GRAPHICAL REPRESENTATIONS OF TOPOLOGIES ON A FINITE SET

by

Emilie-Anne Francis Hruzek, B.S.

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Committee Members:

Eugene Curtin, Chair

David Snyder

Jian Shen
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DEDICATION

I dedicate this thesis to my best friend, Brett Fuller, for letting me go, for letting me create, and for never expecting me to say the right thing. No one else has been worth the return trip. No one else inspires me more. And no one else has made me wish so hard that I could find the words.
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1. INTRODUCTION

As an undergraduate at Texas State University-San Marcos in an introductory topology course, Dr. Sukhjit Singh presented an exercise to the class of finding all the topologies on a three-element set.

**Definition 1.** A topology on a set $X$ is a collection $T$ of subsets of $X$ having the following properties:

1. $\emptyset$ and $X$ are in $T$.
2. The union of elements of any subcollection of $T$ is in $T$.
3. The intersection of the elements of any finite subcollection of $T$ is in $T$. [8]

Most of my classmates tried to use brute force to tackle the problem. The brute force method would require the student to look at $P(P(X)) = P(\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\})$, which has a total of 256 elements to check for the properties of a topology.

Rather than perform such a time-consuming process, I drew a graph, $G$. I found it necessary to make vertices of two types. The first type was the single element subsets of $P(X) - \{\emptyset, X\} = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. So

$$V_1(G) = \{\{a\}\}, \{\{b\}\}, \{\{c\}\}, \{\{a, b\}\}, \{\{a, c\}\}, \{\{b, c\}\}\}.$$ 

The second type was the two element subsets of $P(X) - \{\emptyset, X\}$ where both elements have the same cardinality. So we have that $V_2(G)$ is the set:

$$\{\{\{a\}, \{b\}\}, \{\{a\}, \{c\}\}, \{\{b\}, \{c\}\}, \{\{a, b\}, \{a, c\}\}, \{\{a, b\}, \{b, c\}\}, \{\{a, c\}, \{b, c\}\}\}.$$ 

As this notation is cumbersome, we will often make use of the following representations:
\[{\{a\}} \text{ is represented by } a
\]
\[{\{a, b\}} \text{ is represented by } ab
\]
\[{\{a\}, \{b\}} \text{ is represented by } a : b
\]
\[{\{a, b\}, \{a, c\}, \{b, c\}} \text{ is represented by } ab : ac : bc.
\]

Thus

\[
V(G) = V_1(G) \cup V_2(G)
\]

\[
= \{a, b, c, ab, ac, bc, a : b, a : c, b : c, ab : ac, ab : bc, ac : bc\}.
\]

The edge set of \( G, E(G) \), is define by the rule \( v_1 \leftrightarrow v_2 \) for some \( v_1, v_2 \in V(G) \) if \( v_1 \cup v_2 \cup \emptyset, X \) is a topology on \( X \). For example, we have an edge from \( a \) to \( bc \) because

\[
a \cup bc \cup \emptyset, X = \{a\} \cup \{b, c\} \cup \emptyset, X = \emptyset, \{a\}, \{b, c\}, \{a, b, c\}.\]

My original sketch is seen in Figure 1.

Now consider the topologies on \( X \). The only topology with 2 open sets is the indiscrete topology \( \emptyset, X \). The topologies with three open sets are all accounted for by

\[
\{v \cup \emptyset, X | v \in V_1(G)\}.\]

Topologies with four open sets are represented on the graph by the edges between vertices in \( V_1(G) \). The topologies with five and six open sets are then the edges drawn between vertices in \( V_1(G) \) and \( V_2(G) \) or edges drawn between two vertices of \( V_2(G) \), respectively. The inclusion of any more sets beyond six would force the topology to be the discrete topology.

As this was an exercise to count the number of topologies on a three element set, I first counted the number of edges drawn, arriving at a total of 21 topologies. The elements of \( V_1(G) \), when unioned with \( \emptyset, X \), were also topologies in their own right, as mentioned above. Counting each of these, this added another six topologies for a total of 27. Only the
discrete and indiscrete topologies were not pictured in the graph. These two more topologies brought me to the final total of 29 topologies on a set of three elements.

The effectiveness of this construction for the purpose of counting topologies became an interesting question to me that I would continue to ponder throughout my undergraduate career: would the use of graphical representations be able to aid in the unanswered question of how to count topologies on a finite set? Even if not, what could be gleaned about the construction of topologies from the representative graphs? It was these questions which ultimately led me to a recursive definition for $T_0$ topologies on a finite set.

Figure 1: Original Graph of the Topologies on $X = \{a, b, c\}$
2. TERMINOLOGY AND NOTATIONAL CONVENTIONS

Throughout the course of this paper, we will make use of terminology and notation from the fields of graph theory and topology. When the construction is the work of the author, the definitions will be given in text. Presented here are standard definitions and notation considered background material for the following chapters.

We begin with an exploration of graphical representations. As described by West, a graph $G$ is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints [14]. Early representations used, such as the nucleus and shell graphs of Chapter 3, are undirected graphs. Using the elements of the power set of $X$, or the set of all possible subsets of $X$, we define a vertex set. For the relation which defines our edge set we turn to the definition of a topology, given in Definition 1.

This early work will focus largely on distinct, or labeled, topologies, which is a collection of all of the unique permutations of the elements of $X$ into the open sets of a topology $T$. For example, when considering labeled topologies, the collections $\{\emptyset, \{a\}, \{a,b\}\}$ and $\{\emptyset, \{b\}, \{a,b\}\}$ would be separate topologies on $\{a,b\}$. Later in the paper, we will refine our representations and only consider the equivalent, or unlabeled, topologies on $X$. In these instances, the two topologies above would not be considered distinct. As we progress, we will utilize Hasse diagrams to explore the unlabeled topologies on a finite set. These directed graphs are a rendering of a partially ordered set whose edge set relation is defined by the cover relation of the partially ordered set. These graphs are transitive, meaning that if $x < y$ and $y < z$ for some vertices $x,y,z \in V(G)$ and some cover relation $<$, then $x < z$.

Lastly, we will return to the undirected, loopless graph structure and examine the degrees of a vertex $v$, $d(v)$, or the number of edges incident to $v$. These graphs will be found to be $k$-regular, meaning every $v \in V(G)$ has degree $d(v) = k$. 


3. NUCLEUS-SHELL GRAPHS

Starting my exploration from the graph in Figure 1, I defined my topological graphs by the following rules:

1. For the graph $G_X$, let $V(G_X)$ be some subset of $P(X) - \{\emptyset, X\}$.

2. Let the edges of $E(G_X)$ be defined by $v_1 \leftrightarrow v_2$ if $v_1 \cup v_2 \cup \{\emptyset, X\}$ is a topology on $X$, for some $v_1, v_2 \in V(G_X)$.

Note that the question as to which selection of subsets of $P(X)$ which will serve as $V(G_X)$ is a complicated. As shown in the graph in Figure 1, not all of the 256 possible subsets of $P(X)$ are drawn. This issue is discussed in greater detail below.

When $|X| = 1$ or 2, the graphical representations are trivial given our above notational restrictions. They are, however, detailed below. Once $|X| > 2$, a greater organizational structure is needed. For certain vertices, as discussed in the introduction, $v_i \in V(G_X)$, $v_i \cup \{\emptyset, X\}$ is a topology on $X$. This is not necessarily the case for all vertices however. I defined two concepts in an attempt to simplify this complex problem: the nucleus and shell graphs.

**Definition 2.** In a $G_X$, the nucleus, $N_{|X|}$, is the full subgraph of $G_X$ in which $V(N_n) = \{A | A$ is a non-empty proper subset of $X\}$. Thus $V(N_n)$ consists of all the singleton subsets of $P(X) - \{\emptyset, X\}$. Note that $V(N_n) = V_1(G_n)$, as defined in the introduction.

**Example 1.** For $n = 3$, $V(N_3) = V_1(G_3) = \{a, b, c, ab, ac, bc\}$. Similarly, for $n = 4$, $V(N_4) = \{a, b, c, d, ab, ac, ad, bc, bd, cd, abc, abd, acd, bcd\}$.

Thus every topology with three or four open sets on a given $X_n$ is represented as a vertex or an edge of $N_n$. Note that $|V(N_n)| = 2^n - 2$, as these are the number of non-empty, proper subsets of $X$. 5
Note that we have an edge \((u, v) \in E(N_{|X|})\) if and only if \(u \subset v\), \(v \subset u\), or \(u = v^c\). Thus we can count the number of edges in \(N_n\). The number of pairs \((u, v)\) such that \(u = v^c\) is 
\[
\frac{2^n-2}{2} = 2^{n-1} - 1.
\]
Also, the number of pairs \((u, v)\) with \(u \subset v\) is
\[
\sum_{k=2}^{n-1} \binom{n}{k}(2^k - 2) = 3^n - 3 \cdot 2^n + 3.
\]
Thus the number of edges in \(N_n\) is defined by the mapping 
\[
f(n) = 3^n - 3 \cdot 2^n + 2^{n-1} + 2.
\]

**Example 2.** Let \(X_1 = \{a\}\).

Since \(X_1\)'s only topologies are the discrete and indiscrete topologies, neither of which are drawn in this method, there is no graphical representation. Since there is no graph, the total number of edges is \(f(1) = 0\), to which we add 1 for the discrete and indiscrete topologies, which coincide in this case, giving us the total number of topologies on a set of one element: 1.

**Example 3.** Let \(X_2 = \{a, b\}\).

The graph of topologies on \(X_2 = \{a, b\}\) is the isolated vertices \(a\) and \(b\). An edge drawn between them would represent the topology \(\emptyset, \{a\}, \{b\}, \{a, b\}\), which is the power set of \(X_2\) and therefore is not pictured. However, each of these vertices would form a topology in and of themselves. This graph is \(N_2\), the first instance of a nucleus subgraph.

Thus, the topologies on \(X_2\) are formed by each of the two vertices in the nucleus, plus the discrete and indiscrete topologies, for a total count of 4 topologies.

As I began to examine the graph of the topologies, I developed the concepts of the nucleus and shells of the graph in order to better organize my system in anticipation of attempting to graph the topologies of larger sets. I will first show the nucleus, \(N_3\), then its single shell \(S_3\), and finally the complete graph. It is known that there are 29 labeled topologies on this set [9].
$N_3$, pictured in Figure 2, contains $2^n - 2 = 6$ vertices.

We see in the graph of $N_3$ below the $f(3) = 9$ edges representing the topologies with four sets, six of the edges representing subset relationships between the vertices and three representing vertex pairs which partition $X$.

![Figure 2: Nucleus Graph, $N_3$](image)

These values, plus the fact that each vertex in the nucleus is a topology when combined with $\emptyset$ and $X_3$, gives a total of 15 distinct topologies.

My plan was to define shells around this nucleus by making the $k$th shell, $S^k_n$, consist of some judicious choice of $k$ element subsets of $P(X_n)$. I define $S^2_3$ to be the pairs of open sets of the same order.

![Figure 3: Shell Graph of two open sets, $S^2_3$](image)
Thus, all topologies of three open sets, such as \( \emptyset, \{a\}, X_3 \), are counted by the vertices of the nucleus, all topologies of four open sets are counted by the edges of the nucleus, and all topologies of six open sets are counted by the edges of \( S^2_3 \). Once these two subgraphs are combined together, the edges connecting the nucleus to the shell will give us all the topologies of five open sets.

![Figure 4: Nucleus-Shell Graph on \( X_3 = \{a,b,c\} \)](image)

The nucleus contains, as stated above, 15 topologies; \( S^2_3 \) contains 6. The two are joined together by 6 five set topologies, such as \( a : b \leftrightarrow ab = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \). This gives a total of 27 topologies pictured, plus the discrete and indiscrete topologies, for a total of 29 labeled topologies.

After such a successful representation, I attempted to expand this construction to \( X_4 = \{a, b, c, d\} \).

**Example 4.** In Figure 5, we have the graph \( N_4 \). I chose to render the graph in three dimensions for the sake of clarity. The outer edges in the diagram connect sets \( u \) and \( v \) with \( u \subset v \) and the inner edges connect sets \( u \) and \( v \) with \( u = v^c \). It is also clear that permutations of \( N_3 \) is an often repeated subgraph; in fact, every vertex of \( N_4 \) lies on at least 2 subgraphs isomorphic to \( N_3 \).
Thus the nucleus gives a total of 57 labeled topologies on $X_4$, still far from the 355 known.

When attempting to construct the first shell, $S_4^2$, several difficulties in construction occurred.

Using the same method of choosing subsets of $P(X)$ to serve as the vertices of $S_4^2$, I attempted to determine, before construction, the number of vertices in $S_4^2$. Since $S_4^2$ will contain the pairs of elements of the same cardinality, the vertices of $S_4^2$ are shown in Table 1 below.

---

**Figure 5: Nucleus Graph, $N_4$**
Table 1: Vertices of $S_4^2$

<table>
<thead>
<tr>
<th>Cardinality of sets</th>
<th>Example</th>
<th>Total</th>
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<tr>
<td>1</td>
<td>$a : b$</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>$ab : bc$</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>$ab : cd$</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>$abc : abd$</td>
<td>6</td>
</tr>
</tbody>
</table>

So, prior to construction of $S_4^2$, $V(S_4^2)$ contains 27 vertices. Furthermore, not all topologies on five sets would be captured by this selection of vertices for $S_4^2$.

For example, when constructing the topology

$$\{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, X_4\},$$

which in the Nucleus-Shell system would be a edge connecting a vertex in $N_4$ to a vertex in $S_4^2$, how should the topology be represented? As the edge $a \leftrightarrow ab : abc$, $ab \leftrightarrow a : abc$, or $abc \leftrightarrow a : ab$? It seemed I could not, as I had done before, solely choose vertices of pairs of open sets of the same cardinality.

![Possible edges of $G_{X_4}$ which form the topology $T = \{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, X_n\}$](image)

Figure 6: Possible edges of $G_{X_4}$ which form the topology $T = \{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, X_n\}$

Each choice dramatically altered the overall shape and construction of $S_4^2$, as this choice determined the vertices that would be available to construct the topologies of six open sets whose representative edges would lie wholly in $S_4^2$. In the pursuit of simplicity, I attempted to minimize the number of vertices needed for $S_4^2$. However, in any construction, the shell was not as well organized as the nucleus had been, making the joining of the two graphs,
as had been seen with $N_3$ and $S^2_3$, unenlightening. Another downfall of this construction, which was used initially to count the number of topologies on a three element set, is that for the construction of these graphs for $n \geq 4$ it is necessary to have all the topologies on the set before work begins. No obvious pattern had presented itself to allow me to extrapolate the topologies, or their graph structure, for higher values of $n$.

With the shell construction abandoned, greater attention was paid to the nucleus graph. With this came the realization that all topologies on $X$ were paths on the nucleus graph and that the shell structure was truly redundant. As seen in Figures 7 and 8, the topology $\{\{a\}, \{b\}, \{a, b\}, \{b, c\}\} \cup \{\emptyset, X\}$ corresponds to a path on $N_3$ and on $N_4$, respectively.

![Diagram](image)

Figure 7: $T = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X_3\}$ highlighted on $N_3$

However, while every topology can be drawn as a path on the nucleus graph, not every path is in turn a topology on $X$.

Again, it seemed that the nucleus graph, even with the outer shell structure abandoned, would not prove a fruitful path of inquiry. This led to a question: could the topologies on a set of $n$ elements be used to construct the topologies on a set of $n + 1$ elements? I had already seen that each topology was the union of smaller topologies on the set, or the union of edges on a graph forming a path, and that any topology on $n$ elements had a
correlate on $n + 1$ elements, as seen in Figures 7 and 8. This led to my first attempt at a recursive definition for the construction of the topologies on $n + 1$ elements from the topologies on an $n$ element set.
4. A RECURSIVELY DEFINED SET OF TOPOLOGIES

In the nucleus and shell graphs, needing to have all the topologies on the finite set proved to be a major hurdle in representing the topologies graphically. Finding all the topologies on a set through a brute force method is already tedious on a set of 4 elements, and quickly became infeasible as the number of elements increases. For these reasons, my research focus shifted to trying to find a means of generating the topologies on a set of \( n + 1 \) elements from the topologies on a set of \( n \) elements. Working with the nucleus-shell graphs did present some interesting relationships between topologies on sets differing by one element.

Let \( X_n \) be a set with \( n \) elements, let \( x \notin X_n \) and let \( X_{n+1} = X \cup \{x\} \). Then for any topology \{\emptyset, A_1, \ldots, A_k, X_n\}, where each \( A_i \) is a non-empty, proper subset of \( X_n \), the set \{\emptyset, A_1, \ldots, A_k, X_n, X_{n+1}\} is a topology on a set of \( n + 1 \) elements. Also, \{\emptyset, \{x\}, A_1 \cup \{x\}, \ldots, A_k \cup \{x\}, X_{n+1}\} is a topology on \( X_{n+1} \). If for all sets \( A_i, A_j \) in a topology on \( X_n \) we have that \( A_i \cap A_j \neq \emptyset \), then \{\emptyset, A_1 \cup \{x\}, \ldots, A_k \cup \{x\}, X_{n+1}\} is a topology on the set \( X_{n+1} \) as well. Given these realizations, the following definitions are made.

**Definition 3.** Let \( X_n \) be a finite set, \( T \) a topology on \( X_n \), and and \( A_i \) is a non-empty proper subset of \( X_n \) in \( T \). Define the sets \( xT, \hat{x}T, \) and \( x'T \), for some singleton \( x \notin X_n \) as follows:

\[
xT = \{A_i \cup \{x\} | A_i \in T\} \cup \{\emptyset, X_{n+1}\},
\]

\[
x'T = xT \setminus \{x\},
\]

\[
\hat{x}T = T \cup \{X_{n+1}\}.
\]

It is necessary to make further distinctions on the type of sets which compose the topology
Whether or not these classify topologies is dependent on the presence of non-empty open sets with an empty intersection.

**Definition 4.** $T$ is a **conjoint** topology on $X$ means that for all non-trivial proper subsets $A$ and $B$ of $X$, if $A$ and $B$ are in $T$ then $A \cap B \neq \emptyset$.

**Proposition 1.** If $T$ is a topology on $X_n$, then we have the following:

i) $xT$ is a topology on $X_{n+1}$,

ii) $\hat{x}T$ is a topology on $X_{n+1}$,

iii) $x'T$ is a topology on $X_{n+1}$ if and only if $T$ is conjoint,

iv) $xT \cup \hat{x}T$ is a topology on $X_{n+1}$, 

v) $x'T \cup \hat{x}T$ is a topology on $X_{n+1}$ if and only if $T$ is conjoint.

**Proof.** Let $X_n$ be a finite set, $x \notin X_n$, such that $X_{n+1} = X \cup \{x\}$.

Note first that $\{\emptyset, X_{n+1}\}$ is a subset of $xT, x'T, \text{ and } \hat{x}T$, by construction, and as such, $\{\emptyset, X_{n+1}\}$ is a subset of any possible unions of these sets. Thus, we need only check for each case closure by set union and intersection for each set.

i) Let $B, B' \in xT$. Then

$$B \cup B' = (A_i \cup \{x\}) \cup (A_j \cup \{x\}) \text{ for some } A_i, A_j \in T$$

$$= (A_i \cup A_j) \cup \{x\}$$

and

$$B \cap B' = (A_i \cup \{x\}) \cap (A_j \cup \{x\}) \text{ for some } A_i, A_j \in T$$

$$= (A_i \cap A_j) \cup \{x\}.$$
Since \( A_i \cup A_j, A_i \cap A_j \in T \), then \((A_i \cup A_j) \cup \{x\} \) and \((A_i \cap A_j) \cup \{x\} \in xT\). Thus \( xT \) is closed under unions and intersections. Thus \( xT \) is a topology on \( X_{n+1} \).

ii) Now from the construction of \( \hat{x}T \), it is clear that for any topology \( T \) on \( X_n \), \( T \cup \{X_{n+1}\} \) is a topology on \( X_{n+1} \).

iii) By a similar argument as had been made for \( xT \), the set \( x'T \) is closed under unions and intersections. It need only be shown that in a conjoint topology, there are no sets in \( x'T \) such that their intersection is \( \{x\} \).

Assume that there are sets \( C, C' \) in \( x'T \) such that \( C \cap C' = \{x\} \). Then \( C = A_i \cup \{x\}, C' = A_j \cup \{x\} \), where \( A_i, A_j \neq \emptyset \). Then we have that \((A_i \cup \{x\}) \cap (A_j \cup \{x\}) = (A_i \cap A_j) \cup \{x\} \). So \( C \cap C' = (A_i \cap A_j) \cup \{x\} = \{x\} \). However, \( A_i \cap A_j \neq \emptyset \), since \( T \) is a conjoint topology. Thus \( C \cap C' \neq \{x\} \). So \( x'T \) is a topology on \( X_n \cup \{x\} = X_{n+1} \).

iv) Let \( D, D' \in xT \cup \hat{x}T \).

If \( D, D' \in xT \) or \( D, D' \in \hat{x}T \), then, as shown above, the set \( xT \cup \hat{x}T \) is closed under unions and intersections. Assume, without loss of generality, that \( D \in xT \), \( D' \in \hat{x}T \). Then \( D = A_i \cup \{x\} \) and \( D' = A_j \), for some \( A_i, A_j \in T \). So we have that

\[
D \cup D' = (A_i \cup \{x\}) \cup A_j = (A_i \cup A_j) \cup \{x\} \in xT \subset xT \cup \hat{x}T
\]
and

\[
D \cap D' = (A_i \cup \{x\}) \cap A_j \\
= (A_i \cap A_j) \cup (\{x\} \cap A_j) \\
= A_i \cap A_j \\
\subseteq \hat{x}T \\
\subseteq xT \cup \hat{x}T
\]

so \(xT \cup \hat{x}T\) is a topology on \(X_n \cup \{x\} = X_{n+1}\).

v) By similar argument, we have \(x'T \cup \hat{x}T\) is a topology on \(X_{n+1}\) as long as \(T\) is conjoint, shown in case (ii).

\[
\square
\]

We now investigate whether every topology on an \(n + 1\) element set can be obtained by applying of the the operations \(T \rightarrow xT, T \rightarrow x'T, T \rightarrow \hat{x}T, T \rightarrow xT \cup \hat{x}T, \) or \(T \rightarrow x'T \cup \hat{x}T\) to an \(n\) element set. Letting \(\mathcal{T}_n\) be the collection of all topologies on \(X_n\) and \(T_{n,k}\) be the \(k\)th topology on \(X_n\), the following examples will serve to better illuminate the sets defined above.

**Example 5.** Let \(X_0 = \emptyset\). Then the only topology on \(X_0\) is \(T_{0,1} = \{\emptyset\}\). Let \(X_1 = \{a\}\). Then the only topology on \(X_1\) is \(T_{1,1} = \{\emptyset, \{a\}\}\). We note that \(T_{1,1} = aT_{0,1}\).

Then \(\mathcal{T}_{X_1} = \{T_{1,1}\}\).
Choose \( b \not\in X \). Then we have

\[
\begin{align*}
\hat{b}T_{1,1} &= \{\emptyset, \{b\}, \{a,b\}\} \\
\hat{b}T_{1,1} &= \emptyset, \{a,b\} \\
b'T_{1,1} &= \emptyset, \{a\}, \{a,b\} \\
bT_{1,1} \cup \hat{b}T_{1,1} &= \emptyset, \{b\}, \{a,b\} \\
bT_{1,1} \cup b'T_{1,1} &= \emptyset, \{a\}, \{b\}, \{a,b\}.
\end{align*}
\]

Let \( X_2 = \{a,b\} \). The topologies on \( X_2 \), and their expressions using the above operations, are as follows:

\[
\begin{align*}
T_{2,1} &= \emptyset, \{a,b\} = \hat{b}T_{1,1} \\
T_{2,2} &= \emptyset, \{a\}, \{a,b\} = b'T_{1,1} \\
T_{2,3} &= \emptyset, \{b\}, \{a,b\} = bT_{1,1} \cup \hat{b}T_{1,1} \\
T_{2,4} &= \emptyset, \{a\}, \{b\}, \{a,b\} = bT_{1,1} \cup b'T_{1,1}.
\end{align*}
\]

Note that \( bT_{1,1} \cong b'T_{1,1} \cong bT_{1,1} \cup \hat{b}T_{1,1} \).

Applying our operation to the labeled topologies on \( \{a,b\} \) we obtain the following 17 labeled topologies on \( \{a,b,c\} \).
Table 2: Topologies generated from $X_2$

<table>
<thead>
<tr>
<th></th>
<th>$T_{2,1} = \emptyset, {a, b}$</th>
<th>$T_{2,2} = \emptyset, {a}, {a, b}$</th>
<th>$T_{2,4} = \emptyset, {a}, {b}, {a, b}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$cT_{i,j}$</td>
<td>$\emptyset, {c}, {a, b, c}$</td>
<td>$\emptyset, {c}, {a, c}, {a, b, c}$</td>
<td>$\emptyset, {c}, {a, c}, {b, c}, {a, b, c}$</td>
</tr>
<tr>
<td>$c't_{i,j}$</td>
<td>$\emptyset, {a, b}, {a, b, c}$</td>
<td>$\emptyset, {a}, {a, b}, {a, b, c}$</td>
<td>$\emptyset, {a}, {b}, {a, b}, {a, b, c}$</td>
</tr>
<tr>
<td>$\hat{c}T_{i,j}$</td>
<td>$\emptyset, {a, b}, {a, b, c}$</td>
<td>$\emptyset, {a}, {a, b}, {a, b, c}$</td>
<td>$\emptyset, {a}, {b}, {a, b}, {a, b, c}$</td>
</tr>
<tr>
<td>$cT_{i,j} \cup \hat{c}T_{i,j}$</td>
<td>$\emptyset, {c}, {a, b}, {a, b, c}$</td>
<td>$\emptyset, {a}, {c}, {a, b}, {a, c}, {a, b, c}$</td>
<td>$\emptyset, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}$</td>
</tr>
<tr>
<td>$c't_{i,j} \cup \hat{c}T_{i,j}$</td>
<td>$\emptyset, {a, c}, {a, b, c}$</td>
<td>$\emptyset, {a}, {a, b}, {a, c}, {a, b, c}$</td>
<td>$\emptyset, {a}, {b}, {a, b}, {a, c}, {b, c}, {a, b, c}$</td>
</tr>
</tbody>
</table>

* Note that $\hat{c}T_{2,4}$ and $c'T_{2,4} \cup \hat{c}T_{2,4}$ are not topologies on $X_3 = \{a, b, c\}$ since $T_{2,3}$ is a disjoint topology.
1. \{\emptyset, \{a, b, c\}\} \\
2. \{\emptyset, \{c\}, \{a, b, c\}\} \\
3. \{\emptyset, \{a, b\}, \{a, b, c\}\} \\
4. \{\emptyset, \{a, c\}, \{a, b, c\}\} \\
5. \{\emptyset, \{b, c\}, \{a, b, c\}\} \\
6. \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\} \\
7. \{\emptyset, \{b\}, \{a, b\}, \{a, b, c\}\} \\
8. \{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}\} \\
9. \{\emptyset, \{c\}, \{a, c\}, \{a, b, c\}\} \\
10. \{\emptyset, \{c\}, \{b, c\}, \{a, b, c\}\} \\
11. \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\} \\
12. \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\} \\
13. \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}\} \\
14. \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \\
15. \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}\} \\
16. \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\} \\
17. \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\} \\

This list contains at least one copy of each of the 9 inequivalent topologies possible on a 3
element set, which are listed below.

\[
\begin{align*}
T_{3,1} & = \{\emptyset, \{a,b,c\}\} = c'T_{2,1} \\
T_{3,2} & = \{\emptyset, \{c\}, \{a,b,c\}\} = cT_{2,1} \\
T_{3,3} & = \{\emptyset, \{a,c\}, \{a,b,c\}\} = c'T_{2,2} \\
T_{3,4} & = \{\emptyset, \{c\}, \{a,c\}, \{a,b,c\}\} = cT_{2,2} \\
T_{3,5} & = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\} = c'T_{2,4} \\
T_{3,6} & = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\} = c'T_{2,2} \cup cT_{2,2} \\
T_{3,7} & = \{\emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}\} = cT_{2,2} \cup cT_{2,2} \\
T_{3,8} & = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\} = cT_{2,4} \cup cT_{2,4} \\
T_{3,9} & = \{\emptyset, \{c\}, \{a,b\}, \{a,b,c\}\} = cT_{2,1} \cup cT_{2,1}
\end{align*}
\]

To obtain all the equivalent topologies on \{a,b,c\}, we could apply our operations to \{a,c\} using b, and to \{b,c\} using a.

From this work I hoped to prove that all topologies on a set \(X_{n+1}\) were in fact one of the above defined sets derived from a topology on \(X_n\). However, a counter-example was found when examining the recursion from \(X_3\) to \(X_4\).

Let \(X_4 = \{a,b,c,d\}\), \(X_3 = \{a,b,c\}\) and consider the topology \(T = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}\) on \(X_4\). Let \(T'\) be some topology on \(X_3\).

Since \(\{a\} \in T\), we cannot have \(T = dT'\) or \(T = d'T\). Since \(X_3 \notin T\), and \(X_3 \notin dT'\) for all \(T'\), we cannot have \(T = d'T', T = dT' \cup d'T'\) or \(T = d'T' \cup d'T'\).

This construction would not ultimately work to produce all of the topologies on \(X_{n+1}\) from the topologies on the set \(X_n\).

However, this counter-example provided insight into the way that a topology can be broken down into topologies on a set of cardinality one less. Rather than working from \(X_n\)
to $X_{n+1}$, the opposite direction was pursued in order to determine if there was some pattern to why the recursive definition had failed.
5. PARTITIONING TOPOLOGIES

As the recursive definition of the last section failed to capture all topologies, we study further how topologies on an \( n + 1 \) element set are related to topologies on an \( n \) element subset. The goal is to obtain insight into how to make a recursive definition for producing all the finite topologies.

Given any topology \( T \) on a set \( X \) and an element \( a \in X \) there are two natural ways to obtain a topology on the set \( X - \{a\} \). Observe that the sets in \( T \) which contain \( a \) are closed under unions and intersections, so those sets together with \( \emptyset \) form a sub-topology on \( X \) and we can obtain a topology on \( X - \{a\} \) by deleting \( a \) from every set in the sub-topology. Similarly we note that the sets in \( T \) which do not contain \( a \) are also closed under unions and intersections, so by taking these sets together with \( X - \{a\} \) we obtain a topology on \( X - \{a\} \).

More formally, we define three sets \( S_a, S_{\bar{a}}, \) and \( \bar{S}_a \) as follows: \( S_a = \{ A_i \in \bar{T} | a \in A_i \} \), \( S_{\bar{a}} = \{ A_i \in \bar{T} | a \notin A_i \} \) and finally \( \bar{S}_a = \{ A_i - \{a\} | A_i \in S_a \} \).

**Definition 5.** Let \( T_a = \bar{S}_a \cup \{\emptyset, X\} \) and \( T_{\bar{a}} = S_{\bar{a}} \cup \{\emptyset, X\} \).

We record our initial observations in the following proposition.

**Proposition 2.** Let \( T \) be a topology on a set \( X \) and let \( a \in X \). Then \( T_a \) and \( T_{\bar{a}} \) are topologies on \( X - \{a\} \).

**Example 6.** Let \( T = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c,d\}, X_4\} \). For each choice of an element of \( X_4 \), the partitioning can produce different (though not necessarily inequivalent) topologies on \( X_3 \), some three element set, as shown below:
\[T_a = \{\emptyset, \{b\}, \{cd\}, X_3\} \text{ for } X_3 = \{b, c, d\}\]
\[T_{\bar{a}} = \{\emptyset, \{b\}, X_3\}\]
\[T_b = \{\emptyset, \{a\}, X_3\} \text{ for } X_3 = \{a, c, d\}\]
\[T_{\bar{b}} = \{\emptyset, \{a\}, X_3\}\]
\[T_c = \{\emptyset, \{a, d\}, X_3\} \text{ for } X_3 = \{a, b, d\}\]
\[T_{\bar{c}} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X_3\}\]
\[T_d = \{\emptyset, \{a, c\}, X_3\} \text{ for } X_3 = \{a, b, c\}\]
\[T_{\bar{d}} = \{\emptyset, \{a\}, \{b\}, \{a,b\}, X_3\}.\]

It is easy to see that \(T_{\bar{a}} \cong T_{\bar{b}} \cong T_b, T_c \cong T_d, \text{ and } T_{\bar{c}} \cong T_{\bar{d}}.\)

Next we examined whether there were certain properties of \(T\) which would be preserved by the partitioning process. Most notably the \(T_0\) property was preserved.

**Lemma 1.** Every \(T_0\) topology contains a singleton set.

**Proof.** Let \(T\) be a \(T_0\) topology on a set \(X\) and assume that \(T\) does not have a singleton set.

Let \(A, |A| \geq 2,\) be the non-empty set in \(T\) with the fewest number of elements.

Let \(a_1, a_2 \in A, a_1 \neq a_2.\)

Then there exists a \(B \in T\) such that, without loss of generality, \(a_1 \in B\) and \(a_2 \notin B\) since \(T\) is a \(T_0\) topology.

Then \(a_1 \in A \cap B\) and \(a_2 \notin A \cap B.\) This implies that \(|A \cap B| < |A|,\) which is a contradiction. Thus, \(T\) contains a singleton set.

\(\square\)
Proposition 3. If $T$ is a $T_0$ topology on $X_{n+1}$, then for some $a \in X_{n+1}$, $T_a$ is a $T_0$ topology on $X_n$.

Proof. Let $T$ be a $T_0$ topology on $X_{n+1}$.

By Lemma 1, there exists an $a \in X_{n+1}$ such that $\{a\} \in T$. Then we have from Proposition 2 we have that $T_a$ is a topology on $X_n$. It will suffice to show that $T_a$ is $T_0$ on $X_n$.

Let $u, v \in X_n$. Then $u, v$ are distinct points with respect to $T$. So, there exists an open set $U \in T$ such that, without loss of generality, $u \in U, v \notin U$. Since $\{a\} \in T$ then $W = U \cup \{a\} \in T_a$ and $u \in W - \{a\}$ and $v \notin W - \{a\}$.

Thus $T_a$ is a $T_0$ topology on $X_{n+1} - \{a\} = X_n$.

By a simple counter-example, however, we see that $T_{\bar{a}}$ is not necessarily $T_0$ if $T$ is. Letting $T = \{\emptyset, \{a\}, \{a,b\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}, X_4\}$ we have that $T_{\bar{a}} = \{\emptyset, \{b,c\}, X_3\}$, which is not a $T_0$ topology on $X_3 = \{b,c,d\}$. However, in the case of the topology $T = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{b,c\}, X_3\}$, $T_{\bar{a}} = \{\emptyset, \{b\}, \{b,c\}, X_3\}$ is a $T_0$ topology. Ultimately, the conditions under which $T_{\bar{a}}$ preserved the $T_0$ property led to the definition of a maximal, complete chain, which is explored in the following section and became the central focus of my research from this point forward.

I turned to contemporary research to try and help elucidate some of the reasons why the construction worked for some topologies and not for others. In [5], Erné showed that $T(n)$, the number of topologies on a set of $n$ elements, is asymptotically equal to $T_0(n)$, the number of $T_0$ topologies on $n$ elements. For smaller $n$, Evans et al. [6] and Renteln [10] reduced the computation of $T(n)$ to the number of partially ordered set on $n$ elements. These values are connected by the following formula:
Let $T(n)$ denote the number of distinct topologies on a set with $n$ points. The number of distinct $T_0$ topologies on a set with $n$ points, denoted $T_0(n)$, is related to $T(n)$ by the formula

$$T(n) = \sum_{k=0}^{n} S(n,k) T_0(k)$$

where $S(n,k)$ denotes the Stirling number of the second kind.

Since all the topologies on $X_{n+1}$ are enumerated in terms of all the $T_0$ topologies on $X_0 \ldots X_n$ I decided to focus on just the $T_0$ topologies.
6. MAXIMAL COMPLETE CHAINS

Solely considering $T_0$ topologies on a finite set, I examine their Hasse diagrams.

Uniformly, these Hasse diagrams contained a nested collection of sets whose order increased by one at each level of the diagram, from $\emptyset$ to $X_n$. It was this feature, seen in Figure 10 and 11, which is defined as a maximal, complete chain (or MC-chain).

![Sample Hasse diagrams of topologies on the set $X_5$, [4]](image)

**Figure 9: Sample Hasse diagrams of topologies on the set $X_5$, [4]**

**Definition 6.** Let $T = \{\emptyset, A_1, A_2, \ldots, A_{n-1}, X_n\}$ be a topology on $X_n$. Then $T$ is a maximal, complete chain (or MC-chain) if for all $A_i \in T$, $|A_i| = i$, and $A_i \subseteq A_{i+1}$, where $A_0 = \emptyset$ and $A_n = X_n$.

For example, on the set $X = \{a, b, c, d\}$, the set $\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\}$ is a maximal, complete chain.

**Example 7.** Pictured below are two more examples of MC-chains, highlighted in the Hasse diagrams of two $T_0$ topologies on $X_5$ and $X_6$. The first represents the unlabeled $T_0$ topology

$$\{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, c, d\}, \{a, b, c, d, e\}\}$$

with the MC-chain $\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, \{a, b, c, d, e\}\}$ highlighted. In each Figure, one can see that there are in fact several possible MC-chains in each $T_0$ topology. This property of MC-chains will be discussed more fully later in the section.
Lemma 2. All maximal complete chains are $T_0$ topologies.

Proof. Let $T$ be an MC-chain on the set $X_n$. Then $\emptyset, X_n \in T$. Also, for any $A_i, A_j \in T$, $A_i \not\subseteq A_j$ or $A_j \not\subseteq A_i$. Without loss of generality, assume the former.

Then $A_i \cup A_j = A_j$ and $A_i \cap A_j = A_i$, both sets in $T$. Thus, $T$ is closed under unions and intersections.

Finally, assume that for some $x, y \in X_n$, $x, y \not\in A_i$ and $x, y \in A_{i+1}$. However, this would imply that $|A_i| + 2 = |A_j|$, in which case $T$ is not an MC-chain. Thus, for any $x, y \in X_n$, there exists some $A_{i+1}$ such that $x \in A_{i+1}, y \not\in A_{i+1}$. 

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Thus, $T$ is a $T_0$ topology.

Remark 1. An MC-chain is a minimal $T_0$ topology on a set of $n$ elements.

Letting $T$ be a proper subset of some MC-chain $\mathcal{A}$, we note that if we omit any set $A_i \in \mathcal{A}$, then the two elements in $A_{i+1} - A_{i-1}$ are topologically indistinguishable.

It should be noted that similar constructions have been used in modern research in finite set topology. In [1], the Adamenko and Velichko use a topological quiver, or T-quiver, from $\emptyset$ to $X_n$. While slightly different in its construction from an MC-chain, the T-quiver serves the same purpose - creating a path of open sets in a $T_0$ space. The authors asserts that “[a]t the $k$th level of an arbitrary T-quiver of $T_0$-topology, there are vertices corresponding to $k$-element open sets”, which is, in terms of MC-chains, the fact that $|A_i| = i$. The major difference between the T-quivers and MC-chains is only that some union of MC-chains would make up a T-quiver. They go on to prove that for any topology on an $n$-element set, it can only be $T_0$ if its corresponding T-quiver has $n$ levels.

Adamenko considers the empty set as the zero level of the T-quiver, so this statement is equivalent to Lemma 3 below, that $T$ is $T_0$ if it contains an MC-chain.

Lemma 3. If $T$ is $T_0$ on $X_n$, then $T$ contains a maximal complete chain.

Proof. Proof by induction.

As $n = 1$ is a trivial case, we start with $n = 2$.

So $X_2 = \{x_1, x_2\}$. There are two inequivalent $T_0$ topologies on $X$:

- $\{\emptyset, \{x_1\}, \{x_1, x_2\}\}$
- $\{\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}\}$
The first is a maximal, complete chain (MC-chain) and is also a subset of the second. So for our base case the proposition holds.

Assume that for all \( k < n \) the proposition holds where \( n > 2 \).

Let \( T \) be a \( T_0 \) topology on \( X_n \).

By Lemma 1, \( T \) contains a singleton set \( \{x_1\} \).

Let \( Y = X_n - \{x_1\} \) and define \( T_x = \{A - \{x_1\}|A \in T\} \).

Since \( T \) is \( T_0 \) on \( X \), \( T_x \) is \( T_0 \) on \( X - \{x_1\} = Y \).

Since \( T_x \) is \( T_0 \) on a set \( Y \) of \( n - 1 \) elements and \( n - 1 \geq 2 \), \( T_x \) contains an MC-chain \( \emptyset : x_2 : x_2x_3 : \ldots : x_2\ldots x_n \) on \( Y \), by the induction hypothesis.

This give us the MC-chain \( \emptyset : x_1 : x_1x_2 : x_1x_2x_3 : \ldots : x_1x_2\ldots x_n \) as a subset of \( T \) on \( X_n \).

So we have that each \( T_0 \) topology contains at least one MC-chain. Examination of the Hasse diagrams shows that in fact all sets of a \( T_0 \) topology are a member set of an MC-chain.

Figure 12: \( T_0 \) topology on \( X_5 \), some distinct MC-chains highlighted
**Theorem 1.** Every open set in a $T_0$ topology is an element of an MC-chain.

*Proof.* Let $T$ be a $T_0$ topology on $X_n$.

Then there exists an MC-chain $\mathcal{A} = \{\emptyset, A_1, A_2, \ldots, A_{n-1}, X\}$ that is a subset of $T$.

Let $B \in T$ such that $B \notin \mathcal{A}$.

Then there exists an $A_i \in \mathcal{A}$ such that $B \subset A_i$ and an $A_j \in \mathcal{A}$ such that $A_j \subset B$, even if these are trivially $\emptyset$ and $X$, respectively. Thus, we have already the following subsets of $\mathcal{A}$ as parts of our MC-chain which contains $B$: $\{\emptyset, \ldots, A_j\}$ and $\{A_i, \ldots, X\}$.

Note that $B \cap A_j = B$.

Consider the sequence of sets
$\emptyset, A_1 \cap B, A_2 \cap B, \ldots, A_{n-1} \cap B, B, B \cup A_1, B \cup A_2, \ldots, B \cup A_{n-1}, X_n$. Each set in the sequence is either equal to the previous one, or is the union of the previous set with a singleton. Therefore if we remove the duplicates, we obtain an MC-chain containing $B$.

Since every $T_0$ topology $T$ contains an MC-chain, these are the only minimal $T_0$ topologies. Also, we have that every set in $T$ is in some MC-chain, which is visually apparent in the Hasse diagrams as each node lies on a direct path in the diagram of length $n + 1$ from the empty set to $X$. Thus we may say the following.

**Remark 2.** Every $T_0$ topology is a union of MC chains.

It should be noted that the converse doesn’t hold. Consider the MC-chains on $X_3$:
$\mathcal{A} = \{\emptyset, \{a\}, \{a, b\}, X_3\}$ and $\mathcal{B} = \{\emptyset, \{c\}, \{b, c\}, X_3\}$. Their union is not even a topology on $X_3$, as it lacks the open sets $\{\{b\}, \{a, c\}\}$. 

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Once we have that each $T_0$ topology is the union of MC-chains, and since the set $P(X)$, the power set of $X$, is $T_0$ on $X$, $P(X)$ is the union of, specifically, $n!$ distinct MC-chains. By “distinct”, we mean that for two MC-chains $\mathcal{A}, \mathcal{B}$, there exists at least one set $B \in \mathcal{B}$ such that $B \notin \mathcal{A}$. However, the intersection of the MC-chains need not be empty.

The question then became whether or not there was some method in which these MC-chains could be used to form the topologies on a set $X_n$. The development of MC-chains was for the purpose of finding a recursive definition with which the topologies on a set could be produced. If each set in a $T_0$ topology is part of an MC-chain, could a relationship between these MC-chains be found for the purposes of recursion? It became necessary to make a more rigorous definition for the specific relationship between the MC-chains which whose union forms a $T_0$ topology.

**Definition 7.** Two MC-chains $\mathcal{A}, \mathcal{B}$ are **adjacent** if they differ by only one set.

Note that given any topology $T$, Definition 7 defines a graph structure $G$ on the set of all MC-chains of possible on a set $X$. When $|X| = n$ the graph has $n!$ vertices because MC-chains are in a one-to-one correspondence with the ways to arrange the elements of $X$ in order. Also, two MC-chains are adjacent if and only if the corresponding orderings of $X$ are related by swapping two adjacent spots in the ordering. So the graph has $n!$ vertices and is $(n - 1)$-regular. Given a topology $T$ on $X$ we consider the full sub-graph $G_T$ consisting of the MC-chains and adjacency edges which are in the topology $T$.

**Theorem 2.** For any topology $T$ on a finite set $X$, the graph $G_T$ is connected.

**Proof.** Let $\mathcal{A} \not\subseteq T$ be an MC-chain, where $T$ is a non-minimal $T_0$ topology on $X_n$. Since $T$ is non-minimal, there exists a set $B \in T$ such that $B \notin \mathcal{A}$. By Theorem 1, let $B \in \mathcal{B}$, an MC-chain in $T$. Thus $\mathcal{A}, \mathcal{B}$ are distinct vertices of $G_T$.

Let $A_i \in \mathcal{A}$ and $B_i \in \mathcal{B}$, where $i$ is the smallest value such that $A_i \neq B_i$. We note that $|A_i| = |B_i| = i$. 

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We will consider this problem in cases.

First, assume $A_i \cup B_i = A_{i+1}$. Then $C = \{\emptyset, A_1, A_2, \ldots, A_{i-1}, B_i, A_{i+1}, \ldots, A_{n-1}, X_n\}$ is an adjacent MC-chain to $A$ on the graph $G_n$.

Now, assume $A_i \cup B_i \neq A_{i+1}$. Note that there exists some smallest possible set $A_{i+j} \in A$ such that $B_i \subseteq A_{i+j}$. So $R - A = \{B_i, B_{i+1}, \ldots, B_{i+j-1}\}$, and thus has order $j$. Then consider the MC-chain $C_1 = \{\emptyset, A_1, \ldots, A_i, A_i \cup B_i, A_{i+1} \cup B_i, \ldots, A_{i+j-1} \cup B_i, A_{i+j}, \ldots, X_n\}$. The set $A_{i+j-1} \cup B_i$ is the last set of $C_1$ which is not also an element of $A$. So $C_1 - A = \{A_i \cup B_i, A_{i+1} \cup B_i, \ldots, A_{i+j-1} \cup B_i\}$, and $|C_1 - A| = j - 1$. Thus the process has produced an MC-chain $C_1$ which has one few set of difference from $A$ than $R$ has.

Continue this construction recursively, taking the first set of $C_1$ which is not in $A$ and taking its union with all the sets of higher order in $A - C_1$ to produce $C_2$. Thus we have the following sequence of MC-chains:

\[
\begin{align*}
R &= \{\emptyset, A_1, \ldots, B_i, \ldots, B_{i+j-1}, A_{i+j}, \ldots, A_{n-1}, X_n\} \\
C_1 &= \{\emptyset, A_1, \ldots, A_i, A_i \cup B_i, \ldots, A_{i+j-1} \cup B_i, A_{i+j}, \ldots, A_{n-1}, X_n\} \\
C_2 &= \{\emptyset, A_1, \ldots, A_i, A_{i+1} \cup B_i, \ldots, A_{i+j-1} \cup B_i, A_{i+j}, \ldots, A_{n-1}, X_n\} \\
&\vdots \\
C_k &= \{\emptyset, A_1, \ldots, A_{i+j-1}, A_{i+j-1} \cup B_i, A_{i+j}, \ldots, A_{n-1}, X_n\} \\
A &= \{\emptyset, A_1, \ldots, A_{i+j-1}, A_{i+j}, A_{i+j+1}, \ldots, A_{n-1}, X_n\}
\end{align*}
\]

for some $k \in \mathbb{N}$. This sequence corresponds to a path on $G_{X_n}$ from $A$ to $R$. 

$\square$
Thus we have arrived at the fact that all $T_0$ topologies correspond to a connected subgraph of $G_n$. With this in mind, we revisit the notion of a recursive definition. However, our focus remains on $T_0$ topologies.
7. RECURSIVE DEFINITION OF $T_0$ TOPOLOGIES

In this section we show how an extension of the operations considered in Chapter 4 will lead to generating all of the $T_0$ topologies on a set $X$ with $n + 1$ elements from those on a set with $n$ elements.

We know every topology on an $n + 1$ element set contains the MC-chain $\emptyset : x_1 : \ldots : x_1 \ldots x_n : x_1 \ldots x_n x_{n+1}$. Given some $T_0$ topology $T$ on $X_{n+1}$, let $T^* = \{A \in T \mid x_{n+1} \notin A\}$. Then $T^*$ is a topology on $X_n$.

We note that in fact $T^*$ is $T_0$ on $X_n$. If $u \neq v$ are in $X_n$, then there exists sets $A, B \in T$ such that $u \in A, b \notin A$ or $u \notin A, v \in A$. Without loss of generality, assume the former. But then $A_1 = A \cap x_1 x_2 \ldots x_n \in T^*$ and $u \in A_1$ and $v \notin A_1$.

Now suppose that $U$ and $V$ are in $T^*$ and $U \cup \{x_{n+1}\}, V \cup \{x_{n+1}\} \in T$. Then $(U \cap V) \cup \{x_{n+1}\} = (U \cup \{x_{n+1}\}) \cap (V \cup \{x_{n+1}\})$ is in $T$ also. So the collection of sets $U \in T^*$ such that $U \cup \{x_{n+1}\} \in T$ is closed under intersection. We obtain a minimal such set $W \in T^*$ by taking the intersection of all the sets with this property.

We also note that if we start with any $T_0$ topology $T^*$ on $X_n$, such that $x_{n+1} \notin X_n$ and $W \in T^*$, then $T = T^* \cup \{A \cup \{x_{n+1}\} \mid A \in T^* \text{ and } W \subset A\}$ is a $T_0$ topology on $X_{n+1}$.

In terms of the Hasse diagrams, we may interpret this as follows. We take the Hasse diagrams of the inequivalent topologies on a set with $n$ elements. In each diagram select a vertex $v$. We take the set $U$ of all vertices $u$ such that $v \leq u$. For each $u \in U$, we introduce a new vertex $u'$, which represents the addition of $\{x_{n+1}\}$ to the set represented by $u$ and draw an edge from $u$ to $u'$. We also draw edges from $u'_1$ to $u'_2$ whenever there is an edge from $u_1$ to $u_2$.

In this manner we obtain all the Hasse diagrams for the unlabeled topologies on the set of $n + 1$ elements.
Definition 8. Let $\mathcal{T}_n = \{T_0\text{ topologies on } X_n\}$. Let $S_n = \{(T,A) \mid T \in \mathcal{T}_n \text{ and } A \in T\}$. Define the mapping $L : S_n \to \mathcal{T}_{n+1}$ by

$$L(T,A) = \{B \mid B \in T\} \cup \{B \cup \{x_{n+1}\} \mid B \in T, A \subseteq B\},$$

for some $x_{n+1} \notin X_n$.

Theorem 3. $L$ is onto.

Proof. Given a $T_0$ topology $T$ on a set with $n+1$ elements we follow the notation of the preceding discussion. We find an element $x_{n+1}$ which is the last to occur as an element of a set of some MC-chain, the $T_0$ topology $T^*$ on $X_n = X_{n+1} - \{x_{n+1}\}$ and a minimal set $W \in T^*$ such that $W \cup \{x_{n+1}\} \in T$. Then $L(T^*,W) = T$.

Then we can recover $T$ from $T^*$ and $W$ by the relation $T = T^* \cup \{A \cup \{x_{n+1}\} \mid A \in T^* \text{ and } W \subseteq A\}$.

Thus we have found a recursion such that, given the $T_0$ topologies on $X_n$, all of the $T_0$ topologies on $X_{n+1}$ can be produced. There is necessarily some double counting, as will be shown in the following example.

Example 8. Let $\mathcal{T}_2 = \{T_{2,1}, T_{2,2}, T_{2,3}\}$ be the set of $T_0$ topologies on the set $X_2$, where $T_{2,1} = \{\emptyset, \{a\}, X_2\}, T_{2,2} = \{\emptyset, \{b\}, X_2\}, T_{2,3} = \{\emptyset, \{a\}, \{b\}, X_2\}$ and $X_2 = \{a, b\}$. Choose $c \notin X_2$. I will show that the recursion defined above produces all five inequivalent $T_0$ topologies on $X_3$.

First we will apply the recursion to $T_{2,1}$. Note that since $T_{2,1}$ is homeomorphic to $T_{2,2}$, the $T_0$ topologies produced by $T_{2,2}$ in this process would also be homeomorphic to those produced by $T_{2,1}$. For that reason, the process is not shown for $T_{2,2}$. In the Figure 13 below, we selected in turn each element $A$ of $T_{2,1}$. From there we construct the Hasse
diagrams for a $T_0$ topology on $X_3$ by letting $S = \{ \{c\} \cup B \mid A \subseteq B \in T_{2,1} \}$ and then taking the union of $S$ and $T_{2,1}$.

The process is repeated then repeated for $T_{2,3}$ in Figure 14.

As can be seen, $T_{3,1} \cong T_{3,5} \cong T_{3,6}$, yet all the elements of $\mathcal{S}_3$ are produced.

We also have from our previous formula which relates the number of topologies on a set $X_n$ to the number of $T_0$ topologies on $\{X_0, X_1, \ldots, X_n\}$ that all non-$T_0$ topologies are produced from $T_0$ topologies. It is in this way that the Stirling numbers of the Second Kind $S(n, k)$ are required [7], as $S(n, k)$ is the number of partitions of $n$ elements into $k$ open sets.

**Example 9.** Consider the indiscrete topology on $X_n$, $T = \{\emptyset, X_n\}$. When $n = 1$, $T = \{\emptyset, \{a\}\}$ and $T$ is the power set on $X_1$, therefore $T_0$. It is from this $T_0$ topology that the
indiscrete topology is produced for all \( n > 1 \). There is only one way to partition \( n \) elements into 1 set, since no element can be placed in \( \{\emptyset\} \). Thus, \( S(n, 1) = 1 \) for all \( n \).

As \( n \) increases, the complexity of producing non-\( T_0 \) topologies from \( T_0 \) topologies increases as well.

**Example 10.** Consider the \( T_0 \) topology \( T = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \) on \( X_2 \). To produce topologies on \( X_3 \), we find all the possible partitions of \( \{a, b, c\} \) into two open sets (as no element may be placed in \( \emptyset \) and all elements must be placed in \( X_3 \)). This gives us the \( S(3, 2) = 3 \) following topologies on \( X_3 \):

\[
\begin{align*}
T_1 & = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\} \\
T_2 & = \{\emptyset, \{b\}, \{a, c\}, \{a, b, c\}\} \\
T_3 & = \{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}\},
\end{align*}
\]

which our recursive definition from Chapter 4 failed to produce.

See Appendix A for the Hasse diagrams for the unlabeled topologies on \( X_2, X_3, X_4 \), produced in this fashion, as well as the diagrams for the unlabeled topologies on \( X_5 \) [4].
8. SUMMARY

After some unsuccessful efforts we found a recursive definition for generating all $T_0$ topologies on finite sets. Our initial goal was to obtain a graphical representation of all the topologies for small finite sets.

Mimicking earlier work, the author initially attempted to construct a graph of each $T_0$ topology on $X_3$, with edges defined by $v_1 \leftrightarrow v_2$ if $v_1 \subseteq v_2$ or $v_2 \subseteq v_1$, as seen in Figure 15. As before, $P(X_3)$ is not pictured. For higher values of $n$ this construction quickly becomes too cluttered to be of any value. While there are only 19 inequivalent $T_0$ topologies on a set of three elements, there are 219 on a set of four elements. It is for this reason that this construction was quickly abandoned in favor of a simpler, more compact approach, as the author had done in abandoning the previous shell structures.

![Figure 15: Graph of labeled $T_0$ topologies on $X_3$](image)

For that reason, we return to the graph of all MC-chains described in Definition 7, which we denote here by $G_n$ for chains on a base set with $n$ elements.
Example 11. Let $X_3 = \{a, b, c\}$. Then $|V(G_3)| = 6$, and each element of $V(G_3)$ has degree 2. Pictured below in Figure 16 is $G_3$.

The construction was then extended to $X_4$. The resulting 24-node 3-regular graph is depicted in Figure 17.

While an elegant representation, the graph $G_n$ has some of the same constraints on its analysis as $N_n$ did. Firstly, while every $T_0$ topology is a connected subgraph of $G_n$, not every connected subgraph is in turn a $T_0$ topology. So yet again we encounter the issue of how to specifically construct a subgraph of $G_n$ such that the resulting union of vertices gives a topology. In $G_n$ every vertex represents a minimal $T_0$ topology, which has $n + 1$ open sets. Every pair of adjacent edges represents a topology with $n + 2$ open sets. Any vertex together with any subset of its immediate neighbors gives a representation of a $T_0$ topology obtained by taking the union of all the chains represented by those vertices. The simplest connected subsets of $G_4$ which do not represent topologies are some paths with four vertices. For example, the path $(d, cd, bcd) - (c, cd, bcd) - (c, bc, bcd) - (b, bc, bcd)$ does not represent a topology. The singleton sets with elements $b$ and $d$ are represented,
Figure 17: $G_4$

but not the union of these sets.

The initial goal of presenting graphical representations of topologies quickly proved intractable. However we did find many interesting relationships between the topologies, and graphical structures which illustrated these relationships for small sets. And while we succeeded in giving a recursive rule to generate all finite $T_0$ topologies, we can produce a topology on $n + 1$ elements from one on $n$ elements in many different ways. The enumeration of the $T_0$ topologies remains a challenging problem.
APPENDIX A: UNLABELED HASSE DIAGRAMS OF TOPOLOGIES

A.1 TOPOLOGIES ON $X_2$

A.2 TOPOLOGIES ON $X_3$

A.3 TOPOLOGIES ON $X_4$
A.4 TOPOLOGIES ON $X_5$

1. Draw the Hasse diagrams for all the 139 different topologies for a set of five elements $X = \{a, b, c, d, e\}$.

Solution.

For set of 5 elements, the Hasse diagrams of the 139 different topologies are:
All images for Appendix A.4 were produced by Choo, [4].
REFERENCES


