A NEW INTERPRETATION OF THE MATRIX TREE THEOREM USING WEAK WALK CONTRIBUTORS AND CIRCLE ACTIVATION

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Ellen Beth Robinson

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Ellen Beth Robinson

Thesis Supervisor:

Lucas Rusnak, Ph.D.
Department of Mathematics

Second Reader:

Daniela Ferrero, Ph.D.
Department of Mathematics

Approved

Heather C. Galloway, Ph.D.
Dean, Honors College

Abstract

This thesis provides an alternate proof of the Matrix Tree Theorem by shifting the focus to oriented incidences. We examine the weak walk contributors from the determinant of the Laplacian matrix of oriented graphs and classify them according to similar circle structures attained through circle activation. The members of each of these contribution classes form an alternating rank-signed Boolean lattice in which all members cancel. We then restrict our contributors to those corresponding to a given cofactor L_{ij} and demonstrate that those contributors that no longer cancel are in one-to-one correspondence with the spanning trees of the graph. These results allow for possible extension into examining tree-counts in signed graphs and oriented hypergraphs.

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Finally I would like to thank my parents and loved ones that have been so supportive of my work. They have traveled across the state to see my thesis presentation and I am very thankful to have people surrounding me that continue to support the furthering of my mathematical education.

Contents

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1 Background and Definitions

The following definitions and theorems provide background into the methods used in this paper and possible extensions of the theory. In section 1.1, we begin by defining the specific type of graphs we are going to be working with. In section 1.2, we introduce weak walks and an important graphical structure, trees. In section 1.3, we build the graphic matrices that are used in calculating the Laplacian of a specific graph. In section 1.4, we introduce permutations which will be used to sort objects defined in section 2.

1.1 Graph Definitions

The following definitions restrict the type of graph that will be used in the following paper. Although the form of a graph used is restricted by not allowing loops or hyperedges, our approach allows for the extension to the generalized case of oriented hypergraphs (defined in [7]). The following definitions are adaptations of the definitions in [6] that have been modified to fit this paper.

V is a finite set of elements called *vertices* and E is a finite set of elements called *edges*. An *incidence function* is a function $\iota : V \times E \to \{0, 1\}$ such that a vertex and edge are said to be *incident* if $\iota(v, e) = 1$. The double (v, e) is an *incidence* where $\iota(v, e) = 1$. $\mathcal{I}$ denotes a set of incidences determined by ι. An *incidence orientation* is a function $\sigma : \mathcal{I} \to \{+1, -1\}$. An *oriented graph* G is a quadruple $(V, E, \mathcal{I}, \sigma)$ such that:

1. $\sigma(v, e)\sigma(w, e) = -1$.

2. Each edge appears in exactly 2 incidences.

This notation for a graph has been chosen in order to place an emphasis on the incidences of the graph. The results in section 3 use the incidences to develop a new approach to look at the Matrix Tree Theorem. Note that statement 2 from the definition for an oriented graph disallows hyperedges and loops.

The *degree* of vertex v, denoted by $deg(v)$ is equal to the number of incidences containing v. Two distinct vertices v and w are said to be *adjacent with respect to edge* e if there exist incidences (v, e) and (w, e) such that $(v, e) \neq (w, e)$. An *adjacency* is a triple $(v, w; e)$ where v and w are adjacent with respect to e by incidences (v, e) and (w, e). The *sign of an adjacency* $(v, w; e)$ is defined as: $sgn_e(v, w) = -\sigma(v, e)\sigma(w, e)$. If v and w are not adjacent with respect to e, we say $sgn_e(v, w) = 0$. Notice that in this paper, all adjacencies are positively signed because the definition for an oriented graph requires that $\sigma(v, e)\sigma(w, e) = -1$ and thus $sgn_e(v, w) = 1$ for all $e \in E$. These definitions were adapted from [3]. Figure 1 depicts the three possibilities for the signing of adjacencies.

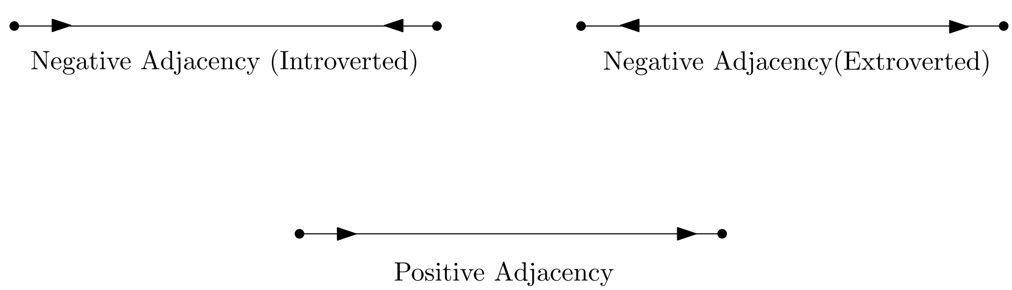

Figure 1: Positive and Negative Signed Adjacencies.

1.2 Walks, Circles, and Trees

In this section, we adapt the definition of a weak walk from [3], and develop the definition of a spanning tree of a graph G. The Matrix Tree Theorem proved in this paper counts the number of spanning trees in a graph. Following the definitions, we begin an example for a graph G_1 and display the spanning trees for this specific graph.

A *weak walk* is a sequence $\widetilde{W} = a_0, i_1, a_1, i_2, a_2, i_3, a_3, \ldots, a_{n-1}, i_n, a_n$ of vertices, edges, and incidences, where $\{a_k\}$ is an alternating sequence of vertices and edges, i_h is an incidence containing a_{h-1} and a_h and $a_0 \in V$, $a_n \in V$. A *walk* is a weak walk in which the condition $i_{2h-1} \neq i_{2h}$ is added. This implies that when leaving a vertex v, you must travel to another vertex w before returning to v. A weak walk of length 1 of the form v, i, e, i, v is called a *backstep* and because the unique incidence (v, e) appears twice, we will represent this by $(v, e)^2$.

The *sign of a weak walk* is:

$$sgn(\widetilde{W}) = (-1)^p \prod_{h=1}^{n} \sigma(i_h), \qquad \text{where } p = \lfloor n/2 \rfloor.$$

Thus the sign of a weak walk is the product of the signs of all incidences in the weak walk times -1 for every pair of incidences in the walk. This signing is a direct result of the signing for the adjacencies in the graph.

We let $\widetilde{w}(v_i, v_j, k)$ denote the number of weak walks of length k from v_i to v_j. The number of positive weak walks of length k from v_i to v_j is $\widetilde{w}^+(v_i, v_j, k)$, and the number of negative weak walks of length k from v_i to v_j is $\widetilde{w}^-(v_i, v_j, k)$. We let $\widetilde{w}^\pm(v_i, v_j, k) = \widetilde{w}^+(v_i, v_j, k) - \widetilde{w}^-(v_i, v_j, k)$.

A *closed weak walk* is a weak walk in which $a_0 = a_n$. This means that the first and last vertex of the walk are the same. A *circle* is a weak walk in which no vertex or edge is repeated except for $a_0 = a_n$. If the final condition is no longer required – thus a_0 does not have to equal a_n – the weak walk is called a *path.* A *connected graph* is a graph in which there exists a path from each vertex to every other vertex. Throughout this paper, all graphs are connected graphs.

A *tree* is a connected circle-free graph. A *spanning tree of G* is a tree that contains all vertices of the graph. It is known from [4] that every connected graph is a tree if and only if it contains $|V| - 1$ edges. Thus, a spanning tree of G contains $|V|$ vertices, $|V| - 1$ edges, and no circles. Consider the oriented graph G_1 in Figure 2.

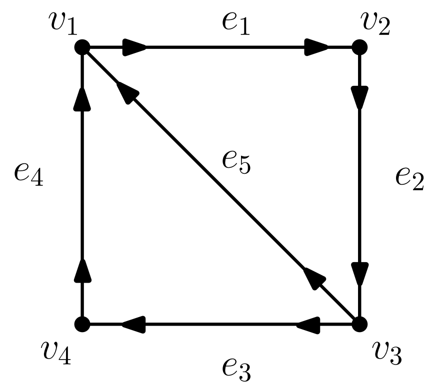

Figure 2: The incidence-oriented graph G_1.

Figure 3 below depicts the eight possible spanning trees of G_1. Notice that each of the trees has been created by removing two of the original edges.

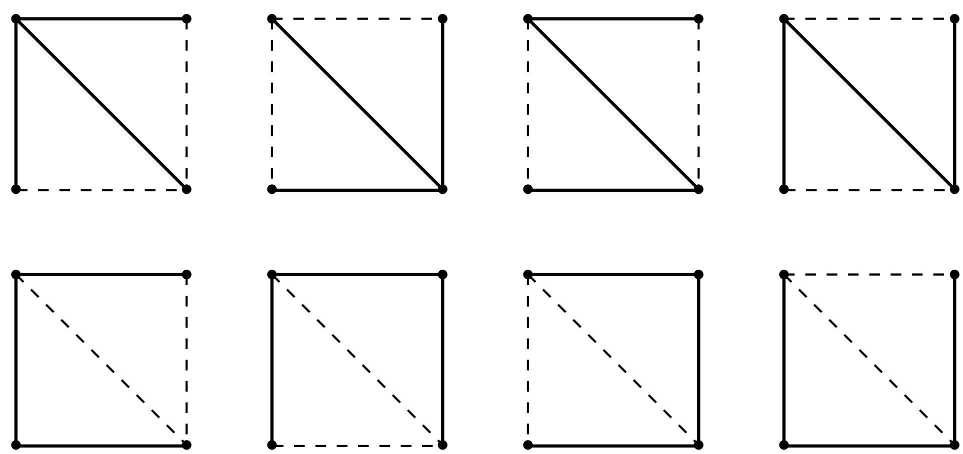

Figure 3: All 8 spanning trees of G_1 (the removed edges are left as dashed lines).

1.3 Graphic Matrices

In this section, we define five graphic matrices and provide examples of the matrices corresponding to the graph G_1 in Figure 2. We also discuss what the entries in certain matrices represent in the graph. This is important because the link between weak walks and the Laplacian matrix is the key idea to the following alternative proof of the Matrix Tree Theorem. The following definitions and theorems are adapted from [6].

The *degree matrix* of a graph G is a $V \times V$ matrix $D_G = diag(deg(v_1), \ldots, deg(v_n))$.

Theorem 1.3.1. *The number of strictly weak-walks of length k in a graph G, from v_i to v_j, is equal to the ij-entry of D_G^k.*

The *adjacency matrix* of a graph G is a $V \times V$ matrix $A_G = [a_{ij}]$, where $a_{ij} = \sum_{e \in E} sgn_e(v_i, v_j)$.

Theorem 1.3.2. *The number of walks of length k in a graph G, from v_i to v_j, is equal to the ij-entry of A^k.*

Example 1.3.3. *For the graph G_1 in Figure 2, we have*

$$D_{G_1} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \quad A_{G_1} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

The *incidence matrix* of an oriented graph G is the $V \times E$ matrix $H_G = [\eta_{ij}]$, where

$$[\eta_{ij}] = \begin{cases} +1, & \text{if } v_i \text{ is the positive end of } e_j, \\ -1, & \text{if } v_i \text{ is the negative end of } e_j, \\ 0, & \textit{otherwise.} \end{cases}$$

The *Laplacian Matrix* of a graph G is $L_G = H_G H_G^T$.

Example 1.3.4. *For the graph G_1 in Figure 2, we have*

$$H_{G_1} = \begin{bmatrix} -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \quad L_{G_1} = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}.$$

Theorem 1.3.5. *If G is an oriented graph, then $L_G = H_G H_G^T = D_G - A_G$*

It is important to understand why the two expressions in Theorem 1.3.5 are equivalent. H_G has elements which are the incidences of the graph G and thus represents all of the possible weak walks of length $k = \frac{1}{2}$ from v_i to e_j. H_G^T has elements which are the incidences of the graph G and thus represents all of the possible weak walks of length $k = \frac{1}{2}$ from e_i to v_j. When $H_G H_G^T$ is calculated, through matrix multiplication all of the possible weak walks of length k from v_i to v_j are calculated. On the other side of the equality, D_G is the number of strictly weak walks of length k from v_i to v_j. Subtracted from that is the matrix A_G which is the number of walks of length k from v_i to v_j where $i \neq j$. Thus, both matrices hold the number of weak walks of length k from v_i to v_j. The signs on these entries, however are off by -1. This issue is resolved by the following corollary.

Corollary 1.3.6. *By Theorem 1.3.1 and 1.3.2, the number of weak walks of length 1 in a graph G, from v_i to v_j, is*

$$\widetilde{w}(v_i, v_j, 1) = \begin{cases} -(L_G)_{ij}, & \text{if} \quad i \neq j \\ (L_G)_{ij}, & \text{if} \quad i = j. \end{cases}$$

The *k-weak-walk matrix* of a graph G is $\widetilde{W}_{G,k} = [w_{ij}]$ where $w_{ij} = \widetilde{w}^{\pm}(a_i, a_j; k)$ and $a_i, a_j \in V$.

Example 1.3.7. *For the graph G_1 in Figure 2, we have*

$$\widetilde{W}_{G_1} = \begin{bmatrix} -3 & 1 & 1 & 1 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -3 & 1 \\ 1 & 0 & 1 & -2 \end{bmatrix}.$$

Theorem 1.3.8 (Introduced in [7] and refined in [3])**.** *For a graph G, $L_G = -\widetilde{W}_{G,1}$.*

This theorem provides an important link between the elements of the Laplacian and what they represent in the graph — weak walks.

1.4 Permutations

A *permutation* of a set X is a bijective function $\pi : X \to X$. If there is a list of distinct elements $i_1, i_2, \ldots i_r$ such that $\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_{r-1}) = i_r, \pi(i_r) = i_1$, then there is a cycle of length r, denoted by $(i_1 i_2 \ldots i_r)$. The product of all cycles in π is called the *cycle decomposition* of π. Every permutation can be written as a unique product of disjoint cycles (see [5]).

Example 1.4.1. *All possible permutations of the* 4 *element set* $\{1, 2, 3, 4\}$*:*

$$e, (12), (13), (14), (23), (24), (34), (123), (132), (234), (243), (134), (143), (124), (142),$$

$$(12)(34), (13)(24), (14)(23), (1234), (1432), (1243), (1342), (1423), (1324).$$

2 Contributors

In this section, we introduce an important structure called a contributor. In section 2.2 we introduce theorems to count the number of contributors a graph has along with the sign of each contributor. In section 2.3 we sort contributors into contribution classes using unpacking along adjacencies and circle activation. In section 2.4 we show that the signs of contributors in any contribution class sum to zero.

2.1 Relevant and Irrelevant Permutations

We will now examine permutations of the vertices of a graph G. We define the *relevant permutations of* G, denoted $R(S_V, G)$, as the set of all permutations π such that every cycle of π is a circle in G. Moreover, the *irrelevant permutations of* G, denoted $I(S_V, G)$, is the set of all permutations π such that some cycle of π is not a circle in G.

Referring to the graph G_1 in Figure 2, (123) is a relevant permutation while (24) is an irrelevant permutation.

Lemma 2.1.1. *If π is an irrelevant permutation of G, then* $\displaystyle\prod_{\substack{i\in[1,|V|]\\ \pi(i)\neq i}} a_{i,\pi(i)} = 0.$

Proof. If π is an irrelevant permutation, then there exists a cycle in π such that its adjacency sequence is not a circle in G, so at least one adjacency has sign 0. Therefore, the product of the adjacencies is 0. □

For the following example, we represent the vertex v_i by its index i. In our example G_1, there is no adjacency between v_2 and v_4, thus the irrelevant permutations are those in which the permutation includes $\pi(2) = 4$ or $\pi(4) = 2$.

Example 2.1.2. *The irrelevant and relevant permutations of G_1:*

$$R(S_V, G_1) = \{e, (12), (13), (14), (23), (34), (123), (132), (234), (134), (143),$$

$$(12)(34), (14)(23), (1234), (1432)\}$$

$$I(S_V, G_1) = \{(24), (243), (124), (142), (13)(24), (1243), (1423), (1342), (1324)\}.$$

2.2 Contributors

Each permutation π of the set V of G creates objects called contributors, where a *contributor* is an incidence structure consisting of weak walks that form circles corresponding to the disjoint cycles from π and one backstep from each fixed element in π. The *sign of a contributor* is $sgn(c) = (-1)^{\psi(c)}$ where $\psi(c)$ is the number of circles in contributor c. We define the set $\mathfrak{C}$ to be the set of all contributors.

Each permutation can produce multiple contributors. For each fixed element (i) in π there are $deg(v_i)$ many backsteps possible at v_i, while for each non-fixed element

j in π, there are $a_{j,\pi(j)}$ many walks of length 1 possible from v_j to $v_{\pi(j)}$. From this observation, we produce the following theorem.

Theorem 2.2.1. *Each permutation* π *produces* $\prod_{\substack{i\in[1,|V|] \\ \pi(i)\neq i}} a_{i,\pi(i)} \prod_{\substack{i\in[1,|V|] \\ \pi(i)=i}} \deg(v_i)$ *many contributors.*

By Lemma 2.1.1, the irrelevant permutations produce no contributors, so contributors only come from the relevant permutations. Thus, the total number of contributors is the sum of the number of contributors produced by each relevant permutation. This is represented in the following theorem.

Theorem 2.2.2. *The number of contributors is*

$$\sum_{\pi\in R(S_V,G)} \prod_{\substack{i\in[1,|V|] \\ \pi(i)\neq i}} a_{i,\pi(i)} \prod_{\substack{i\in[1,|V|] \\ \pi(i)=i}} \deg(v_i).$$

Example 2.2.3. *For the graph* G_1*, the number of contributors created by the permutation* (23) *is*

$$a_{2,3} \cdot a_{3,2} \cdot deg(v_1) \cdot deg(v_4) = 1 \cdot 1 \cdot 3 \cdot 2 = 6.$$

If this is applied to every relevant permutation in the graph G_1*, the total number of contributors is* 76*.*

Figure 4 displays all of the contributors of the graph G_1 sorted by their permutations. When only one incidence is drawn with a 2 above it, this refers to a backstep from that vertex along the corresponding incidence. Notice that the permutations that contain non-trivial cycles create contributors with the corresponding circles. For this reason, the terms cycle and circle can be used interchangeably.

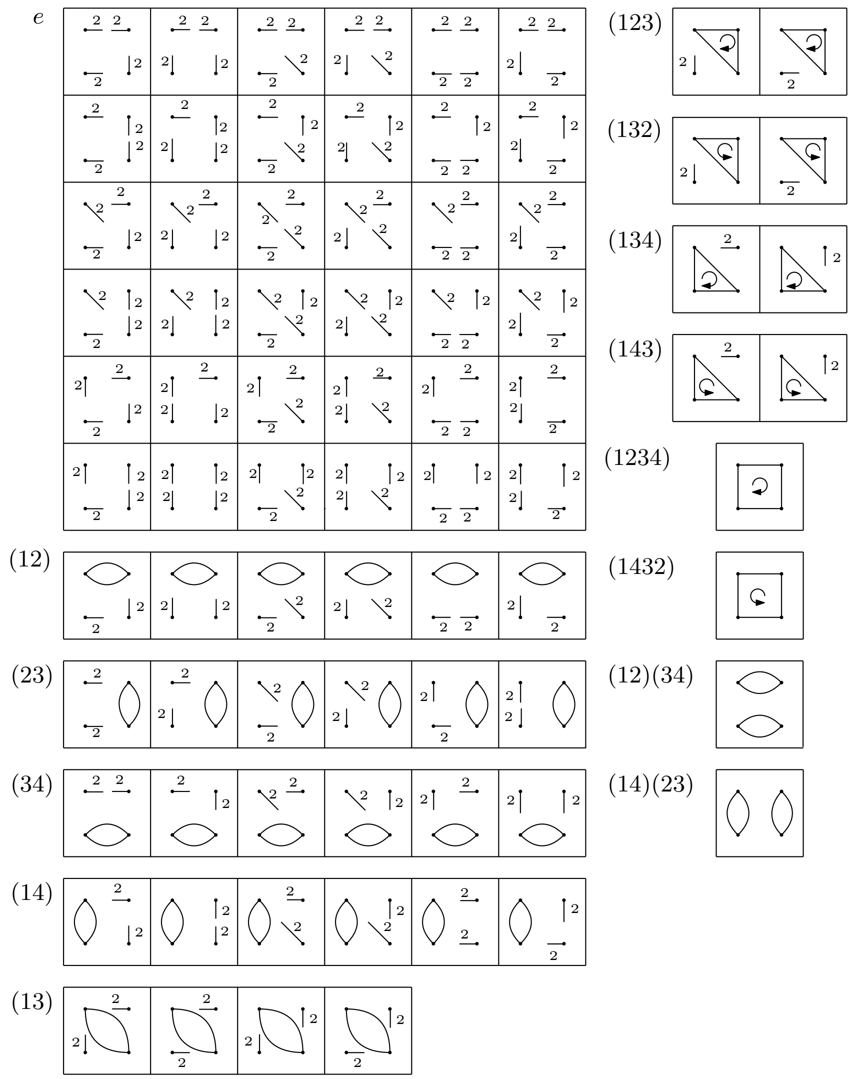

Figure 4: Contributors of the graph G_1.

2.3 Contribution Classes

This section introduces the method used for grouping contributors together into classes rather than sorting them by their permutations. This method of grouping is proved to be a partition of all contributors.

Unpacking a backstep along an adjacency is a function $\rho : (v,e)^2 \to (v,w;e)$ such that the adjacency $(v,w;e)$ exists. Given an adjacency, *packing* is the function $\rho^{-1} : (v,w;e) \to (v,e)^2$ such that in the permutation, $\pi(v) = w$. *Activating a circle* *of a contributor* is the minimal sequence of unpackings that completes a circle in G and such a circle is called *active*. *Deactivating a circle of a contributor* is the minimal sequence of packings such that no adjacencies in the given circle remain and such a circle is called *inactive*.

Let R be the relation $c_i R c_j$ if the contributors c_i and c_j can be made identical through a sequence of activating or deactivating circles. These contributors are then said to be in the same *contribution class* C.

Figure 5 depicts all of the contributors again, but also highlights three examples of contribution classes and the contributors contained in them.

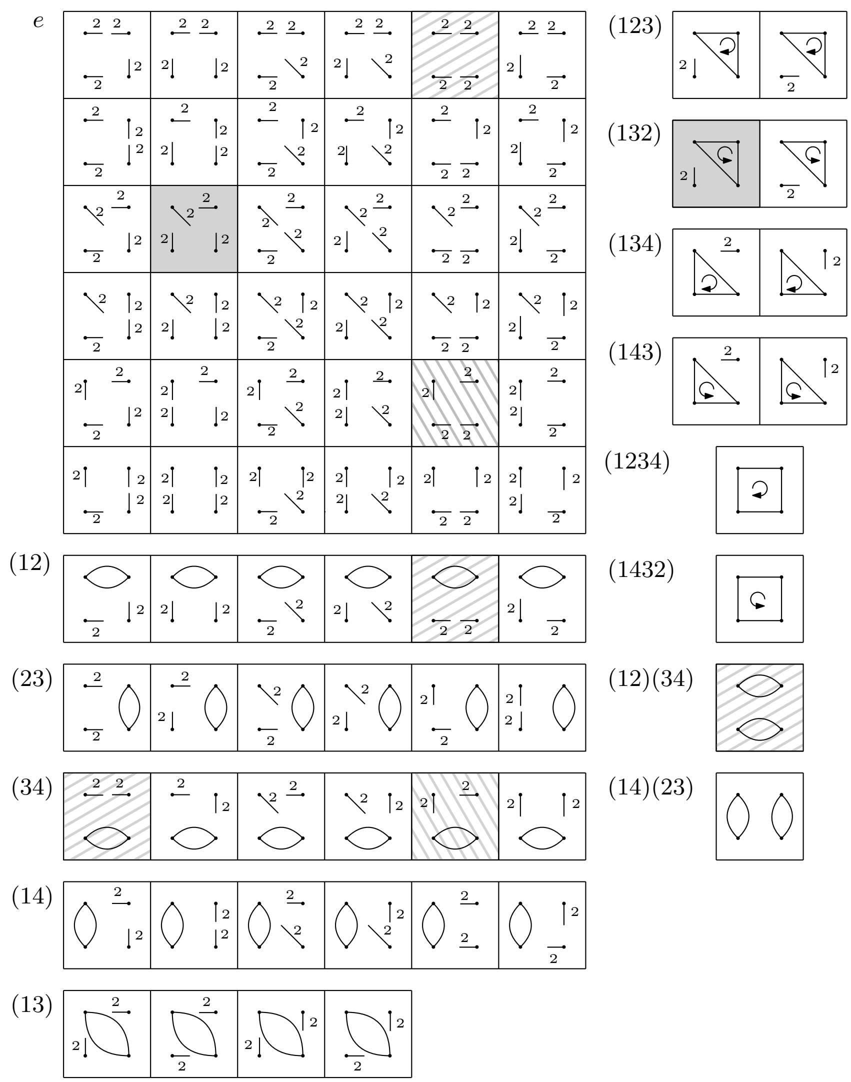

Figure 5: Contributors of the graph G_1 with 3 contribution classes shaded.

The highlighted contributors are grouped into their corresponding contribution classes, as shown in Figure 6. We organize each contribution class into a boolean lattice which is ranked by $\psi(c)$ (the number of active circles in a given contributor) and ordered by subsets of active circles. The height of each lattice is $h(C)$ which is the rank of the largest element in the lattice. The highest level of each contribution class is the contributor with all possible circles active, and the bottom level is the contributor with all circles inactive.

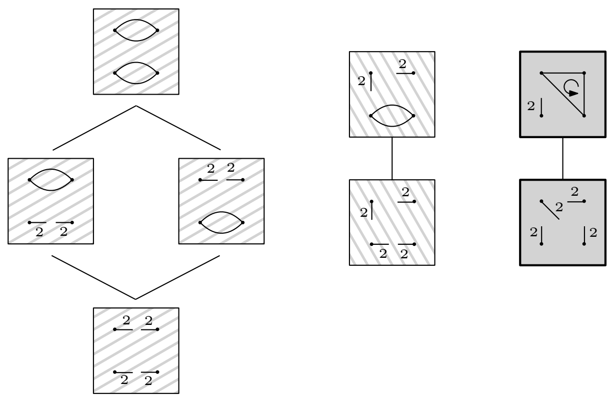

Figure 6: Contribution classes in Boolean Lattices.

Theorem 2.3.1. *R is an equivalence relation, where each contribution class is an equivalence class.*

The proof of this is trivial. Theorem 2.3.1 then implies that R forms a partition of the contributors. Let $\mathcal{C}$ be the set of all contribution classes.

Theorem 2.3.2. *The number of contribution classes in a graph G is $\prod_{v \in V} \deg(v)$.*

Proof. The $\mathbf{0}$ element of each boolean lattice is a contributor created from the identity permutation. Thus, there are the same number of contributors from the identity permutation as contribution classes. By Theorem 2.2.1, there are $\prod_{v \in V} \deg(v)$ many contribution classes. $\square$

Example 2.3.3. *For the graph G_1, the number of contribution classes is $3 \cdot 2 \cdot 3 \cdot 2 = 36$.*

Notice that this is the number of contributors from the identity permutation in Figure 4.

2.4 Counting Each Contribution Class

Theorem 2.4.1. *Each contributor in its completely unpacked form contains at least 1 circle.*

Proof. In the completely unpacked form, each contributor resembles a graph with n vertices and n edges. By a known theorem, a graph with n vertices and n edges contains a cycle. □

We now let r_C equal the number of circles in a completely activated contributor of class C, and recall that $\psi(c)$ equals the number of circles in a given contributor c.

Theorem 2.4.2. *The number of contributors in each contribution class C is 2^{r_C}.*

Proof. The contributors arise from all relevant permutations, so within each contribution class, all possible combinations of cycles appear, thus every contributor has a distinct boolean representation where 1's represent cycles that are active and 0's represent cycles that are inactive. Because of this boolean nature, every contribution class has 2^{r_C} members. □

Theorem 2.4.3. *Each contribution class has at least 2 members.*

Proof. Each contribution class has at least one cycle in its completely unpacked member. Thus, there are at least 2^1 members. □

We know $\begin{pmatrix} r_C \\ \psi(c) \end{pmatrix}$ is the number of contributors in C that have exactly $\psi(c)$ circles. Thus, $\begin{pmatrix} r_C \\ 0 \end{pmatrix} + \begin{pmatrix} r_C \\ 2 \end{pmatrix} + \begin{pmatrix} r_C \\ 4 \end{pmatrix} + \ldots = \begin{pmatrix} r_C \\ 1 \end{pmatrix} + \begin{pmatrix} r_C \\ 3 \end{pmatrix} + \begin{pmatrix} r_C \\ 5 \end{pmatrix} + \ldots$

shows that in each contribution class, the number of contributors with an even number of circles equals the number of contributors with an odd number of circles. Those with an even number of circles are signed positively and those with an odd number of circles are signed negatively. The highlighted contribution classes are represented again, in Figure 7, with the signing function $sgn(c)$ applied.

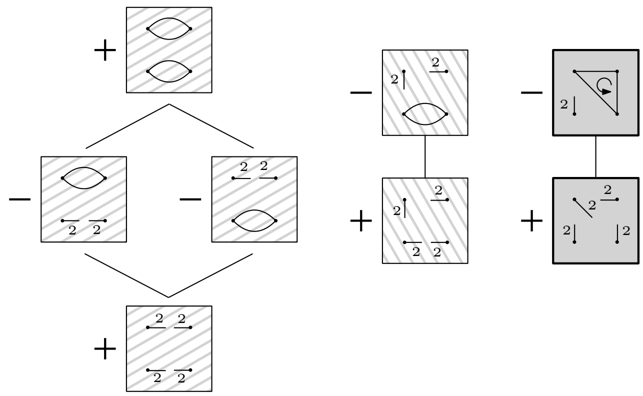

Figure 7: Contribution classes in Boolean Lattices with signs determined by $sgn(c)$.

Since each contribution class is boolean and the alternating sum of binomial coefficients is zero, we have the following theorem.

Theorem 2.4.4. *Given a graph G, for each contribution class C,* $\sum_{c \in C} sgn(c) = 0$.

3 The Determinant of the Laplacian Matrix

In this section we introduce determinants and examine what happens when we take the determinant of the Laplacian matrix. This is an important step in proving the Matrix Tree Theorem in section 4.

3.1 Determinants

The *determinant* of a matrix A is

$$det(A) = \sum_{\pi \in S_n} sgn(\pi) \prod_{i=1}^{n} a_{i,\pi_i}.$$

Where the sum is computed over all permutations π of $\{1, 2, 3, ..., n\}$. Where the sign of a permutation is $sgn(\pi) = (-1)^{N(\pi)}$ and $N(\pi)$ is the number of inversions in π. An inversion is an instance where numbers are out of their natural order. We define $ec(\pi)$, $oc(\pi)$, and $tc(\pi)$ to be the number of even, odd, and total (non-trivial) cycles in π, respectively.

Theorem 3.1.1. *For a given permutation π, $sgn(\pi) = (-1)^{ec(\pi)}$*

Proof. Every transposition (cycle with 2 elements) contains one inversion. Every odd cycle is the product of an even number of transpositions. Every even cycle is the product of an odd number of transpositions. Thus, the number of even cycles is the number of inversions in π, and $sgn(\pi) = (-1)^{ec(\pi)}$. □

Theorem 3.1.2. *For a given permutation π, $\prod_{i=1}^{n} \ell_{i,\pi(i)} = (-1)^{oc(\pi)} \prod_{i=1}^{n} |\ell_{i,\pi(i)}|$ where $\ell_{i,\pi(i)}$ is the $(i, \pi(i))$ entry of the Laplacian matrix.*

Proof. In $L_G = D_G - A_G$, the sign of $\ell_{i,\pi(i)}$ where $i = \pi(i)$ is positive because entries in D_G are positive. The sign of $\ell_{i,\pi(i)}$ where $i \neq \pi(i)$ is negative because the entries in A_G are positive but they are each subtracted from 0 in the calculation of the Laplacian. Thus the sign of each entry of the Laplacian is $(-1)^{\alpha(\pi)}$ where $\alpha(\pi)$ is the number of adjacencies in π. Each odd cycle in π contains an odd number of adjacencies. Each even cycle in π contains an even number of adjacencies. Thus, the number of odd cycles is the number of adjacencies in π. Therefore, $\prod_{i=1}^{n} \ell_{i,\pi(i)} = (-1)^{oc(\pi)} \prod_{i=1}^{n} |\ell_{i,\pi(i)}|$. □

3.2 Taking the Determinant of the Laplacian

In this section we show that calculating $det(L_G)$ is equivalent to summing all of the signs of the contributors. Using Theorem 2.4.4, and the fact that the contribution classes form a partition of the contributors, we prove the following theorem:

Theorem 3.2.1. *For an oriented graph G,*

$$det(L_G) = 0.$$

This theorem is a known result. However, the proof below utilizes contribution classes which will aid in the final proof of the Matrix Tree Theorem.

Proof. From the definition we have

$$\begin{aligned}
\det(L_G) &= \sum_{\pi \in S_V} sgn(\pi) \prod_{i \in [1,|V|]} \ell_{i,\pi(i)} \\
&= \sum_{\pi \in S_V} (-1)^{ec(\pi)} \prod_{i \in [1,|V|]} \ell_{i,\pi(i)} \\
&= \sum_{\pi \in S_V} (-1)^{ec(\pi)} (-1)^{oc(\pi)} \prod_{i \in [1,|V|]} \left|\ell_{i,\pi(i)}\right| \\
&= \sum_{\pi \in S_V} (-1)^{tc(\pi)} \prod_{i \in [1,|V|]} \left|\ell_{i,\pi(i)}\right|.
\end{aligned}$$

We now separate the summation by relevant and irrelevant permutations, and get

$$= \sum_{\pi \in R(S_V,G)} (-1)^{tc(\pi)} \prod_{i \in [1,|V|]} \left|\ell_{i,\pi(i)}\right| + \sum_{\pi \in I(S_V,G)} (-1)^{tc(\pi)} \prod_{i \in [1,|V|]} \left|\ell_{i,\pi(i)}\right|.$$

By Lemma 2.1.1, the second summation equals 0. Simplifying the expression to:

$$= \sum_{\pi \in R(S_V,G)} (-1)^{tc(\pi)} \prod_{i \in [1,|V|]} \left|\ell_{i,\pi(i)}\right|.$$

We now separate the product into off-diagonal and diagonal elements,

$$\begin{aligned}
&= \sum_{\pi \in R(S_V, G)} (-1)^{tc(\pi)} \prod_{\substack{i \in [1,|V|] \\ \pi(i) \neq i}} \left|\ell_{i,\pi(i)}\right| \prod_{\substack{i \in [1,|V|] \\ \pi(i) = i}} \left|\ell_{i,\pi(i)}\right| \\
&= \sum_{\pi \in R(S_V, G)} (-1)^{tc(\pi)} \prod_{\substack{i \in [1,|V|] \\ \pi(i) \neq i}} a_{i,\pi(i)} \prod_{\substack{i \in [1,|V|] \\ \pi(i) = i}} d_{i,\pi(i)} \\
&= \sum_{\pi \in R(S_V, G)} (-1)^{tc(\pi)} \prod_{\substack{i \in [1,|V|] \\ \pi(i) \neq i}} a_{i,\pi(i)} \prod_{\substack{i \in [1,|V|] \\ \pi(i) = i}} \deg(v_i).
\end{aligned}$$

From Theorem 2.2.2 we sum over all contributors,

$$= \sum_{c \in \mathfrak{C}} sgn(c) \cdot 1.$$

Summing over all contributors within their disjoint contribution classes gives us:

$$= \sum_{C \in \mathcal{C}} \sum_{c \in C} sgn(c).$$

By Theorem 2.4.4,

$$\begin{aligned}
&= \sum_{C \in \mathcal{C}} 0 \\
&= 0.
\end{aligned}$$

Thus, $det(L_G) = 0$. $\square$

4 Matrix Tree Theorem

In this section, we introduce the Matrix Tree Theorem and then provide an alternative way of showing that it is true. This method is useful because it generalizes the theorem by breaking it down to the smallest pieces possible, incidences.

Theorem 4.0.1 (Matrix Tree Theorem). *The number of spanning trees of a graph G is the value of any cofactor of the Laplacian matrix.*

One proof of this theorem is found in [2] where Seth Chaiken uses directed arcs and also provides a count of objects when multiple rows and columns are deleted. In [8], Tutte uses spanning aboresences, darts and the conductances of link-darts to prove this theorem.

4.1 Calculating a Determinant by Expansion by Minors

The method of taking a determinant using expansion by minors will allow us to see what contributors are present when we examine a cofactor of the Laplacian matrix. The determinant is defined as follows:

$$det(A) = \sum_{i=1}^{k} (-1)^{i+j} a_{ij} M_{ij}$$

such that M_{ij} is a minor of A (determinant of A with row i and column j crossed out). A *cofactor* C_{ij} of A is a signed version of a minor where $C_{ij} = (-1)^{i+j} M_{ij}$.

Each part of the sum consists of the entry a_{ij}, a signing function, and the minor M_{ij}. So when we examine a cofactor of the matrix L_G we will rather examine the contributors of G that consist of the weak walk from v_i to v_j.

4.2 Partitioning Contribution Classes

Fix vertex v_r, let $M_0(v_r, C)$ be the maximal element in contribution class C where v_r appears in no active circle, and let $m_1(v_r, C)$ be the minimal element in contribution class C where v_r is contained in an active circle. The *downset* of a contributor c is $\downarrow c = \{x \in C \mid x \leq c\}$ and the *upset* of a contributor c is $\uparrow c = \{x \in C \mid x \geq c\}$ (see [1]).

Lemma 4.2.1. *If $m_1(v_r, C)$ exists and C has height n, then:*

1. *$rk(m_1(v_r, C)) = 1$.*

2. *$m_1(v_r, C)$ and $M_0(v_r, C)$ are complements.*

3. *$\downarrow M_0(v_r, C)$ and $\uparrow m_1(v_r, C)$ are sub-boolean lattices of C, both with height $n-1$.*

4. *$\downarrow M_0(v_r, C)$ and $\uparrow m_1(v_r, C)$ bipartition the elements of C into sub-boolean lattices (one may be empty).*

Proof. The order of these proofs is important because the conclusions from each part build upon each other to prove part 4.

part 1: By definition, $m_1(v_r, C)$ is the contributor in C with only one circle activated, and thus has rank 1.

part 2: By the definition of $m_1(v_r, C)$ and $M_0(v_r, C)$, $m_1(v_r, C) \wedge M_0(v_r, C) = \mathbf{0}$ since they have no circles in common. Also, $m_1(v_r, C) \vee M_0(v_r, C) = \mathbf{1}$ because all possible circles not in $m_1(v_r, C)$ are in $M_0(v_r, C)$ and thus their join contains all possible circles. Therefore, $m_1(v_r, C)$ and $M_0(v_r, C)$ are complements.

part 3: The upset and downset of any element of a boolean lattice is still a boolean lattice. Thus, $\downarrow M_0(v_r, C)$ and $\uparrow m_1(v_r, C)$ are sub-boolean lattices of C. $M_0(v_r, C)$ has corank 1 so $h(\downarrow M_0(v_r, C)) = n - 1$. Also, $m_1(v_r, C)$ has rank 1 so $h(\uparrow m_1(v_r, C)) = n - 1$.

part 4: It follows from parts $1, 2$ and 3 that there is no element in both $\downarrow M_0(v_r, C)$ and $\uparrow m_1(v_r, C)$. Also, each contributor in the contribution class C is either in $\downarrow M_0(v_r, C)$ or $\uparrow m_1(v_r, C)$ because v_r must either be in an active circle or not in an active circle. In addition, if $m_1(v_r, C)$ does not exist, the two sets still create a bipartition with $\uparrow m_1(v_r, C)$ empty.

□

We define the following subsets of $\mathfrak{C}$:

$$\mathfrak{C}_0(v_r) = \{c \in \mathfrak{C} \mid v_r \nsim v, \{c\} = \downarrow M_0(v_r, C) \text{ for some } c \in \mathcal{C}\}$$

$$\mathfrak{C}_{1,k}(v_r) = \{c \in \mathfrak{C} \mid v_r \sim v_k, \{c\} = \uparrow m_1(v_r, C) \text{ for some } c \in \mathcal{C}\}.$$

Recall from Theorem 2.4.3 that each boolean lattice has at least 2 contributors. However, the sub-boolean lattices $\downarrow M_0(v_r, C)$ and $\uparrow m_1(v_r, C)$ may only contain 1 contributor. The subsets $\mathfrak{C}_0(v_r)$ and $\mathfrak{C}_{1,k}(v_r)$ are the sets of contributors which were in height 1 boolean lattices that have now been split into two height 0 sub-boolean lattices and thus their signs no longer cancel within their respective sub-boolean lattices. All of the contributors not contained in these subsets belong to sub-boolean lattices of height greater than or equal to 1 and therefore, the sum of the signs of these contributors is 0 within their respective sub-boolean lattices.

Example 4.2.2. *Figure 8 is an adaptation of Figure 6 in which each contribution class has been partitioned by* $\downarrow M_0(v_r, C)$ *and* $\uparrow m_1(v_r, C)$*.*

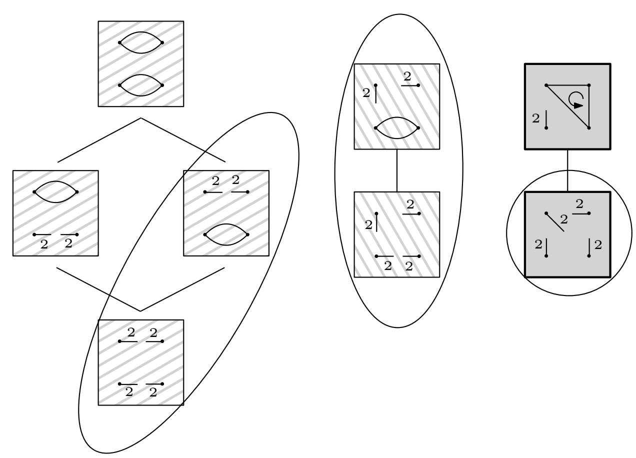

Figure 8: The sub-boolean lattices corresponding to $\downarrow M_0(v_1, C)$.

The elements which are circled in this figure are in the set $\downarrow M_0(v_1, C)$ *and those which are not circled are in the set* $\uparrow m_1(v_r, C)$*. In the first contribution class* $\downarrow M_0(v_1, C)$ *is a height* 1 *sub-boolean lattice and* $\uparrow m_1(v_r, C)$ *is a height* 1 *sub-boolean* *lattice. In the second contribution class* $\downarrow M_0(v_1, C)$ *is a height* 1 *sub-boolean lattice and* $\uparrow m_1(v_r, C)$ *is empty. Therefore, because the height of these sub-boolean lattices is greater than or equal to* 1*, the sum of the signs of the contributors in each of these sub-boolean lattices is* 0*. In the third contribution class* $\uparrow m_1(v_r, C)$ *is a height* 0 *sub-boolean lattice and* $\downarrow M_0(v_r, C)$ *is a height* 0 *sub-boolean lattice so* $m_1(v_r, C) = \uparrow m_1(v_r, C)$ *and* $M_0(v_r, C) = \downarrow M_0(v_r, C)$*, moreover, these contributors are in* $\mathfrak{C}_{1,k}(v_r)$ *and* $\mathfrak{C}_0(v_r)$*, respectively.*

Theorem 4.2.3. *We can rewrite the determinant of the Laplacian using the sets* $\mathfrak{C}_0(v_r)$ *and* $\mathfrak{C}_{1,k}(v_r)$ *as follows:*

$$det(L_G) = deg(v_r)\left[\frac{1}{deg(v_r)}\sum_{c\in\mathfrak{C}_0(v_r)} 1\right] + \sum_{k\in[1,|V|]\smallsetminus\{r\}} -a_{r,k}\left[\frac{1}{-a_{r,k}}\sum_{c\in\mathfrak{C}_{1,k}(v_r)} -1\right].$$

Proof. We continue from a step in the proof of Theorem 3.2.1.

$$\begin{aligned} \det(L_G) &= \sum_{c\in\mathfrak{C}} sgn(c) \\ &= \sum_{C\in\mathcal{C}}\sum_{c\in C} sgn(c). \end{aligned}$$

By Lemma 4.2.1, each contribution class can be partitioned as follows:

$$\begin{aligned} &= \sum_{C\in\mathcal{C}}\left[\sum_{c\in\downarrow M_0(v_r,C)} sgn(c) + \sum_{c\in\uparrow m_1(v_r,C)} sgn(c)\right] \\ &= \sum_{C\in\mathcal{C}}\sum_{c\in\downarrow M_0(v_r,C)} sgn(c) + \sum_{C\in\mathcal{C}}\sum_{c\in\uparrow m_1(v_r,C)} sgn(c). \end{aligned}$$

The first pair of sums is the signed number of contributors where v_r is not adjacent to any other vertex, while the second pair of sums is the signed number of contributors where v_r is adjacent to some other vertex. By Theorem 2.4.4 each inner sum is 0 if, and only if, $\downarrow M_0(v_r, C)$ or $\uparrow m_1(v_r, C)$ is a boolean lattice with height greater than 0.

Moreover, in a height 1 contribution class C with non-empty partitions $\downarrow M_0(v_r, C)$ and $\uparrow m_1(v_r, C)$, the contributors $M_0(v_r, C)$ and $m_1(v_r, C)$ are the elements $\mathbf{0}$ and $\mathbf{1}$ of the lattice C, respectively. Thus,

$$= \sum_{c \in \mathfrak{C}_0(v_r)} sgn(c) + \sum_{k \in [1,|V|] \smallsetminus \{r\}} \sum_{c \in \mathfrak{C}_{1,k}(v_r)} sgn(c).$$

In the first summation, $sgn(c) = 1$ for all contributors because none of these contributors contain circles. In the second summation, $sgn(c) = -1$ for all contributors because they are all top elements in height 1 boolean lattices. Therefore,

$$= \sum_{c \in \mathfrak{C}_0(v_r)} 1 + \sum_{k \in [1,|V|] \smallsetminus \{r\}} \sum_{c \in \mathfrak{C}_{1,k}(v_r)} -1.$$

The determinant of the Laplacian can now be written as follows:

$$det(L_G) = deg(v_r)\left[\frac{1}{deg(v_r)} \sum_{c \in \mathfrak{C}_0(v_r)} 1\right] + \sum_{k \in [1,|V|] \smallsetminus \{r\}} -a_{r,k}\left[\frac{1}{-a_{r,k}} \sum_{c \in \mathfrak{C}_{1,k}(v_r)} -1\right].$$

□

4.3 Proving the Matrix Tree Theorem

We will now examine the expressions from Theorem 4.2.3 within the brackets. Below we show that each expression represents the number of spanning trees in the graph G and is a cofactor of the Laplacian matrix of G.

Theorem 4.3.1. *The number of spanning trees in a graph G, $T(G)$, is represented by:*

$$T(G) = \frac{1}{deg(v_r)} \sum_{c \in \mathfrak{C}_0(v_r)} 1 = \frac{1}{-a_{r,k}} \sum_{c \in \mathfrak{C}_{1,k}(v_r)} -1$$

for some $k \in [1, |V|] \smallsetminus \{r\}$.

Proof. The first expression is the number of contributors in $\mathfrak{C}_0(v_r)$ divided by $deg(v_r)$, thus those contributors in $\mathfrak{C}_0(v_r)$ that are identical when the weak walk corresponding to $deg(v_r)$ is eliminated will be grouped as one contributor. The remaining contributors in the first expression now contain $|V|$ vertices and $|V|-1$ weak walks (which via unpacking are in one-to-one correspondence with edges). These contributors when unpacked also do not contain a circle by the definition of $\mathfrak{C}_0(v_r)$. Therefore, these unpacked contributors are the spanning trees of the graph G.

Similarly, the second expression is negative the number of contributors in $\mathfrak{C}_{1,k}(v_r)$ divided by $-a_{r,k}$, thus those contributors in $\mathfrak{C}_{1,k}(v_r)$ that are identical when the weak walk corresponding to $a_{r,k}$ is eliminated will be grouped as one contributor. The remaining contributors in the second expression now contain $|V|$ vertices and $|V|-1$ weak walks (which via unpacking are in one-to-one correspondence with edges). These contributors when unpacked also do not contain a circle because the definition of $\mathfrak{C}_{1,k}(v_r)$ requires that they each had only one circle containing the weak walk from v_r to v_k which has now been eliminated. Therefore, these unpacked contributors are the spanning trees of the graph G.

□

Example 4.3.2. *Figure 9 is an adaptation of Figure 5 where the contributors that contain a weak walk from v_1 to v_1 are outlined in bold. The contributors that contain an active or inactive circle after the weak walk from v_1 to v_1 has been eliminated are in height* 1 *sub-boolean lattices while the others are in height* 0 *sub-boolean lattices. The contributors that are shaded alike are examples of contributors that are identical when the weak walk from v_1 to v_1 is eliminated. After grouping the identical contributors that are in height* 0 *sub-boolean lattices, only* 8 *remain. These* 8 *contributors when completely unpacked are the spanning trees in Figure 3.*

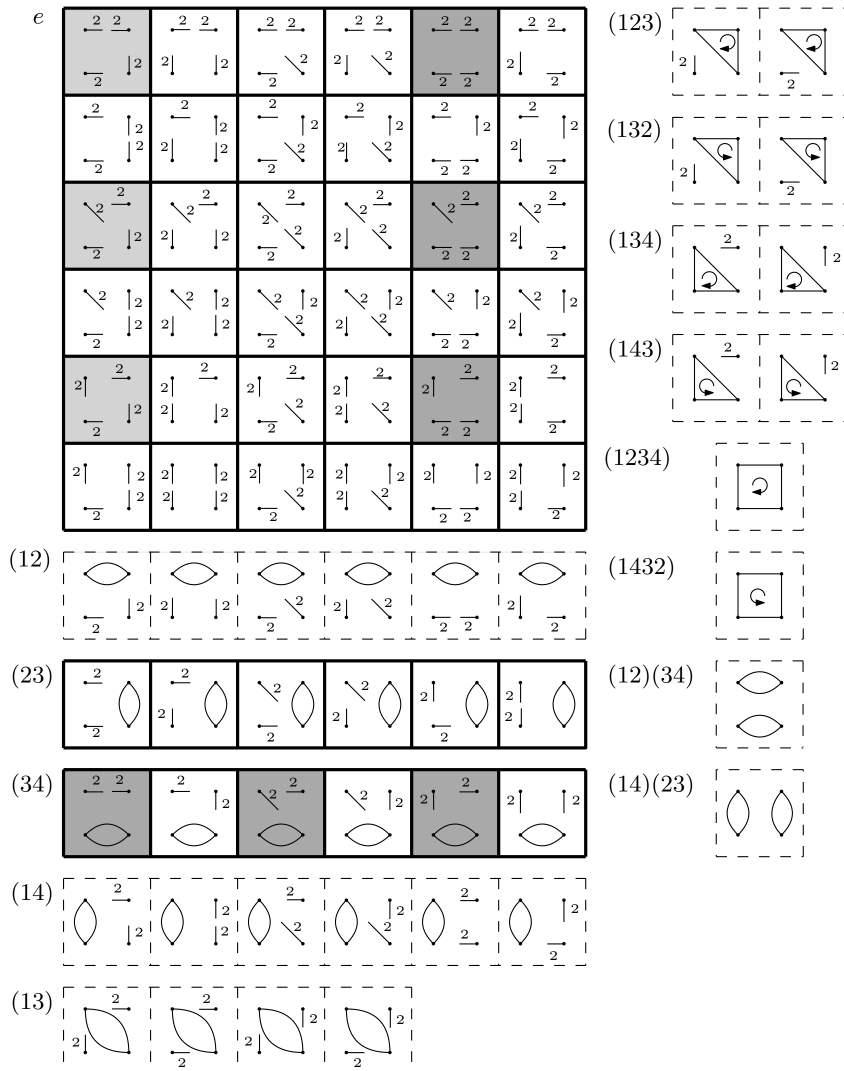

Figure 9: Contributors of the graph G_1 with a weak walk from v_1 to v_1.

Now we must prove that the expressions from Theorem 4.3.1 are the cofactors of L_G, thus completing the proof of the Matrix Tree Theorem.

Theorem 4.3.3. *For the Laplacian matrix for a graph G the cofactors are*

$$C_{rr} = \frac{1}{deg(v_r)} \sum_{c \in \mathfrak{C}_0(v_r)} 1$$

$$C_{rk} = \frac{1}{-a_{r,k}} \sum_{c \in \mathfrak{C}_{1,k}(v_r)} -1 \qquad \textit{for some } k \in [1, |V|] \smallsetminus \{r\}.$$

Proof. Recall from the proof of Theorem 3.2.1

$$det(L_G) \quad = \quad \sum_{\pi \in R(S_V,G)} (-1)^{tc(\pi)} \prod_{\substack{i\in[1,|V|] \\ \pi(i)\neq i}} a_{i,\pi(i)} \prod_{\substack{i\in[1,|V|] \\ \pi(i)=i}} \deg(v_i).$$

To calculate this determinant using expansion by minors across row r (the matrix can expanded, across any row or column) we have the following:

$$\det(L_G) = \deg(v_r) \sum_{\substack{\pi \in R(S_V,G) \\ \pi(r)=r}} (-1)^{tc(\pi)} \prod_{\substack{i\in[1,|V|] \\ \pi(i)\neq i}} a_{i,\pi(i)} \prod_{\substack{i\in[1,|V|]\smallsetminus\{r\} \\ \pi(i)=i}} \deg(v_i) \tag{4.1}$$

$$+ \sum_{k\in[1,|V|]\smallsetminus\{r\}} -a_{r,k} \sum_{\substack{\pi \in R(S_V,G) \\ \pi(r)=k}} (-1)^{tc(\pi)-1} \prod_{\substack{i\in[1,|V|]\smallsetminus\{r\} \\ \pi(i)\neq i}} a_{i,\pi(i)} \prod_{\substack{i\in[1,|V|] \\ \pi(i)=i}} \deg(v_i). \tag{4.2}$$

The signing function in Equation 4.2 has changed because the adjacency $a_{r,k}$ has been factored out and every adjacency in an oriented graph is negative so a negative sign has been removed from the summation as well.

Thus, the cofactors of L_G are

$$C_{rr} = \sum_{\substack{\pi \in R(S_V,G) \\ \pi(r)=r}} (-1)^{tc(\pi)} \prod_{\substack{i\in[1,|V|] \\ \pi(i)\neq i}} a_{i,\pi(i)} \prod_{\substack{i\in[1,|V|]\smallsetminus\{r\} \\ \pi(i)=i}} \deg(v_i)$$

$$C_{rk} = \sum_{\substack{\pi \in R(S_V,G) \\ \pi(r)=k}} (-1)^{tc(\pi)-1} \prod_{\substack{i\in[1,|V|]\smallsetminus\{r\} \\ \pi(i)\neq i}} a_{i,\pi(i)} \prod_{\substack{i\in[1,|V|] \\ \pi(i)=i}} \deg(v_i)$$

for $k \in [1, |V|] \smallsetminus \{r\}$.

The expression in Equation 4.1 is the sum of the signs of all contributors containing a weak walk from v_r to v_r, which is equivalent to the number of elements in $\mathfrak{C}_0(v_r)$ because the sum of the signs of the contributors that are in sub-boolean lattices of height greater than 0 will equal 0. All of the remaining contributors are positively signed because they do not contain circles, so

$$\deg(v_r)\sum_{\substack{\pi\in R(S_V,G)\\ \pi(r)=r}}(-1)^{tc(\pi)}\prod_{\substack{i\in[1,|V|]\\ \pi(i)\neq i}}a_{i,\pi(i)}\prod_{\substack{i\in[1,|V|]\smallsetminus\{r\}\\ \pi(i)=i}}\deg(v_i)=\sum_{c\in\mathfrak{C}_0(v_r)}1. \tag{4.3}$$

Similarly, the expression in Equation 4.2 is the sum of the signs of all contributors containing a weak walk from v_r to v_k for some $k\in[1,|V|]\smallsetminus\{r\}$, which is equivalent to -1 times the number of elements in $\mathfrak{C}_{1,k}(v_r)$ because the sum of the signs of the contributors that are in sub-boolean lattices of height greater than 0 will equal 0. All of these contributors are negatively signed because they each contain one circle, so

$$-a_{r,k}\sum_{\substack{\pi\in R(S_V,G)\\ \pi(r)=k}}(-1)^{tc(\pi)-1}\prod_{\substack{i\in[1,|V|]\smallsetminus\{r\}\\ \pi(i)\neq i}}a_{i,\pi(i)}\prod_{\substack{i\in[1,|V|]\\ \pi(i)=i}}\deg(v_i)=\sum_{c\in\mathfrak{C}_{1,k}(v_r)}-1. \tag{4.4}$$

for some $k\in[1,|V|]\smallsetminus\{r\}$.

Therefore, by dividing both sides of Equation 4.3 by $deg(v_r)$ and by dividing both sides of Equation 4.4 by $a_{r,k}$ we have

$$\sum_{\substack{\pi\in R(S_V,G)\\ \pi(r)=r}}(-1)^{tc(\pi)}\prod_{\substack{i\in[1,|V|]\\ \pi(i)\neq i}}a_{i,\pi(i)}\prod_{\substack{i\in[1,|V|]\smallsetminus\{r\}\\ \pi(i)=i}}\deg(v_i)=\frac{1}{deg(v_r)}\sum_{c\in\mathfrak{C}_0(v_r)}1$$

$$\sum_{\substack{\pi\in R(S_V,G)\\ \pi(r)=k}}(-1)^{tc(\pi)}\prod_{\substack{i\in[1,|V|]\smallsetminus\{r\}\\ \pi(i)\neq i}}a_{i,\pi(i)}\prod_{\substack{i\in[1,|V|]\\ \pi(i)=i}}\deg(v_i)=\frac{1}{-a_{r,k}}\sum_{c\in\mathfrak{C}_{1,k}(v_r)}-1$$

for some $k\in[1,|V|]\smallsetminus\{r\}$.

We have now shown that any cofactor of the Laplacian matrix for a graph G is the number of spanning trees in the graph. This completes the proof of the Matrix Tree Theorem. □

5 Conclusion

This new interpretation of the Matrix Tree Theorem shifts the focus to oriented incidences in order to provide a representation of the incidence structures that appear in the determinant of the Laplacian matrix for a graph. This generalization can possibly aid future investigations into the tree-count for signed graphs and oriented hypergraphs. Evaluating tree-counts using the contributor method for the duals of graphs may lead to further conclusions about oriented hypergraphs and their tree structures.

Future investigations may also include the examination of cycle covers and their relationship to contributors. In addition, we have begun to investigate what the eigenvalues for Laplacian matrices represent in relation to the incidence structure of the graph. Hopefully this new interpretation will be a valuable tool when attempting to answer these questions.

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