $P - Z(G)$ is a union of orbits with size divisible by p . Thus, $|Z(P)| \equiv 0 \pmod{p}$ and, since $1 \in Z(P)$, $|Z(P)| \geq p$.

If $Z(P) = P$ we are done, so suppose $|Z(G)| = p$. Then there is an element $x \in P - Z(P)$ and $\langle x \rangle \cap Z(P) = 1$. Therefore, the group $\langle x \rangle$ is a complement of $Z(G)$ and $P = Z(G) \rtimes_{\varphi} \langle x \rangle$ for some $\varphi \in \text{Aut}(Z(P))$. But, for $z \in Z(P)$ we must have $z^x = z$. Then φ is the identity and $P = Z(P) \times \langle x \rangle$. Since both $Z(P)$ and $\langle x \rangle$ are abelian, we must conclude that $Z(P) = P$. \square

Now, we get to a more sweeping characterization:

Theorem 5.3. *Let G be a finite group and p be an odd prime. The group G is abelian of order $p+1$ or generalized dihedral of order $2p^2$ if and only if $e = p$.*

Proof. Suppose that G is abelian of order $p+1$. Then $c(x) = p+1$ for all $x \in G$. This gives that $k(x) = 1$ for all $x \in G$ and we may conclude that $e = (c(x) - 1) \cdot \sqrt{k(x)} = p$.

For the other case, let G be generalized dihedral of order $2p^2$. Take $t \in G$ to be an involution. Since G is generalized dihedral, the conjugacy class size of t is p^2 by Proposition 2.8 since p is odd. This conjugacy class is as large as it can be and so corresponds to the minimal value of e . Then $e = (c(t) - 1) \cdot \sqrt{k(x)} = \sqrt{k(x)}$. But, $k(x)$ is the size of t 's conjugacy class, p^2 , and so $e = p$.

All that remains to be shown is that no other classes of groups have such an e . Let $e = p$. Since e is an integer, we know that $k(x)$ must be a perfect square, less we arrive at a contradiction (this holds due to the elementary result that the product of a rational and irrational number must be irrational). Then either $c(x) = p+1$ and $k(x) = 1$ or $c(x) = 2$ and $k(x) = p^2$. In the former case, we know that the group is of order $p+1$. Since e is minimal, we know that $c(x)$ is the size of the smallest centralizer. Then every element centralizes every other element and the group must

be abelian. In the latter case, we know that the group must be of order $2p^2$ and have a self-centralizing involution. But, this implies that there are p^2 involutions in G . Furthermore, G must have a Sylow p -subgroup P and it must be disjoint from these involutions as p is an odd prime. Since P has prime-power order p^2 , it must be abelian by Lemma 5.2. Then G is generalized dihedral. \square

This result is much stronger than any similar one for the character theoretic e . Snyder specifically mentioned in [14] that arguments classifying groups with particular values of e did not generalize. That is not the case here.

It is possible to bound the order of a group by e . For Snyder's original parameter, much work has been done to make a better bound ([9], [6], [12]). In our case, the answer is much easier and the bound much better:

Theorem 5.4. *Let G be a non-trivial finite group. Then $|G| \leq 2e^2$.*

Proof. We begin with the fact that $k(x) = |G|/c(x)$. This gives the following equation for some $x \in G$:

$$\begin{aligned} e &= (c(x) - 1) \cdot \sqrt{k(x)} \\ &= \frac{(c(x) - 1)}{\sqrt{c(x)}} \cdot \sqrt{|G|} \end{aligned}$$

If we square this equation we get:

$$e^2 = \frac{(c(x) - 1)^2}{c(x)} \cdot |G|$$

from which we may deduce that:

$$|G| = e^2 \cdot \frac{c(x)}{(c(x) - 1)^2}.$$

Since $c(x)/(c(x)-1)^2$ decreases as $c(x)$ increases, we need only look at the minimal value of $c(x)$. Our group G is non-trivial and thus all of its centralizers have at least 2 elements. Then $c(x) \geq 2$. This yields the following inequality:

$$2e^2 \geq e^2 \cdot \frac{c(x)}{(c(x)-1)^2} = |G|$$

which completes the proof. □

A natural follow-up to this theorem is the question: Which groups attain the bound? Before we answer this, let us review the following basic fact:

Lemma 5.5 ([10]). *Let $|G| = 2n$ for an odd positive integer n . Then G has a normal subgroup of index 2.*

Proof. Consider $\pi : G \rightarrow \text{Sym}(G)$, the permutation representation of the right multiplication action of G on itself. By Cauchy's Theorem, we can find an element $t \in G$ of order two. Under the aforementioned action, t has no fixed points. So $\pi(t)$, the permutation of $\text{Sym}(G)$ such that $g \cdot \pi(t) = gt$ for $g \in G$, consists of $|G|/2 = n$ 2-cycles. Since n is odd, t is an odd permutation.

Let H be the elements of G that induce even permutations on G . Take $x \in G - H$. Since x induces an odd permutation, $\pi(xt)$ is an even permutation. Then $xt \in H$ and so $x \in Ht$. Thus $Ht = G - H$. We conclude that H has index 2 and must be normal. □

We now classify groups attaining the bound mentioned above.

Theorem 5.6. *Let G be a finite group. The group G is generalized dihedral of order not divisible by 4 if and only if $|G| = 2e^2$.*

Proof. Suppose G is generalized dihedral with order not divisible by 4. Such a group has $|G|/2$ involutions that are all conjugate. Then there exists an $x \in G$ such

that $c(x) = 2$. Since this is the largest possible conjugacy class, it corresponds to the smallest possible e . Then:

$$e^2 = \frac{(c(x) - 1)^2}{c(x)} \cdot |G| = \frac{1}{2} \cdot |G|$$

and so $|G| = 2e^2$.

Now, suppose a finite group G has order $2e^2$. We know from the proof of Theorem 5.4, that this implies $c(x) = 2$ for some $x \in G$.

Then x has conjugacy class size of $e^2 = n/2$. Also, e^2 cannot be divisible by 2 lest $x \in G$ be contained in an abelian subgroup of order p^2 by Lemma 5.2. Now, we are in the situation of Lemma 5.5 and may conclude that G has a normal subgroup B of index 2. Since x is conjugate to e^2 involutions (including itself) and none of these can be contained in B (its order is not divisible by 2), $G - B$ must be exactly x^G . Thus, G is generalized dihedral of order not divisible by 4. \square

This theorem can be generalized fairly faithfully.

Theorem 5.7. *Let G be a group such that the smallest prime dividing $|G|$ is p and p^2 does not divide $|G|$. Then $|G| = (p/(p-1)^2)e^2$ if and only if G is Frobenius with complement C_p .*

To prove this we will need the following facts about automorphism groups:

Theorem 5.8 ([8]). *Let G be a finite group and H be a subgroup. Then $C_G(H) \triangleleft N_G(H)$ and $N_G(H)/C_G(H)$ is isomorphic to a subgroup of $\text{Aut}(H)$, the automorphism group of H .*

Lemma 5.9 ([10]). *Let G be a group of prime order p . Then $|\text{Aut}(G)| = p - 1$.*

Proof of 5.7. Let $|G| = (p/(p-1)^2)e^2$. Then G contains an element of order p that generates a Sylow p -subgroup H . The subgroup H must either be normal or it must

intersect its conjugates trivially. If H is normal, then $G/C_G(H) \leq \text{Aut}(H)$ by Theorem 5.8. Since H has prime order, Lemma 5.9 gives $|G : C_G(H)| \leq p - 1$. No prime less than p divides $|G|$, so $C_G(H) = G$ and $H \leq Z(G)$. This implies that G cannot have a centralizer of order p and hence $|G| \neq (p/(p-1)^2)e^2$. Then H must intersect its conjugates trivially.

Take $g \in N_G(H) - H$ and let $A = \langle g \rangle$ act on the non-identity elements of H by conjugation. Since $g \notin C_G(H)$, there is some $h \in H - \{1\}$ such that $h^g \neq h$. Let $h \in H - \{1\}$ be such an element. Then $1 < |A : A_h| = |\mathcal{O}_h| \leq p - 1$ where A_h is the stabilizer of h . This implies that a prime less than p divides $|A|$ and so must also divide $|G|$. This contradicts that p is the smallest prime dividing $|G|$ which lets us conclude that $N_G(H) = H$. Then H is self-normalizing. By Theorem 4.6, G is Frobenius with complement $H = C_p$.

Now we need only show that if G is a Frobenius group with complement $H = C_p$ it satisfies $|G| = (p/(p-1)^2)e^2$. By Theorem 4.6, we know that H is disjoint from its conjugates and is its own normalizer. Let $x \in H$ be a non-trivial element. Then $C_G(x) \leq H$ and, in fact, $C_G(x) = H$. The conjugacy class of x must be the largest possible within G . Thus, $e = (p-1)\sqrt{|G : H|}$ which lets us conclude that $|G| = (p/(p-1)^2)e^2$. □

6. AN ANALOG TO DURFEE'S RELATIVE PARAMETER

In this section, we consider a generalization of the parameter e that incorporates a normal subgroup. For $x \in G$ and $N \triangleleft G$, let $c_N(x) = |C_G(x) : C_N(x)|$ and $k_N(x) = |G : NC_G(x)|$. We then define e_N , the relative parameter, as follows:

$$e_N = \min\{(c_N(x) - 1) \cdot \sqrt{k_N(x)} : x \in G\}.$$

If $N = 1$, then $c_N(x) = |C_G(x)|$ and $k_N(x) = |G : C_G(x)|$ which yields $e_N = \min\{(|C_G(x)| - 1) \cdot \sqrt{|G : C_G(x)|}\} = e$. This new parameter thus generalizes our previous analog. Note that e_N is distinct from our parameter e for the factor group. An important difference is that e_N can be zero for non-trivial groups (see Theorem 6.3). Therefore, there are groups G for which G/N has a non-zero e but G has $e_N = 0$ for the same normal subgroup (this is true for all pairs (G, N) such that $e_N = 0$). For example, take $G = S_4$ and $N = A_4$. We know that the 3-cycles are self-centralizing in S_4 (see Appendix A) and, since these are even permutations, they are contained in A_4 . If we let $x = (1\ 2\ 3) \in S_4$, then we have $c_N(x) = 1$ and $k_N(x) = 2$ which forces us to conclude that $e_N = 0$. Also, we have that S_4/A_4 has $e = 1$ which further illustrates that e_N is distinct from e for the factor group.

We have the following relationship between e_N and e :

Theorem 6.1. *Let N be a normal subgroup of G . If e_N is the relative parameter for (G, N) and e is the non-relative parameter for G/N , then $e_N \leq e$.*

Proof. Using the “bar convention”, consider the relationship between $\overline{C_G(\bar{x})}$ and $\overline{C_G(x)}$ for $x \in G$. If $xg = gx$ ($g \in G$), then $\bar{x}\bar{g} = \bar{g}\bar{x}$. Thus, $\overline{C_G(x)} \leq \overline{C_G(\bar{x})}$. Note

that

$$\begin{aligned} c_N(x) &= |C_G(x) : C_N(x)| \\ &= |C_G(x)/(C_G(x) \cap N)| \\ &= |\overline{C_G(x)}|. \end{aligned}$$

Since $c(\bar{x}) = |\overline{C_G(\bar{x})}|$, $c_N(x) \leq c(\bar{x})$.

Now, the function $g(n) = (n-1)/\sqrt{n}$ is increasing over the positive real numbers since its derivative, $g'(n) = (n+1)/2n^{3/2}$, is positive for $n > 0$. Then

$$g(c_N(x)) = \frac{c_N(x) - 1}{\sqrt{c_N(x)}} \leq \frac{c(\bar{x}) - 1}{\sqrt{c(\bar{x})}} = g(c(\bar{x})).$$

We can now show that $e_N \leq e$. Let $y \in G$ be such that $e = (c(\bar{y}) - 1)\sqrt{k(\bar{y})}$.

Then

$$\begin{aligned} e &= g(c(\bar{y})) \cdot \sqrt{|G : N|} \\ &\geq g(c_N(y)) \cdot \sqrt{|G : N|} \\ &= (c_N(y) - 1) \cdot \sqrt{k_N(y)}. \end{aligned}$$

Finally,

$$e_N = \min\{(c_N(x) - 1)\sqrt{k_N(x)} : x \in G\} \leq (c_N(y) - 1)\sqrt{k_N(y)} \leq e$$

and so $e_N \leq e$. □

Before we continue to analyze e_N , we introduce a definition used by J. Britnell and M. Wildon [3].

Definition Let G be a finite group and $N \triangleleft G$. A conjugacy class x^G of G is *non-split* if $x^G = x^N$.

The following lemma establishes a useful result regarding the behavior of $c_N(x)$ and $k_N(x)$.

Lemma 6.2. *Let G be a finite group with normal subgroup N . For $x \in G$, $c_N(x)$ and $k_N(x)$ are positive integers. The conjugacy class of x is non-split if and only if $k_N(x) = 1$.*

Proof. The first part of the lemma follows directly from the fact that both $c_N(x)$ and $k_N(x)$ are indices of finite groups.

For the next part, consider $|NC_G(x)|$. It satisfies the following identity:

$$|NC_G(x)| = \frac{|N| \cdot |C_G(x)|}{|C_N(x)|}.$$

Now

$$\begin{aligned} k_N(x) &= |G : NC_G(x)| \\ &= |G| \cdot \frac{|C_N(x)|}{|N| \cdot |C_G(x)|} \\ &= \frac{|G : C_G(x)|}{|N : C_N(x)|} \end{aligned}$$

and thus $k_N(x) = 1$ if and only if $|x^N| = |x^G|$. But, since $x^N \subseteq x^G$, it must be the case that $x^N = x^G$ and thus x^G is non-split. \square

We begin by noting that $e_N \geq 0$ since $c_N(x)$ and $k_N(x)$ are both greater than 1. This brings us to our first result:

Theorem 6.3. *Let G be a finite group with normal subgroup N . The parameter $e_N = 0$ if and only if $C_G(x) \leq N$ for some $x \in G$. In particular, $e_N > 0$ if N contains no centralizers $C_G(x)$ for any $x \in G$.*

Proof. Suppose $e_N = 0$. Then there exists an $x \in G$ such that $(c_N(x) - 1) \cdot \sqrt{k_N(x)} = 0$. Since $k_N(x) \geq 1$, we must have $c_N(x) = 1$. Then $C_G(x) = C_N(x)$ and $C_G(x) \leq N$. Reversing this chain of logic gives the converse. \square

Zero is the minimum value e_N can take since both $c_N(x)$ and $k_N(x)$ are greater than or equal to 1 and e_N is non-decreasing with respect to both $c_N(x)$ and $k_N(x)$ (see the previous chapter for a discussion of this; the argument is the same). The next smallest value of e_N is 1. For $e_N > 0$, we must have $c_N(x) \geq 2$. Since e_N is non-decreasing with respect to $k_N(x)$, we may conclude that $e_N = 1$ is the smallest non-zero value e_N can take. This mirrors the situation with the non-relative e in that the values e_N are restricted to only a countably infinite subset of the reals (the values of e_N are indexed by pairs of integers).

There are many pairs (G, N) with $e_N = 0$. Since the parameter is defined as a minimum, if N contains any centralizers, it must fall into this class of groups. Notably this includes Frobenius groups which were ubiquitous in the last chapter. This gives further evidence that the relative e_N is distinct from the non-relative e .

The next theorem continues our classification of specific values of e_N and considers the case when $e_N = 1$.

Theorem 6.4. *The parameter $e_N = 1$ if and only if $|G : N| = 2$ and x^G is non-split.*

Proof. Suppose $e_N = 1$. Then there exists an $x \in G$ such that $(c_N(x) - 1) \cdot \sqrt{k_N(x)} = 1$. Because $c_N(x) - 1$ is always an integer and $k_N(x) \geq 1$, it must follow that $c_N(x) = 2$ and $k_N(x) = 1$. By Lemma 6.2 then, x^G is non-split.

Also, we have $|G : N| = |C_G(x) : C_N(x)| = 2$ since

$$\begin{aligned} 1 = k_N(x) &= |G : NC_G(x)| \\ &= \frac{|G : N|}{|C_G(x) : C_N(x)|}. \end{aligned}$$

Now, let $|G : N| = 2$ and $x^G = x^N$. Then $k_N(x) = 1$ by Lemma 6.2. We may infer then that $2 = |G : N| = |C_G(x) : C_N(x)| = c_N(x)$. Thus, $e_N = 1$. \square

One example giving $e_N = 1$ is $G = N \times \langle t \rangle$ for any group N and where $t^2 = 1$. Then t is in the center of G and so, for any $x \in G$, $x^N = x^{Nt}$. We infer from this that $x^N = x^G$ and so all the conjugacy classes of G are non-split. Since $|G : N| = 2$, the previous theorem gives $e_N = 1$ for (G, N) .

The next value of interest is $\sqrt{2}$. Let $G = D_8$ and $N = Z(D_8)$. Then $|G : N| = 4$. Since all conjugacy classes of G are either of size 1 or size 4, we know that each non-central element of G has centralizer of order 4. Then we may conclude that $c_N(x) = 2$ for $x \in G - N$ and $e_N = \sqrt{2}$ for (G, N) .

The relative parameter can also take on the value $\sqrt{3}$. Let $G = A_4$, the alternating group on 4 letters, and let $N = \langle (12)(34) \rangle$. Here $|G : N| = 6$. There are two conjugacy class sizes other than 1: 3 and 4 [2]. The latter classes are larger but must have centralizers of order 3 which cannot intersect N non-trivially. The former class, let us denote it x^G consists of all ‘‘double transpositions’’ of A_4 and yields $c_N(x) = 2$. Given that A_4 has no classes of size 6, $c_N(x) = 2$ is minimal. Then, for (G, N) , $e_N = \sqrt{3}$.

There are no other possible values between 1 and 2. Theorem 6.4 handles the case where $c_N(x) = 1$ and $k_N(x) = 1$. The preceding paragraphs handle the $c_N(x) = 2$, $k_N(x) = 2$ and the $c_N(x) = 2$, $k_N(x) = 3$ cases respectively. All other choices for these terms imply $e_N \geq 2$.

Theorem 6.5. *If $e_N = p$ for a prime p , then either:*

1. $|G : N| = p + 1$ and some conjugacy class x^G of G is non-split.

2. $|G : N| = 2p^2$ and some conjugacy class x^G of G is split.

Proof. Let $e_N = p$. There exists an $x \in G$ such that $(c_N(x) - 1) \cdot \sqrt{k_N(x)} = p$. Since $c_N(x) - 1$ is an integer and $k_N(x) \geq 1$, both $c_N(x) - 1$ and $\sqrt{k_N(x)}$ must be factors of p (i.e., integers whose product is p). Then we may infer, because p is prime, that either:

1. $c_N(x) - 1 = p$ and $\sqrt{k_N(x)} = 1$ or

2. $c_N(x) - 1 = 1$ and $\sqrt{k_N(x)} = p$.

Case 1

Suppose $c_N(x) = p + 1$ and $k_N(x) = 1$. By Lemma 6.2, $x^G = x^N$. All that remains to be shown is the index of N in G . We have

$$1 = k_N(x) = \frac{|G : N|}{|C_G(x) : C_N(x)|} = \frac{|G : N|}{p + 1}.$$

Then $|G : N| = p + 1$.

Case 2

Suppose $c_N(x) = 2$ and $k_N(x) = p^2$. Consider $k_N(x)$:

$$k_N(x) = \frac{|G : N|}{c_N(x)}.$$

Then $|G : N| = k_N(x) \cdot c_N(x) = 2p^2$. Also, since $k_N(x) > 1$, $x^G \neq x^N$ and x^G is split. □

The former case is satisfied by certain direct products. Let $G = N \times A$ where N can be any group and A is abelian of order $p+1$ for a prime p . Then $|G : N| = p+1$ and $A \leq Z(G)$. Thus, for $x \in G$, $x^N = x^{Na}$ for $a \in A$. Because all cosets of N produce the same orbit via conjugation, $x^G = x^N$. We know then that all conjugacy classes of G are non-split with respect to N and so $e_N = p$ for (G, N) .

For the latter case, we consider another construction. Let $G = N \times H$ where N is abelian and H is generalized dihedral of order $2p^2$. Then $N \leq Z(G)$. We see that $x^H = x^{Hn}$ for $x \in G$ and $n \in N$ and conclude, as before, that $x^H = x^G$ for all $x \in G$. Note that $|(hn)^H| = |h^H|$ for $h \in H$ and $n \in N$. Without loss of generality then, we can assume that for some $h \in H$, $e_N = (c_N(h) - 1)\sqrt{k_N(h)}$. Also, $C_G(h) = NC_H(h)$ so

$$c_N(h) = |C_G(h) : C_N(h)| = |NC_H(h) : N| = |C_G(h)|.$$

Now,

$$\begin{aligned} e_N &= (c_N(h) - 1) \cdot \sqrt{\frac{|G : N|}{c_N(h)}} \\ &= (|C_H(h)| - 1) \cdot \sqrt{|H : C_H(h)|}. \end{aligned} \tag{6.1}$$

We recognize (6.1) as being equivalent to the non-relative parameter for H . By Theorem 5.3 then, $e_N = p$.

APPENDIX A: The Largest Conjugacy Class of the Symmetric Group

Conjugation in the symmetric group is well-understood. Here we build off the conjugation properties of S_n and show that its largest conjugacy class is that of the $(n-1)$ -cycles.

To begin, we review a few basic facts about S_n .

Theorem A.1 ([13]). *Disjoint cycles of S_n commute.*

Theorem A.2 ([13]). *Every permutation in S_n has a cycle decomposition that is unique up to ordering of the cycles and up to a cyclic permutation of the elements within each cycle.*

Theorem A.3 ([13]). *Suppose $\sigma \in S_n$, and let m_1, m_2, \dots, m_r be the distinct integers (including 1 if applicable) in the cycle type of σ , and let there be k_i cycles of order m_i in σ . (Thus $\sum k_i m_i = n$.) Then σ has conjugacy class of size:*

$$\frac{n!}{\prod_{i=1}^r (k_i! m_i^{k_i})}$$

The previous theorem tells us precisely the size of each conjugacy class in S_n . However, its form does not make it amenable to standard analysis techniques. Instead, we take another approach.

Lemma A.4. *Let $\sigma \in S_n$. Then $|C_{S_n}(\sigma)| \geq n-1$.*

Proof. If σ is the identity, it has centralizer equal to S_n . Assume then that $\sigma \in S_n$ is a non-identity element. Then the permutation σ has a unique cycle decomposition

into m cycles $\sigma = \sigma_1\sigma_2\dots\sigma_m$ (not including 1-cycles) where each σ_i is disjoint from σ_j for all $j \neq i$. Let k_i denote the cycle length of σ_i and $\kappa = \sum_{i=1}^m k_i$. Note that $\kappa \leq n$. Consider the elements that commute with σ : Each subgroup $\langle \sigma_i \rangle$ commutes with σ as does each cycle disjoint from σ . Together then, we have

$$|C_{S_n}(\sigma)| \geq \left(\sum_{i=1}^m k_i - 1 \right) + (n - \kappa) = n - m$$

where the summation does not count identity elements. But then σ also commutes with

$$\begin{aligned} &\sigma_1\sigma_2\dots\sigma_m, \\ &\sigma_1\sigma_2\dots\sigma_{m-1}, \\ &\vdots \\ &\sigma_1\sigma_2 \end{aligned}$$

which yields another $m - 1$ elements not already counted. Thus $|C_{S_n}(\sigma)| \geq n - 1$. \square

Corollary A.5. *The symmetric group S_n can have no conjugacy class larger than $n!/(n - 1)$.*

Theorem A.6. *The largest conjugacy class in S_n is of size $n!/(n - 1)$.*

Proof. Let $\sigma \in S_n$ be an $(n - 1)$ -cycle. Its cycle type is $[(n - 1), 1]$ and it has only one cycle of each order. Then σ has class size:

$$\frac{n!}{(1!(n - 1)^1)(1!1^1)} = \frac{n!}{n - 1}$$

by Theorem A.3. By Corollary A.5, this is the largest conjugacy class. \square

BIBLIOGRAPHY

- [1] Y. Berkovich. “Groups with few characters of small degrees”. In: *Israel J. Math.* 110 (1999), pp. 325–332.
- [2] H. U. Besche, B. Eick, and E. O’Brien. *The Small Groups Library*. http://www.icm.tu-bs.de/ag_algebra/software/small/.
- [3] J. Britnell and M. Wildon. “On the distribution of conjugacy classes between the cosets of a finite group in a cyclic extension”. In: *Bull. London Math. Soc.* 40 (2008), pp. 897–906.
- [4] D. Chillag. “On Blichfeldt’s like congruences and other close characters-conjugacy classes analogs”. In: *Ischia Group Theory 2008*. Hackensack, NJ: World Sci. Publ., 2009, pp. 42–55.
- [5] C. Durfee. “Groups with an irreducible character of large degree relative to a fixed normal subgroup”. PhD thesis. University of Wisconsin-Madison, 2012.
- [6] C. Durfee and S. Jensen. “A bound on the order of a group having a large character degree”. In: *J. Algebra* 338 (2011), pp. 197–206.
- [7] D. Gorenstein. *Finite Groups*. Chelsea Publishing Company, 1980.
- [8] I.M. Isaacs. *Algebra: A Graduate Course*. Amer. Math. Soc., 2009.
- [9] I.M. Isaacs. “Bounding the order of a group with a large character degree”. In: *J. Algebra* 38 (2011), pp. 264–275.
- [10] I.M. Isaacs. *Finite Group Theory*. Amer. Math. Soc., 2008.

- [11] M. Larsen, G. Malle, and P.H. Tiep. “The largest irreducible representations of simple groups”. In: *arXiv:101.5628v1 [math.RT]* (2010).
- [12] M. Lewis. “Bounding group orders by large character degrees: A question of Snyder”. In: *arXiv: 1206.4334 [math.GR]* (2012).
- [13] rm50. *conjugacy classes in the symmetric group S_n* .
<http://planetmath.org/conjugacyclassesinthesymmetricgroupsn>.
- [14] N. Snyder. “Groups with a character of large degree”. In: *Proc. Amer. Math. Soc.* 136 (2008), pp. 1893–1903.

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