BRUHAT ORDER AND COXETER HYPERPLANE ARRANGEMENTS

by

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DEDICATION

To my parents Meredith and Alex, and to Rowan, for their continual and enduring support.

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ABSTRACT

In the paper "Bruhat order, rationally smooth Schubert varieties, and hyperplane arrangements," [8] S. Oh and H. Yoo studied Schubert varieties in generalized flag manifolds by linking them with a certain hyperplane arrangement coming from the reflection synmmetries of a Weyl group. They made two conjectures. The first relates to a curious property of maximal parabolic quotients of finite Weyl groups. The second states that for an element $w \in W$ of a finite Coxeter group, the generating function $R_w(q)$ of its hyperplane arrangement coincides with the rank-generating function $P_w(q)$ of its lower interval $[e, w]$ in the Bruhat order, if and only if $[e, w]$ is rank-symmetric. Here, we prove the first conjecture. We use this result to prove the second conjecture in the case of Weyl groups. Two chapters of background material on Coxeter systems and hyperplane arangements are provided.

CHAPTER 1

INTRODUCTION

The material which follows gives a combinatorial solution to a conjecture of Suho Oh and Hwanchul Yoo regarding finite Weyl groups [8], and uses this result to prove another conjecture regarding hyperplane arrangements. It is assumed that the reader has basic familiarity with group theory and combinatorics, at the level of a first year graduate student. We do, however, give a thorough (if minimal) exposition of all necessary background material, so that the concepts and propositions employed in the proofs to come may be found within this document. We will give examples along the way, to aid the reader in understanding the proof methods and the background concepts.

The topic of this paper originated when Oh, Postnikov, and Yoo studied Schubert varieties by linking them with certain hyperplane arrangements [7]. Prior to this, Peterson and Carrell had shown that we can check whether the rational locus of a Schubert variety is smooth by examining its Poincaré polynomial:

Theorem 1.1. [1] For any element of a finite Weyl group $w \in W$, the Schubert variety X_w is rationally smooth if and only if the Poincaré polynomial $P_w(q)$ is palindromic; that is, if $P_w(q) = q^{\ell(w)} P_w(q^{-1})$.

Oh and Yoo conjectured that for finite Weyl groups, the Poincaré polynomial $P_w(q)$ coincides with another polynomial, the distance-enumerating function $R_w(q)$ of the chambers of a certain hyperplane arrangement corresponding to the inversions of $w \in W$, if and only if the Schubert variety X_w is rationally smooth. The proof involves factoring the polynomial $P_w(q)$ in a manner laid out by Billey and Postnikov [2], and by then exploiting a curious property of parabolic quotients which they discovered. They conjectured that this property is true for all finite Weyl groups:

Conjecture 1.2. [8] Let W be a finite Weyl group, and let J be a maximal proper subset of the simple roots. Then v has palindromic lower interval $[e, v]_J$ in W^J if and only if the interval is isomorphic to a maximal parabolic quotient of some Weyl group.

Finite Weyl groups are a special ("crystallographic") case of finite Coxeter groups, which themselves arise as the reflective and rotational symmetries of objects living in finite dimensional Euclidean space [6]. Although we do not have Schubert varietes for non-crystallographic Coxeter groups, we can nonetheless generalize the definition of Poincaré polynomial for elements of a finite Coxeter group to be the rank-generating function of their lower Bruhat interval; and instead of talking about rational smoothness, we can use the fact that $P_w(q)$ is palindromic if and only if $[e, w]$ is rank-symmetric. In these terms, Oh and Yoo made the following conjecture:

Conjecture 1.3. [8] Let W be a finite Coxeter group. Let $w \in W$. Then $[e, w]$ is rank-symmetric if and only if its Poincaré polynomial coincides with the distance-enumerating polynomial for the regions of its inversion hyperplane arrangement: $P_w(q) = R_w(q)$.

In this paper, we prove Conjecture 1.2 and Conjecture 1.3 in the case where W is a Weyl group. In Chapter II, we give necessary background on Coxeter systems, especially those of finite type. First, we see a terse but complete definition of a Coxeter group. Special emphasis is given to the combinatorial basics of reduced words and descents. We look at two related partial orderings on a Coxeter group: the strong and weak Bruhat orders. We examine parabolic subgroups — that is, Coxeter groups generated by a subset of a Coxeter group's generators — and their systems of unique minimal length left/right coset representatives. Finally, we classify and characterize the finite, irreducible Coxeter systems.

In Chapter III, we discuss hyperplane arrangements and their chambers. We connect them as the "mirrors" corresponding to the reflectional symmetries of finite reflection groups. We give the alternate description of mirror systems in terms of sets of their normal vectors, called *root systems*. Lastly, we define the subarrangements of interest to the paper.

In Chapter IV, we prove Conjecture 1.2 in a purely combinatorial, case-by-case fashion. Finally, in Chapter V, we give a proof of the Conjecture 1.3 in the case of Weyl groups, using the results of Chapter IV. We conclude by discussing the limitations of what we have accomplished, and we posit a few open questions.

CHAPTER 2

COXETER SYSTEMS

2.1 Coxeter Groups

One way to describe a group is via its *presentation* $G = \langle S|R\rangle$, where S denotes a set of elements (called the *generators*) such that each group element x can be written as a product of integral powers of the generators, and R denotes a set of *relations* on these generators. Usually, we take a group presentation to describe a group which is subject only to the relations R and any other relations that R entails. For example, the presentation $\langle S|\varnothing\rangle$ describes the free group on the set of letters S.

We proceed to define a Coxeter group.

Definition 2.1. Let S be a set. Let $m\colon S \times S \to \mathbb{N}_+ \cup \{\infty\}$. We call m a *Coxeter matrix* if it satisfies the following properties:

i. $m(s, s') = m(s', s)$ for all $s, s' \in S$.

ii. $m(s, s') = 1$ iff $s = s'$.

Let W be a group generated by S subject to the relations that $(ss')^{m(s,s')} = e$ for all $s, s' \in S$ such that $m(s, s')$ is finite. W is called a *Coxeter group*, and the pair (W, S) is a *Coxeter system*. The generators are also known as *simple reflections*

The Coxeter matrix is often conveniently encoded in a labelled graph known as the *Coxeter diagram*, whose vertices are the simple reflections. Two distinct vertices s and s' are connected by an edge if $m(s, s') \geq 3$ - that is, if $ss' \neq s's$. If $m(s, s') \geq 4$, the edge is labeled with the value of $m(s, s')$. Thus, the edges encode the *braid relations* of the generators; that is,

$$\underbrace{ss'ss'\dots s[s']}_{m(s,s')} = \underbrace{s'ss's\dots s'[s]}_{m(s,s')} \tag{1}$$

where the placement of the rightmost square brackets is to account for which letter appears last, depending on whether $m(s, s')$ is even or odd.

If $m(s, s') \in \{1, 2, 3, 4, 6\}$ for all $s, s' \in S$, then W is called a *Weyl group*, due to its appearance in the theory of Lie algebras. If $m(s, s') \in \{1, 2, 3\}$, for all $s, s' \in S$, then W is called *simply laced*. The group $\{e\}$ is called the *trivial* group, and $(\{e\}, \varnothing)$ the *trivial Coxeter system*. The cardinality of S is the *rank* of the Coxeter group. We call a Coxeter group W *reducible* if there exist nontrivial Coxeter groups W_1 and W_2 such that $W \cong W_1 \times W_2$; in the same manner, a Coxeter group which is not reducible is called *irreducible*. The Coxeter diagram of an irreducible Coxeter group is connected, and the connected components of the Coxeter diagram of W are precisely the Coxeter diagrams of the irreducible Coxeter groups W_i in the decomposition $W \cong \prod_i W_i$. The various irreducible Coxeter systems may be described. Two kinds have been completely classified: finite Coxeter groups, where $|W| < \infty$, and *affine Coxeter groups*: in which W is infinite but contains a normal abelian subgroup N such that $|W/N| < \infty$.

Example 2.2. Take $W \cong \mathfrak{S}_3 \times \mathfrak{S}_3$ to be the subgroup of $\mathfrak{S}_6$ which fixes the sets $\{1, 2, 3\}$ and $\{4, 5, 6\}$. This can be minimally generated by the adjacent transpositions $s_1 = (1\ 2)$, $s_2 = (2\ \ 3)$, $s_3 = (4\ \ 5)$, and $s_4 = (5\ \ 6)$. It has Coxeter diagram as in Figure 1. The Coxeter matrix, with row and column numbers corresponding to the indices of the generators, is

$$m = \begin{bmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 2 & 2 \\ 2 & 2 & 1 & 3 \\ 2 & 2 & 3 & 1 \end{bmatrix} \tag{2}$$

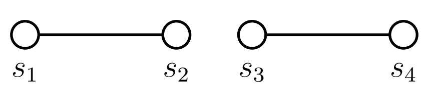

Figure 1: Coxeter diagram for $W \cong \mathfrak{S}_3 \times \mathfrak{S}_3$

2.2 Reduced Words, Reflections, and Descents

We now develop some vital terminology regarding Coxeter systems. Let (W, S) be a Coxeter system where $S = \{s_1, \ldots, s_n\}$. Suppose $w = s_{i(1)} \ldots s_{i(k)}$ and k is minimal for all possible indexings $i(j)$. Then we say that $s_{i(1)} \ldots s_{i(k)}$ is a *reduced expression* or a *reduced word* for w and write that its *length* is $\ell(w) \stackrel{\text{def}}{=} k$.

Proposition 2.3. [3] Let (W, S) be a Coxeter system. For all $s \in S$ and $w \in W$, the following are true:

i. $\ell(s) = 1$.

ii. $\ell(sw) = \ell(w) \pm 1$ and $\ell(ws) = \ell(w) \pm 1$.

iii. $\ell(w^{-1}) = \ell(w)$.

Coxeter groups have a couple definitive properties: the Exchange Property, and the Deletion Property. Here, a caret above a generator indicates deletion from an expression.

1. *The Exchange Property*: Let $w = s_{a(1)} \ldots s_{a(k)}$ be a reduced expression. Let $s \in S$. If $\ell(sw) < \ell(w)$, then $sw = s_{a(1)} \ldots \hat{s}_{a(j)} \ldots s_{a(k)}$ for some $j \in [k]$.

2. *The Deletion Property*: If $w = s_{a(1)} \ldots s_{a(k)}$ and $\ell(w) < k$, then $w = s_{a(1)} \ldots \hat{s}_{a(i)} \ldots \hat{s}_{a(j)} \ldots s_{a(k)}$ for some $1 \leq i < j \leq k$.

In fact, these two properties serve as cryptomorphic definitions of a Coxeter system:

Theorem 2.4. [3] Let $W = \langle S \rangle$ be a group and S a set of generators of order 2. The following are equivalent:

i. (W, S) is a Coxeter system.

ii. (W, S) has the Exchange Property.

iii. (W, S) has the Deletion Property.

There some immediate consequences of the Deletion Property. Firstly, any expression for a Coxeter element contains a reduced expression for itself obtainable by deleting an even number of letters. Secondly, the set of letters used in a reduced expression for a Coxeter element is invariant of the choice of expression. Thirdly, S is a minimal generating set for W.

The *reflections* of a Coxeter group - that is, the elements of order two - are all conjugates of the simple reflections:

$$T \stackrel{\text{def}}{=} \bigcup_{w \in W} wSw^{-1}. \tag{3}$$

For a given Coxeter element $w \in W$, we define two sets called (respectively) the *left/right associated reflections*:

$$T_L(w) \stackrel{\text{def}}{=} \{t \in T \colon \ell(tw) < \ell(w)\} \text{ and } T_R(w) \stackrel{\text{def}}{=} \{t \in T \colon \ell(wt) < \ell(w)\}. \tag{4}$$

It is easy to see that $T_L(w) = T_R(w^{-1})$ for any $w \in W$ (and vice versa). From the associated reflections, we define the *left/right associated descent sets*:

$$D_L(w) \stackrel{\text{def}}{=} T_L(w) \cap S \text{ and } D_R(w) \stackrel{\text{def}}{=} T_R(w) \cap S. \tag{5}$$

We may actually strengthen the statement of the Exchange Property to the following:

Theorem 2.5. [3] Let (W, S) be a Coxeter system. Let $w = s_{a(1)} \dots s_{a(k)}$ be a reduced expression. If $t \in T_L(w)$, then $tw = s_{a(1)} \dots \hat{s}_{a(j)} \dots s_{a(k)}$ for some $j \in [k]$.

Having defined the associated reflections and descents, several useful properties follow as corollaries of the above *Strong Exchange Property*:

Corollary 2.6. [3] $|T_L(w)| = \ell(w) = |T_R(w)|$.

Corollary 2.7. [3] Suppose $s \in S$ and $w \in W$. Then s is in the left descent set of w if and only if some reduced expression for w begins with s. Similarly, s is in the right descent set of w if and only if some reduced expression for w ends with s.

2.3 The Finite Irreduclble Coxeter Systems

The finite, irreducible Coxeter systems have been fully classified into several families and a few exceptional types. The families are the types $A_{n\geq 1}$, $B_{n\geq 2}$, $D_{n\geq 4}$, and $I_2(n \geq 3)$. The exceptional cases are the types E_6, E_7, E_8, F_4, G_2, H_3, and H_4. The crystallographic Coxeter systems are A_n, B_n, D_n, E_6, E_7, E_8, F_4, and G_2. Note that there is some redundancy here, as $A_2 \cong I_2(3)$, $B_2 \cong I_2(4)$, and $G_2 \cong I_2(6)$.

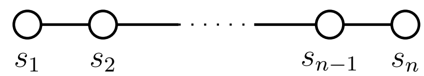

Figure 2: Type A_n Coxeter system

Let $n \geq 1$ be an integer. Let $\mathfrak{S}_{n+1}$ denote the *symmetric group* on $n+1$ letters; that is, the group of bijections $w\colon [n+1] \to [n+1]$ (called *permutations*) closed under composition. Let $S = \{s_1, \ldots, s_n\}$ be the subset of $\mathfrak{S}_{n+1}$ consisting of the adjacent transpositions $s_i = (i, i+1)$ for $i \in [n]$. $(\mathfrak{S}_{n+1}, S)$ is a Coxeter system of type A_n with diagram shown in Figure 2. Its group order is $(n+1)!$, and it has $\binom{n+1}{2}$ reflections. Geometrically, this describes the rotational and reflective symmetries of a regular n-simplex. For $w \in \mathfrak{S}_{n+1}$, the length of w as a Coxeter word coincides with the *inversion* statistic of w:

$$\ell_A(w) = \mathrm{inv}(w) = \#\{(i,j) \in [n+1]^2 \colon i < j \text{ and } w(i) > w(j)\}. \tag{6}$$

Because of this, it is often more convenient to specify a permutation $w \in \mathfrak{S}_n$ in bracket notation, as — $[w(1), w(2), \ldots, w(n)]$ — rather than writing it in disjoint cycle notation. We may also write it in a shortened form by omitting the brackets and commas, when the meaning is clear. For example, the longest element w_0 of $\mathfrak{S}_4$ could be written either as $w_0 = (1\,4)(2\,3)$, as $w_0 = [4,3,2,1]$, or as $w_0 = 4321$.

Let $n \geq 2$. Denote by $\mathfrak{S}_n^B$ the group of "signed permutations" w of $-[n] \cup [n]$ such that $w(-k) = -w(k)$ for every $-n \leq k \leq n$. Let $S = \{s_0^B, s_1, \ldots, s_{n-1}\}$, where $s_0^B = (1, -1)$ and $s_i = (i, i+1)(-i, -i-1)$ for $i \in [n-1]$. Then $(\mathfrak{S}_n^B, S)$ is a type B_n Coxeter

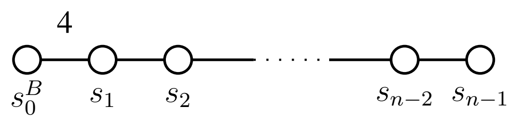

Figure 3: Type B_n Coxeter system

system with diagram as in Figure 3. Its group order is $2^n n!$, and it contains n^2 reflections. Note that $\langle S \setminus \{s_0^B\}\rangle$ is a type A_{n-1} Coxeter system, since its action on $[n]$ is the same as that of $\mathfrak{S}_n$. Geometrically, a type B_n Coxeter group describes the rotational and reflective symmetries of an n-cube and its dual polytope, the n-dimensional hyperoctahedron. For this reason, $\mathfrak{S}_n^B$ is also called a *hyperoctahedral group*. When abbreviating the bracket notation of a signed permutation, we write negative numbers with a horizontal bar above them, rather than to the left. For example, the longest element w_0 of $\mathfrak{S}_4^B$ could be written $[-1,-2,-3,-4]$ or $\bar{1}\bar{2}\bar{3}\bar{4}$, and $s_0^B \in \mathfrak{S}_4^B$ could be written $s_0^B = [-1,2,3,4]$ or $s_0^B = \bar{1}234$.

The length of $w \in \mathfrak{S}_n^B$ as a Coxeter element coincides with the *B-inversion* statistic. This statistic is defined in terms of the inversion number from (6); nsp, the number of negative sum pairs (8); and neg, the number of negative entries in the bracket notation (7). The following are defined for all $w \in \mathfrak{S}_n^B$:

$$\mathrm{neg}(w) \overset{\text{def}}{=} \#\{i \in [n] \colon w(i) < 0\}. \tag{7}$$

$$\mathrm{nsp}(w) \overset{\text{def}}{=} \#\{(i,j) \in [n]^2 \colon i < j \text{ and } w(i) + w(j) < 0\}. \tag{8}$$

$$\ell_B(w) = \mathrm{inv}_B(w) \overset{\text{def}}{=} \mathrm{inv}(w) + \mathrm{neg}(w) + \mathrm{nsp}(w). \tag{9}$$

We have another useful way to enumerate the length:

$$\mathrm{inv}_B(w) = \mathrm{inv}(w) - \sum_{i \in [n] \colon w(i) < 0} w(i). \tag{10}$$

Let $n \geq 4$. Let $\mathfrak{S}_n^D \overset{\text{def}}{=} \{w \in \mathfrak{S}_n^B \colon \ \mathrm{neg}(w) \text{ is even}\}$. Take $s_1, \ldots, s_{n-1}$ to be the same as described for $\mathfrak{S}_n^B$ and let $s_0^D = [-2,-1,3,\ldots,n]$. Let $S = \{s_0^D, s_1, \ldots, s_{n-1}\}$. Then

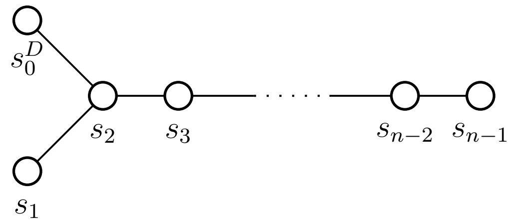

Figure 4: Type D_n Coxeter system

$(\mathfrak{S}_n^D, S)$ is a Coxeter system of type D_n with diagram as in Figure 4. Its group order is $2^{n-1}n!$, and it contains $n^2 - n$ reflections. The length of an element $w \in \mathfrak{S}_n^D$ in terms of elements of S is given by what we call D-inversions.

$$\ell_D(w) = \mathrm{inv}_D(w) \stackrel{\text{def}}{=} \mathrm{inv}(w) + \mathrm{nsp}(w). \tag{11}$$

Take care to note that $\mathrm{inv}_B(w) = \mathrm{inv}_D(w)$ if and only if $\mathrm{neg}(w) = 0$.

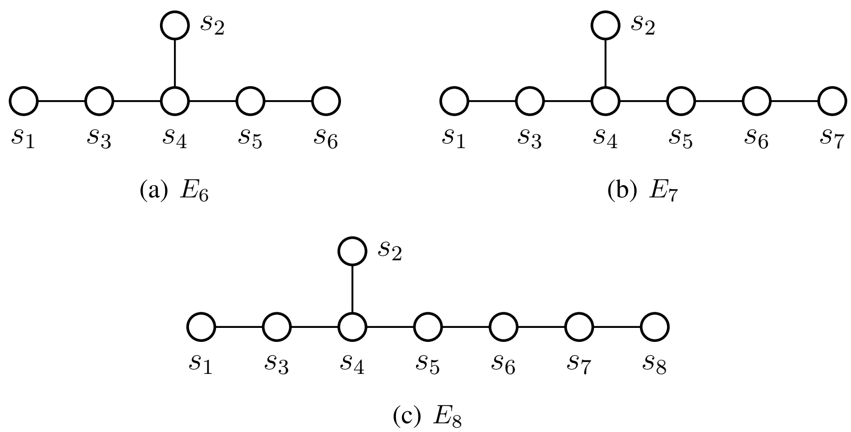

Figure 5: Finite type E_n Coxeter systems

The type E_6 Weyl group is of order $2^7 \cdot 3^4 \cdot 5$ and has 36 reflections. The type E_7 Weyl group is of order $2^{10} \cdot 3^4 \cdot 5 \cdot 7$ and has 63 reflections. The type E_8 Weyl group is of order $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ and has 120 reflections. These three exceptional types do not arise as the symmetries of any regular polytope, but they are of interest in the study of Lie algebras.

The exceptional type F_4 Coxeter system (diagram shown in Figure 6) describes the symmetries of the regular 24-cell, a four dimensional polytope. It has group order 1152,

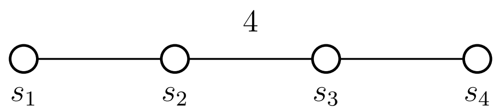

Figure 6: Type F_4 Coxeter system

and it contains 24 reflections.

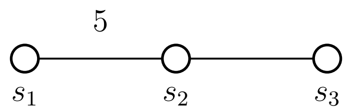

Figure 7: Type H_3 Coxeter system

The exceptional type H_3 Coxeter system (diagram shown in Figure 7) describes the symmetries of the regular icosahedron and its dual polytope, the regular dodecahedron. It has group order 120, and it contains 15 reflections.

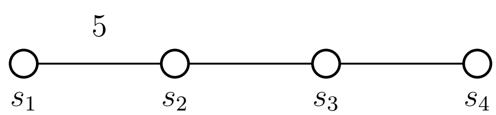

Figure 8: Type H_4 Coxeter system

The exceptional type H_4 Coxeter system (diagram shown in Figure 8) describes the symmetries of the regular 600-cell and its dual polytope, the regular 120-cell. It has group order 14400, and it contains 60 reflections.

Let $m \geq 3$. The type $I_2(m)$ Coxeter group (diagram shown in Figure 9) is called a *dihedral group*, because it describes the reflectional and rotational symmetries of a regular m-gon. It has group order $2m$, and it contains m reflections. In the case of $m = 6$, $I_2(6)$ is also referred to as the exceptional Weyl group G_2 to emphasize the fact that it arises from an exceptional Lie algebra. $I_2(3)$ and $I_2(4)$ are usually referred to as A_2 and B_2, respectively.

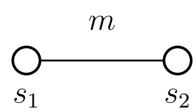

Figure 9: Type $I_2(m)$ Coxeter system

2.4 Strong Bruhat Order

There are two partial orderings on Coxeter groups which are immensely useful. The primary one is called the *strong Bruhat order*. We first define what a partially ordered set is and introduce some useful terminology. The reader who desires to know more about posets, and in particular, lattices (which we refrain from discussing here) is referred to Stanley[9].

Definition 2.8. A partially ordered set, or *poset*, is a set P combined with a *partial ordering* $\preceq$; that is, a relation that obeys the following three properties of reflexivity, antisymmetry, and transitivity:

i. $x \preceq x$ for all $x \in P$.

ii. $x \preceq y$ and $y \preceq x$ if and only if $x = y$, for all $x, y \in P$.

iii. If $x \preceq y$ and $y \preceq z$, then $x \preceq z$ for all $x, y, z \in P$.

We write $x \prec y$ to denote that $x \preceq y$ and $x \neq y$. A *chain* of length k is a sequence of elements $x_0, \ldots, x_k \in P$ such that $x_0 \prec x_1 \prec \cdots \prec x_k$. If there is no $u \in P$ such that for some index $i \in [k]$, $x_{i-1} \prec u \prec x_i$, then we say the chain is *saturated*. If there is no chain $C \subset P$ such that $\{x_i\}_{i=0}^k \subsetneq C$, then we say $x_0 \prec \cdots \prec x_k$ is a *maximal chain*. If $x, y \in P$ such that $x \prec y$ is a saturated chain, then we say that y *covers* x and write $x \lhd y$ For $x, y \in P$, the *interval* $[x, y]$ is the set $\{z \in P \colon x \preceq z \preceq y\}$. If P has a unique element $\hat{0}$ such that $\hat{0} \preceq x$ for all $x \in P$, then we call that element the *minimal* element and often denote it as $\hat{0}$. If P has a unique element $\hat{1}$ such that $x \preceq \hat{1}$ for all $x \in P$, then we say P is a *directed poset* having *maximal* element often denoted just as $\hat{1}$. For a graded poset P, we have a *rank function* $\rho \colon P \to \mathbb{N}$ satisfying

- $\rho(\hat{0}) = 0$ if $\hat{0}$ is minimal.

- $\rho(x) = \rho(y) + 1$ if $y \lhd x$.

A map of posets $\phi\colon P \to Q$ is *order-preserving* if for all $s, t \in P$, $s \preceq_P t$ implies $\phi(s) \preceq_Q \phi(t)$. A *poset isomorphism* is an order-preserving bijection $\phi\colon P \to Q$ whose inverse ϕ^{-1} is also order-preserving; so,

$$s \preceq_P t \Longleftrightarrow \phi(s) \preceq_Q \phi(t) \text{ for all } s, t \in P \tag{12}$$

When such a map exists, we say the posets P and Q are *isomorphic* and write $P \cong Q$. A poset isomorphism $P \to P$ is called an *automorphism*. If $Q \subseteq P$ are sets with partial orderings such $s \preceq_Q t$ if and only if $s \preceq_P t$ for all $s, t \in Q$, then we say Q is an *induced subposet* of P.

Definition 2.9. Let (W, S) be a Coxeter system.

i. Denote by $u \lhd v$ that there exists a reflection $t \in T$ such that $v = tu$ and $\ell(v) = \ell(u) + 1$.

ii. Let $u \leq nv$ denote that there exists a sequence $\{u_i\}_{i=0}^{k}$ in W such that $u = u_0 \lhd u_1 \lhd \ldots \lhd u_k = v$.

iii. Then $\leq$ gives a partial ordering on the set W called the *strong Bruhat order*, or simply *Bruhat order* when there is no ambiguity.

Note that we could just as well have taken the covering relation to be $u \lhd' v$ if and only if $v = ut$ for some $t \in T$ and $\ell(v) = \ell(u) + 1$. The transitive closure if the covering relation $\lhd'$ actually coincides with the transitive closure of the covering relation $\lhd$ seen above; hence there is no distinction of a "left" versus a "right" strong Bruhat order. W with the Bruhat ordering is a graded poset whose rank function is the length function ℓ giving the length of reduced expressions of Coxeter elements in terms of the simple reflections. For finite W, there exists a unique maximal element $w_0 \in W$. Furthermore, $w \mapsto w_0 w$ and $w \mapsto w w_0$ are anti-automorphisms of the strong Bruhat order.

Example 2.10. The strong Bruhat order on a type A_2 Coxeter systems has Hasse diagram as in Figure 10.

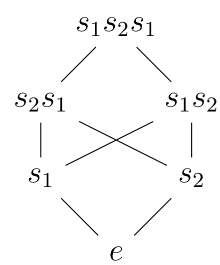

Figure 10: Strong Bruhat order on A_2

2.5 Parabolic Subgroups & Quotients

A subgroup $W_J \overset{\text{def}}{=} \langle J \rangle \subseteq W$ generated by a subset $J \subseteq S$ of the simple reflections is called a *parabolic subgroup*.

Proposition 2.11. [3] The following are true for all $I, J \subseteq S$:

i. (W_J, J) is a Coxeter system.

ii. The length function for W_J is the restriction of the length function for W.

iii. $W_I \cap W_J = W_{I \cap J}$.

iv. $\langle W_I \cup W_J \rangle = W_{I \cup J}$.

v. $W_I = W_J$ implies that $I = J$.

A finite, parabolic subgroup W_J has a unique maximal element which we denote $w_0(J)$. By convention we say that $w_0(\varnothing) = \{e\}$.

A *right quotient* for $J \subseteq S$ is the set $W^J \stackrel{\text{def}}{=} \{w \colon D_R(w) \subseteq S \setminus J\}$. So, $w \in W^J$ if and only if $w \lhd ws$ for all $s \in J$. Similarly, a *left quotient* is the set

$$^J W \stackrel{\text{def}}{=} \{w \colon D_L(w) \subseteq \ S \ \setminus J\}. \tag{13}$$

The following proposition will play an important role later on:

Proposition 2.12. [3] Let $J \subseteq S$. Let $w \in W$. Then w has a unique factorization $w = w^J \cdot w_J$ where $w^J \in W^J$, $w_J \in W_J$, and $\ell(w) = \ell(w_J) + \ell(w^J)$.

Although we must distinguish between left and right quotients, most every construction and proposition which applies to right quotients can be mirrored and applied to left quotients and their cosets, etc. For example, w has also a unique factorization $w = w_J \cdot {}^J w$, where $w_J \in W_J$, ${}^J w \in^J W$, and $\ell(w) = \ell(w_J) + \ell({}^J w)$. Also, just as it is true that for any v, $v \in W^J$ if and only if no reduced expression for v ends with a letter from J, so it is true that $v \in {}^J W$ if and only if no reduced expression for v begins with a letter from J.

The following characterization of these quotients follows as a corollary of the above proposition:

Corollary 2.13. [3] Each left coset wW_J has a unique representative of minimal length. In fact, W^J is precisely the system of these unique minimal left coset representatives.

The next proposition tells us that the strong Bruhat order on W^J is an induced subposet of the strong Bruhat order on W:

Proposition 2.14. [3] The projection map $W \ \rightarrow \ W^J$ given by $w \ \mapsto \ w^J$ is an order preserving map of posets.

This means that W^J is also a directed, graded poset whose length function is the restriction of the length function of W, and that if $|W| < \infty$, then W^J has a unique maximal element denoted w_0^J. For this element,

- $w_0 = w_0^J w_0(J)$.

- $\ell(w_0) = \ell(w_0^J) + \ell(w_0(J))$.

The following proposition is of high importance to us, for it tells us that finite Bruhat quotients are *rank-symmetric* or *palindromic*; i.e. if $n = \ell(w_0^J)$, then the sequence $(a_k)_{k=0}^n$ defined by $a_k = \#\{w \in W^J : \ell(w) = k\}$ has the property that $a_k = a_{n-k}$ for every $0 \leq k \leq n$.

Proposition 2.15. [3] For (W, S) a finite Coxeter system and $J \subseteq S$, the map $\alpha : W^J \to W^J$ given by $x \mapsto w_0 x w_0(J)$ is an anti-automorphism of the strong Bruhat order; that is, $x \leq y$ if and only if $\alpha(y) \leq \alpha(x)$.

Example 2.16. Let (W, S) be a type B_4 Coxeter system with simple reflections labeled as in Figure 3. Let $J = S \setminus \{s_0\}$. The Hasse diagram of the Bruhat order on the quotient W^J is shown in Figure 2.16

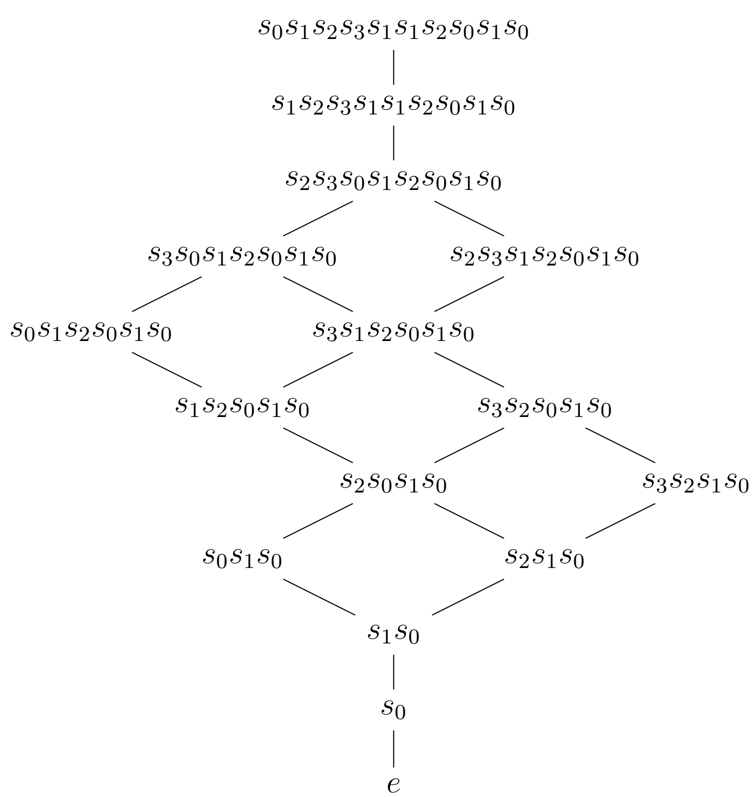

Figure 11: Bruhat order on B_4/A_3

2.6 Weak Bruhat Order

There are a couple partial order relations which are weakened versions of the strong Bruhat order: the right and left weak Bruhat order.

Definition 2.17. [3] Let (W, S) be a Coxeter system and $u, w \in W$.

1. Write $u \leq_R w$ to denote that there exists a sequence $s_1, \ldots, s_k \in S$ such that $w = us_1 \ldots s_k$ and $\ell(us_1 \ldots s_i) = \ell(w) + i$ for all $0 \leq i \leq k$. This is the *right weak Bruhat order*, or *right order* for short.

2. Write $u \leq_L w$ to denote that there exists a sequence $s_1, \ldots, s_k \in S$ such that $w = s_k \ldots s_1 u$ and $\ell(s_i \ldots s_1 u) = \ell(u) + i$ for all $0 \leq i \leq k$.

The weak orders are both proper subposets of the strong Bruhat order:

$$u \leq_R w \text{ or } u \leq_L w \text{ implies } u \preceq w. \tag{14}$$

$[u, w]_R$ will denote an interval in the right order, and $[u, w]_L$ an interval in the left order. We list a few facts about the right weak order (analogous facts exists for the left order):

- Reduced decompositions of w are in bijection with maximal chains in $[e, w]_R$.
- $u \leq_R w$ iff $\ell(u) + \ell(u^{-1}w) = \ell(w)$.
- For $|W| < \infty$, $w \leq_R w_0$.
- The *prefix property*: $u \leq_R w$ if and only if there exists a reduced expression $u = s_1 \ldots s_k$ and $w = s_1 \ldots s_k s'_1 \ldots s'_q$.
- Weak order is a graded poset ranked by the usual length function: $\ell : W \to \mathbb{N}$.
- Suppose $s \in D_L(u) \cap D_L(w)$. Then $u \leq_R w$ iff $su \leq_R sw$.

$$
\begin{tikzcd}
 & s_1s_2s_1 \arrow[dl, dash] \arrow[dr, dash] & \\
s_2s_1 \arrow[d, dash] & & s_1s_2 \arrow[d, dash] \\
s_1 \arrow[dr, dash] & & s_2 \arrow[dl, dash] \\
 & e &
\end{tikzcd}
$$

Figure 12: Left weak Bruhat order on A_2

Example 2.18. The left weak Bruhat order on a type A_2 Coxeter systems has Hasse diagram as in Figure 12; compare with strong Bruhat order in Figure 10.

CHAPTER 3

ROOT SYSTEMS AND HYPERPLANE ARRANGEMENTS

3.1 Hyperplane Arrangements

Although hyperplanes may be defined in any finite-dimensional vector space over any field, we restrict our discussion to real vector spaces here. Let V be an n-dimensional real vector space. A *hyperplane* H is a linear subspace of *codimension* 1 — that is, of dimension $n-1$. A hyperplane may be defined as the set of all points perpendicular to a nonzero *normal* vector α: $H = \{x \in V\colon (\alpha|x) = 0\}$ where $(\cdot|\cdot)$ denotes the scalar product. An *affine hyperplane* H' is an affine subspace of codimension 1. Often, we omit the word "affine" and simply say "hyperplane" to refer to both linear and affine subspaces of codimension 1. It may be described by $H' = \{x \in V\colon (x|\alpha) = c\}$ for some $c \in \mathbb{R}$. A *hyperplane arrangement* $\mathcal{A}$ is a finite collection of hyperplanes in V. If $\bigcap_{H\in\mathcal{A}} H \neq \varnothing$, then we call $\mathcal{A}$ a *central* arrangement. For a central arrangement, we can assume without loss of generality that the point contained in the intersection of every hyperplane is the origin.

A hyperplane H partitions V into two half spaces: $V = V_H^+ \uplus H \uplus V_H^-$. When H has been defined in terms of a normal vector α, we may use V_α^+ to denote the set of all $x \in V$ such that $(x|\alpha) > 0$, and V_α^- to denote the set of all x such that $(x|\alpha) < 0$. If a and b are points such that $a \in V_H^+$ and $b \in V_H^-$, then we say that H *separates* a and b. The *chambers* (or *regions*) of an arrangement $\mathcal{A}$ are the components of $V \setminus \bigcup_{H\in\mathcal{A}} H$. They are open, convex subsets of V. We may denote the chambers, or regions, of $\mathcal{A}$ as $\mathcal{R}(\mathcal{A})$.

A *panel* of a chamber $C \in \mathcal{R}(\mathcal{A})$ is a face of codimension 1 intersecting the boundary of the chamber. A panel is a subset of a hyperplane, which is called a *wall* of C. If two chambers C and C' share a common panel, then that panel comes from a unique hyperplane; we say that C and C' are *adjacent*. A *gallery* Γ is a sequence $C_0, C_1, \ldots, C_\ell$ of chambers such that C_{i-1} is adjacent to C_i for every $i \in [\ell]$. We say Γ has length ℓ and endpoints

C_0 and C_ℓ. A gallery is *geodesic* if it has minimal length among all galleries connecting its endpoints. The *distance* $d(C, D)$ between two chambers C and D is the length of a geodesic gallery connecting them. Any two chambers in a hyperplane arrangement can be connected by a gallery. A hyperplane which separates the endpoints of a gallery $C_0, \ldots, C_\ell$ must give a common panel between two chambers C_{i-1} and C_i for some $i \in [\ell]$. For further discussion on hyperplane arrangements, refer to Borovik, Gelfand and White [4].

3.2 Mirrors and Reflections

Recall that in the space $\mathbb{R}^n$ equipped with the Euclidean norm $\|\cdot\|\colon \mathbb{R}^n \to [0, \infty)$, the orthogonal group $\mathfrak{o}_n(\mathbb{R})$ is the group of origin-preserving isometries: that is, bijective linear transformations $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\|f(x) - f(y)\| = \|x - y\|$ for every x, y in $\mathbb{R}^n$. A *reflection* in a real Euclidean space is a nontrivial isometry $s \in \mathfrak{o}_n(\mathbb{R})$ which fixes a hyperplane H, which we call the *mirror* of s. We may variously find it helpful to write s_H for the reflection fixing the mirror H. A *closed system of mirrors* is a central arrangement of hyperplanes such that for every $H_1, H_2 \in \mathcal{A}$, $s_{H_2}(H_1) \in \mathcal{A}$. The corresponding set of reflections, T, is called a closed system of reflections. It may be easily verified that T is a closed system of reflections if and only if $t^{-1}st \in T$ for every $s, t \in T$. The mirrors of symmetry of a solid $\Delta \subset \mathbb{R}^N$ — namely, the hyperplanes $\mathcal{A}$ such that $s_H(\Delta) = \Delta$ for every $H \in \mathcal{A}$. A finite closed system of reflections generates a finite group of isometries, called a *finite reflection group*.

Proposition 3.1. [4] For a finite reflection group $W \subset \mathfrak{o}_n(\mathbb{R})$, there is a point $\alpha \in \mathbb{R}^n$ such that $w(\alpha) = \alpha$ for every $w \in W$. If $\mathcal{A}$ is the corresponding closed, finite system of mirrors, then $\bigcap_{H \in \mathcal{A}} H \neq \varnothing$.

As a consequence of the above proposition, we may assume without loss of generality that any finite closed system of mirrors is a central arrangement.

3.3 Root Systems

Suppose H is a hyperplane with normal vector α. The corresponding reflection can be written

$$t_\alpha(\beta) \stackrel{\text{def}}{=} \beta - \frac{2(\beta|\alpha)}{(\alpha|\alpha)}\alpha. \tag{15}$$

Definition 3.2. A *root system* is a finite set $\Phi \subset V$ in a Euclidean vector space, satisfying

i. $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$.

ii. $t_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$.

If $\mathcal{A}$ is a finite closed system of mirrors $\{H_\alpha\}$, then the collection of all the defining normal vectors $\{\alpha\}$ forms a root system. Root systems induce finite reflection groups:

Lemma 3.3. [4] Let Φ be a root system. Then $W = \langle t_\alpha \colon \alpha \in \Phi\rangle$ is finite.

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a linear functional such that $f(\alpha) \neq 0$ for all $\alpha \in \Phi$. Then we can partition $\Phi = \Phi^+ \uplus \Phi^-$ evenly into a set of *positive roots* and a set of *negative roots*, respectively, where

$$\Phi^+ \stackrel{\text{def}}{=} \{\alpha \in \Phi \colon f(\alpha) > 0\}, \quad \Phi^- \stackrel{\text{def}}{=} \{\alpha \in \Phi \colon f(\alpha) < 0\}. \tag{16}$$

Note that $\Phi^- = -\Phi^+$.

Let $\Gamma = \sum_{\alpha \in \Phi^+} \mathbb{R}_{\geq 0}\alpha$ be the convex polyhedral cone spanned by the positive roots. We call the positive roots directed along the edges of Γ the *simple roots* Π. Clearly, Γ is minimally spanned by Π.

Proposition 3.4. [4] Every system of simple roots is linearly independent. In particular, every root $\beta \in \Phi$ can be written as a unique linear combination $\beta = \sum_{\alpha \in \Pi} c_\alpha \alpha$, where the c_α's are either all non-positive (when $\beta \in \Phi^-$) or all nonnegative (when $\beta \in \Phi^+$).

Corollary 3.5. [4] Every simple system of roots for Φ has the same cardinality.

3.4 Inversion Hyperplane Arrangements

Let (W, S) be a finite Coxeter system. We define the *inversion set* of w:

$$\Delta_w \stackrel{\text{def}}{=} \{\alpha \in \Phi^+ \colon w(\alpha) \in \Phi^-\}. \tag{17}$$

The *inversion hyperplane arrangement* consists of the hyperplanes corresponding to the roots in the inversion set and can be described as

$$\mathcal{A}_w \stackrel{\text{def}}{=} \{H_\alpha \colon \alpha \in \Delta_w\}, \text{ where } H_\alpha = \{x \colon (x|\alpha) = 0\}. \tag{18}$$

We define the *fundamental chamber* $r_0 \in \mathcal{R}(\mathcal{A}_w)$ as the (unique) chamber satisfying that $(\alpha|x) > 0$ for all $x \in r_0$. From here, we define the *distance-enumerating function*, where distance $d(r_0, r)$ denotes the length of a geodesic gallery connecting the fundamental region r_0 to the chamber r:

$$R_w(q) \stackrel{\text{def}}{=} \sum_{r \in \mathcal{R}(\mathcal{A}_w)} q^{d(r_0, r)}. \tag{19}$$

Note that $R_w(q) = R_{w^{-1}}(q)$ for all $w \in W$ [8].

Let $\mathcal{A}' \subset \mathcal{A}_w$ be a subarrangement. Let $c \in \mathcal{R}(\mathcal{A}')$. The *chamber graph* of c with respect to $\mathcal{A}_w$ is defined as a directed graph $G = (V, E)$ where

- The vertex set V consists of vertices representing each chamber of $\mathcal{A}_w$ contained in c.
- We have an edge directed from $c_1 \to c_2$ if c_1 and c_2 are adjacent and $d(r_0, c_1) + 1 = d(r_0, c_2)$.

We will say that $\mathcal{A}_w$ is *uniform* with respect to $\mathcal{A}'$ if for all chambers of $\mathcal{A}'$, chamber graphs with respect to $\mathcal{A}_w$ are mutually isomorphic. It is easy to see that if $\mathcal{A}_u \subset \mathcal{A}_w$ and $\mathcal{A}_w$ is uniform with respect to $\mathcal{A}_u$, then $R_u(q) | R_w(q)$.

Let $w_0 \in W$ be the maximal element of W. Then $\mathcal{A}_{w_0}$ is the entire *Coxeter [hyperplane]* *arrangement* for the finite system of mirrors of the finite reflection group W. It can be readily seen that the chamber graph of $\mathcal{A}_{w_0}$ looks like the weak Bruhat graph of W.

CHAPTER 4

A PROPERTY OF PARABOLIC QUOTIENTS

The key part of proving the main result involves a property of maximal parabolic quotients which was observed and partially proven by Oh and Yoo [8], and which we prove here for finite Coxeter groups. We will state these results and prove them for right quotients, partly because Björner and Brenti give nice combinatorial descriptions for right quotients [3], but every result here can be mirrored for left quotients.

4.1 Type A_n/A_{n-1} Quotients

Proposition 4.1. Suppose (W, S) is a Coxeter system of type A_n with simple reflections labeled as in Figure 2, and suppose that $J = \{s\}$ where s is a leaf in the Coxeter diagram. Then W^J is a chain of length $n + 1$. Furthermore, for every $v \in W^J$, there is a subset $I \subseteq S$ such that $[e, v]_J = W_I^{I\cap J}$, where $W_I^{I\cap J}$ is a type $A_{\ell(v)}/A_{\ell(v)-1}$ quotient.

Proof. Take $J = S \setminus \{s_1\}$. Then W^J is the chain

$$e < s_1 < s_2 s_1 < \ldots < s_n s_{n-1} \cdots s_2 s_1. \tag{20}$$

For any $v \in W^J$, take $I = \{s_1, \ldots, s_{\ell(v)}\}$. Then $[e, v]_J = W_I^{I\cap J}$.

The proof for $J = S \setminus \{s_n\}$ is similar. □

4.2 Type B_n/B_{n-1} and B_n/A_{n-1} Quotients

Assume by default in this section that (W, S) is a type B_n Coxeter system with simple reflections labelled as in Figure 3. Due to the asymmetry of the Coxeter diagram, we have two different types of leaf quotient to examine. Taking $J = S \setminus \{s_0\}$, we examine the type B_n/B_{n-1} case:

Proposition 4.2. Let (W, S) be a type B_n Coxeter system with simple reflections labelled as in Figure 3. Let $J = S \setminus \{s_{n-1}\}$. Then W^J is a chain poset of length $2n$.

Proof. Every element of W^J can be written as a tail of the reduced expression of the maximal element:

$$v_0 = s_{n-1}s_{n-2}\cdots s_1s_0s_1\cdots s_{n-1} \tag{21}$$

$\square$

Example 4.3. Pictured in Figure 13 is a type B_3/B_2 quotient.

$$\begin{array}{c} s_2s_1s_0s_1s_2 \\ | \\ s_1s_0s_1s_2 \\ | \\ s_0s_1s_2 \\ | \\ s_1s_2 \\ | \\ s_2 \\ | \\ e \end{array}$$

Figure 13: Type B_3/A_2 quotient

To tackle the other case when $J = S \setminus \{s_0\}$, we identify W with $\mathfrak{S}_n^B$.

Lemma 4.4. Let (W, S) be a type B_n Coxeter system and $J = S \setminus \{s_0\}$. Let $u, v \in W^J$ such that $u \leq v$. Then $u(i) \geq v(i)$ for every $i \in [n]$.

Proof. Björner and Brenti [3] give the following description of our quotient:

$$W^J = \{v\colon v(1) < v(2) < \cdots < v(n)\} \tag{22}$$

Let $u \in W^J$. For $i \in [n-1]$, $u < s_iu$ if and only if $u^{-1}(i) < 0$ and $u^{-1}(i+1) > 0$. Assuming this is true, then

$$s_iu(k) = \begin{cases} -i-1 & \text{if } u(k) = i \\ i & \text{if } u(k) = i+1 \\ u(k) & \text{otherwise} \end{cases} \tag{23}$$

whence $u(k) \geq s_i u(k)$ for every $k \in [n]$. The rest follows inductively. $\square$

Define $M(n)$ to be the set of subsets of $[n]$ endowed with the partial ordering $\preceq$ defined as follows: let $A, B \subseteq [n]$. Write $A = \{a_1 < \cdots < a_j\}$ and $B = \{b_1 < \cdots < b_k\}$. Then $A \preceq B$ denotes that $j \leq k$ and $a_{j-i} \leq b_{k-i}$ for every $0 \leq i \leq k-1$.

Lemma 4.5. Let (W, S) be a type B_n Coxeter system and $J = S \setminus \{s_0\}$. Then $W^J \cong M(n)$.

Proof. Define $\phi\colon W^J \to M(n)$ such that $\phi(v) = \{i \in [n]\colon v^{-1}(i) < 0\}$. Since each $v \in W^J$ is determined precisely by the choice of numbers $N \subseteq [n]$ which appear with a negative sign in the window notation $[v(1), \ldots, v(n)]$, it follows that ϕ is a bijection. That ϕ and its inverse are order-preserving is simple to verify.

$\square$

We now have the tools to efficiently prove the following property for the type B_n/A_{n-1} case.

Proposition 4.6. Let (W, S) be a type B_n Coxeter system with simple reflections labelled as in Figure 3 and let $J = S \setminus \{s_0\}$. Let $v \in W^J$. If $[e, v]_J$ is rank-symmetric, then either $[e, v]_J$ is a chain poset or there exists an $I \subseteq S$ such that $[e, v]_J = W_I^{I \cap J}$ where $W_I^{I \cap J}$ is a type B_k/A_{k-1} quotient for an integer $k \in [n]$ such that $\ell_B(v) = \binom{k+1}{2}$.

Proof. For $v \in W^J$, define

$$U(v) \stackrel{\text{def}}{=} \{w \in W^J \colon v \lhd w\} \tag{24}$$

$$D(v) \stackrel{\text{def}}{=} \{w \in W^J \colon w \lhd v\} \tag{25}$$

Let $\phi\colon W^J \to M(n)$ be the poset isomorphism defined in Lemma 4.5. Let k be an integer such that $2 \leq k \leq n$. Let $I = \{s_0, s_1, \ldots, s_{k-1}\}$. By Lemma 4.5, $W_I^{I \cap J} \cong M(k)$. Thus, for w_0 the unique maximal element of $W_I^{I \cap J}$, $[e, w_0]_J$ is rank-symmetric, and is not

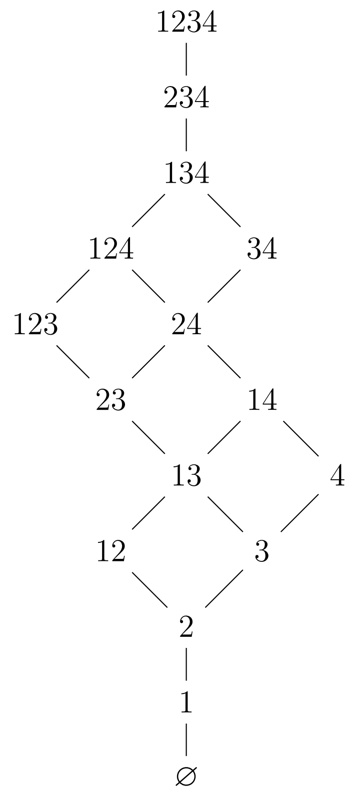

Figure 14: The distributive lattice $M(4) \cong B_4/A_3$

a chain type for $k \geq 3$. To show that these are the only elements with rank-symmetric lower intervals that are not chains, we look to $M(n)$. The corresponding element in $M(n)$ is $\phi(w_0) = [k]$, with $[\varnothing, [k]] = M(k)$.

Assume for the sake of contradiction that there is a subset $A \subset [n]$ such that $[\varnothing, A]$ is a rank-symmetric non-chain interval, and so that $A \neq [k]$ for any $k \leq n$. We take $n \geq 3$, since $M(2) \cong [3]$. Examine the bottom ranks of $M(n)$. The unique minimal element is $\varnothing$. $U(\varnothing) = \{1\}$. $U(U(\varnothing)) = U^2(\varnothing) = \{2\}$. Thus, $\#U(\varnothing) = 1$, $\#U^2(\varnothing) = 1$. Since $[\varnothing, A]$ is rank-symmetric, then $\#D(A) = \#U(\varnothing) = 1$ and $\#D^2(A) = \#U^2(\varnothing) = 1$. Since $\#D(A) = 1$, then A must be a succession of positive integers $A = \{j, j+1, \ldots, k-1, k\}$ for some $1 < j < k \leq n$. To see why this must be, consider a subset $B = \{2, 4\} \subset [n]$, which has a greater-than-unit gap between two consecutive integers. $D(B) = \{14, 23\}$. Similarly, if $C = \{1, 3\} \subset [n]$, then $D(C) = \{3, 12\}$. So, having a "gap" in a succession of integers is the necessary and sufficient condition for $\#D(B) > 1$ for any $B \subset [n]$.

Unfortunately, since $A = \{j, \ldots, k\}$ with $j > 1$, $D(A) = \{\{j-1, j+1, \ldots, k\}\}$. By our observation regarding gaps in integer successions, $\#D^2(A) > 1$, meaning $D^2(A) \neq U^2(\varnothing)$. This contradicts the assumption that $[\varnothing, A]$ is rank-symmetric. Therefore, if $A \subseteq [n]$ such that $[\varnothing, A]$ is rank-symmetric and not a chain, then $A = [k]$ for some $3 \leq k \leq n$. This result carries to W^J via the isomorphism ϕ^{-1}. $\square$

Example 4.7. Look at the poset $M(4) = [\varnothing, 1234]$ in Figure 4.2. Notice how $M(1) = [\varnothing, 1]$, $M(2) = [\varnothing, 12]$, and $M(3) = [\varnothing, 123]$. In fact, this chain continues on infinitely as $M(1) \subset M(2) \subset M(3) \subset \cdots$, whence by isomorphism (Lemma 4.5):

$$B_2/A_1 \subset B_3/A_2 \subset B_4/A_3 \subset \cdots, \tag{26}$$

and we may consider every $M(n)$ as living inside the infinite poset $M(\infty)$, which is the set of positive integers $\mathbb{N}$ endowed with the same partial order relation $\preceq$ described in this section. The only elements with non-chain, rank-symmetric lower intervals are 1234, and 123. If we define $F_S(x)$ to be the rank-generating function for $[\varnothing, S]$ where $S \subseteq \mathbb{N}$, then we see that for $|S| \geq 2$, $F_S(x) = 1 + x + x^2 + \cdots$. Then for $n \geq 2$,

$$F_{[n]}(x) = [4]_x \prod_{k=3}^{n} (1 + x^k) = 1 + x + x^2 + \cdots + x^{\binom{n}{2}-2} + x^{\binom{n}{2}-1} + x^{\binom{n}{2}}.$$

The dots here simply indicate the typographical omission of the terms whose coefficients we are not looking at.

Looking again at Figure 4.2, notice how $F_{34}(x) = 1 + x + x^2 + 2x^3 + 2x^4 + 2x^5 + x^6 + x^7$ is not palindromic because $D(34) = \{24\}$, but $D^2(34) = D(24) = \{14, 23\}$, whereas $D^2(\varnothing) = \{2\}$, which is why the coefficients of the second and fifth degree terms in $F_{34}(x)$ differ. This is an example of an argument in Proposition 4.6 at work.

4.3 Type D_n/D_{n-1} and D_n/A_{n-1} Quotients

We continue to represent a type D_n Coxeter group W as the subgroup $\mathfrak{S}_n^D \subset \mathfrak{S}_n^B$ of signed permutations w where $\mathrm{neg}(w)$ is an even number, if needed.

We call the following leaf-removed quotient a type D_n/D_{n-1} quotient.

Lemma 4.8. Let W be a Coxeter group of type D_n, with $n \geq 4$. Let $J = S \setminus \{s_{n-1}\}$. Every proper, rank-symmetric interval $[e, v]_J$ is an embedded poset $W_I^{I \cap J}$ for some $I \subset S$.

Proof. Every element having only s_{n-1} as a right descent can be written as a right tail of one of two reduced expressions of the maximal element: $s_{n-1}s_{n-2}\ldots s_2s_1s_0s_2\ldots s_{n-1} = s_{n-1}s_{n-2}\ldots s_2s_0s_1s_2\ldots s_{n-1}$. The two reduced expressions are equivalent because s_0 and s_1 are commuting generators. Hence, we have exactly one element of length k for $0 \leq k \leq 2n-2$ and $k \neq n-1$, and exactly two elements of length $n-1$. Seeing the Hasse diagram of this lattice makes it immediately clear that the only proper, rank-symmetric lower intervals are the chains $[e, v]_J$ where $\ell(v) \leq n-1$. All of these elements with rank-symmetric lower interval have that $[e, v]_J = W_I^{I \cap J}$. For $\ell_D(v) < n-1$, we take $I = \{s_{n-\ell_D(v)}, \ldots, s_{n-1}\}$; and for $\ell(v) = n-1$, we take either $I = S \setminus \{s_0\}$ or $I = S \setminus \{s_1\}$. $\square$

Example 4.9. Let (W, S) be a type D_5 Coxeter system with simple reflections labelled as in Figure 4. Let $J = S \setminus \{s_4\}$. The Hasse diagram of W^J is shown in Figure 15.

When $J = S \setminus \{s_0\}$ or $J' = S \setminus \{s_1\}$, we call $W^J \cong W^{J'}$ a type D_n/A_{n-1} quotient. The following proposition was stated as a fact by Stanley; we give a proof merely as a courtesy to the reader:

Proposition 4.10. Let W be a Coxeter system of type D_n for $n \geq 4$. Let $J = S \setminus \{s_0\}$ or $J = S \setminus \{s_1\}$. Then $W^J \cong M(n-1)$.

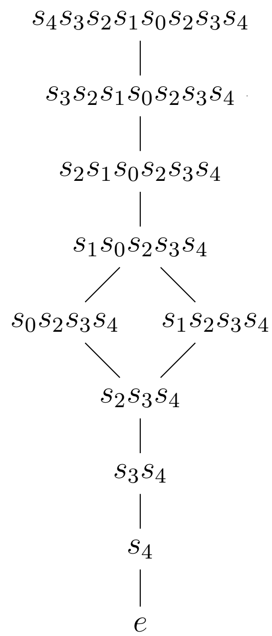

Figure 15: Type D_5/D_4 quotient

Proof. By symmetry of the Coxeter diagram, we may simply examine W^J where $J = S \setminus \{s_0\}$. Define $M_2(n)$ to be the induced subposet of $M(n)$ whose elements are the subsets of $[n]$ with even cardinality. Define $\phi\colon W^J \to M_2(n)$ to be the poset isomorphism $v \mapsto \{-v(k)\colon k \in [n], v(k) < 0\}$. Define $\psi\colon M_2(n) \to M(n-1)$ such that $\psi(A) = \{a-1\colon a \in A\} \setminus \{0\}$ for $A \subseteq [n]$ and $|A|$ even. It is simple to verify that this map is bijective and order-preserving, with its inverse map given by

$$\psi^{-1}(B) = \begin{cases} \{b+1\colon b \in B\}, & \text{if } |B| \text{ is even;} \\ \{b+1\colon b \in B\} \cup \{1\}, & \text{if } |B| \text{ is odd.} \end{cases}$$

Therefore, ψ, and more importantly, $\psi \circ \phi\colon W^J \to M(n-1)$, are poset isomorphisms. □

The lemma for this case follows naturally.

Lemma 4.11. Let W be a type D_n Coxeter group and let $J = S \setminus \{s_0\}$ or $S \setminus \{s_1\}$. Let $v \in W^J$. If $[e, v]_J$ is rank-symmetric, then either $[e, v]_J$ is a chain or there is an $I \subseteq S$

such that $[e,v]_J = W_I^{I\cap J}$ where $W_I^{I\cap J}$ is a type D_k/A_{k-1} quotient for an integer $k \in [n]$ such that $\ell_D(v) = \binom{k}{2}$.

Proof. By using Proposition 4.10, the proof is almost identical to that of Lemma 4.8. $\square$

4.4 Exceptional Type Quotients

The exceptional type leaf quotients may be named as follows: E_6/A_5, E_6/D_5, E_7/A_6, E_7/D_6, E_7/E_6, E_8/A_7, E_8/D_7, E_8/E_7, F_4/B_3, and G_2/A_1. The leaf quotients of G_2 are easily seen to be chain posets. As for all the other types, they have been checked by computer [10,11]. It has been verified that for W any of these exceptional Weyl groups, if J is a maximal, leaf-removed set of generators, and $v \in W^J$ such that $[e,v]_J$ is rank-symmetric, then either the interval is a chain poset or there exists a subset $I \subseteq S$ such that v is the top element of $W_I^{I\cap J}$.

Example 4.12. Let (W,S) be a type E_7 Coxeter system with simple reflections labeled as in Figure 5. Let $J = S \setminus \{s_2\}$. The only elements which have proper, rank-symmetric lower intervals are the elements s_2, s_4s_2, $s_3s_4s_2$ and $s_5s_4s_2$, $s_1s_3s_4s_2$ and $s_6s_5s_4s_2$, and $s_7s_6s_5s_4s_2$ which are all top elements of embedded A_n/A_{n-1} type quotients; $s_2s_4s_3s_5s_4s_2$, which is the top of an embedded D_4/A_3 quotient; $s_3s_4s_5s_6s_2s_4s_3s_5s_4s_2$ and $s_5s_4s_3s_1s_2s_4s_3s_5s_4s_2$ which are the top elements of embedded D_5/A_4 quotients; and $s_2s_4s_3s_5s_4s_6s_5s_7s_6s_2s_4s_3s_5s_4s_2$, which is the top element of an embedded D_6/A_5 quotient.

4.5 The General Property

As a result of the lemmas in the previous sections of this chapter, we conclude with the overall result:

Theorem 4.13. Let (W,S) be a finite, crystallographic Coxeter system and $J = S \setminus \{s\}$, where s corresponds to a leaf in the Coxeter diagram. Suppose $v \in W^J$ such that $[e,v]_J$ is rank-symmetric. Then either $[e, v]_J$ is a chain poset, or there is a subset I of S containing s such that $[e, v]_J$ is the embedded subposet $W_I^{I \cap J}$ having v as its top element.

CHAPTER 5

THE MAIN RESULT

5.1 Billey-Postnikov Decompositions

Recall that the *Poincaré polynomial* of an element $w \in W$ is the rank-generating function of its lower interval:

$$P_w(q) \stackrel{\text{def}}{=} \sum_{v \leq w} q^{\ell(v)}. \tag{27}$$

We call $P_w(q)$ *palindromic* if $q^n P_w(q^{-1}) = P_w(q)$.

Let $J \subset S$. Recall that for each $w \in W$, there exists unique $u \in W_J$ and $v \in {}^J W$ such that $w = uv$ and $\ell(w) = \ell(u) + \ell(v)$. The following is due to van den Hombergh [5]:

Theorem 5.1. For any $w \in W$ and subset J of simple roots, W_J has a unique maximal element below w.

We denote this element $m(w, J)$. The next theorem of Billey and Postnikov gives a useful factorization of the Poincaré polynomial:

Theorem 5.2. [2] Let $J \subset S$. Assume $w \in W$ has parabolic decomposition $w = uv$ where $u \in W_J$ and $v \in {}^J W$. If $u = m(w, J)$, then

$$P_w(q) = P_u(q) P_v^J(q) \tag{28}$$

where P_v^J is the Poincaré polynomial for the quotient ${}^J W$.

We will call a maximal subset of simple reflections $J = S \setminus \{s\}$ *leaf-removed* is s corresponds to a node of degree one in the Coxeter diagram.

Theorem 5.3. [2] Let $w \in W$ such that $[e, w]$ is rank-symmetric. Then there exists a maximal proper subset $J = S \setminus \{s\}$ of simple reflections, such that

i. We have a decomposition of w or w^{-1} as in Theorem 5.3, and

ii. s corresponds to a leaf in the Coxeter diagram of W.

We will call a parabolic decomposition satisfying the conditions of the above theorem a *Billey-Postnikov decomposition*, or *BP-decomposition*.

5.2 Behavior of $R_w(q)$ with Respect to BP-decompositions

Using the notations of Theorem 4.13, our first step is to prove that every reflection formed by simple reflections in $I \cap J$ is in $T_R(u)$. We need the following lemma to prove it:

Lemma 5.4. [8] Let $w \in W$ such that $[e, w]$ is rank-symmetric and $w = uv$ be a BP-decomposition. Then every simple reflection in J appearing in the reduced word of v is a right descent of u.

Actually, we can state much more about u in terms of simple reflections in J appearing in v. Remember, as a consequence of the Deletion Property (Theorem 2.4), the set of simple reflections of any reduced expression of v is invariant.

Lemma 5.5. [8] Let $w = uv$ be a BP-decomposition with respect to J. Let I be the set of simple reflections which appear in reduced words of v. Then every reflection formed by simple reflections in $I \cap J$ is a right inversion reflection of u. In fact, there is a minimal length decomposition $u = u' u_{I \cap J}$ where $u_{I \cap J}$ is the longest element of $W_{I \cap J}$.

The above lemma tells us that for each $w \in W$ having rank-symmetric lower interval, we can decompose w or w^{-1} to $u' u_{I \cap J} v$ where uv is the BP-decomposition with respect to J, $u = u' u_{I \cap J}$, and $u_{I \cap J}$ is the maximal element of $W_{I \cap J}$. Recall that Δ_w denotes the inversion set of w. For $I \subseteq S$, Δ_I will denote the set of roots corresponding to the reflections of W_I. We have a decomposition

$$\Delta_w = \Delta_{u'} \uplus u' \Delta_{u_{I \cap J}} u'^{-1} \uplus u \Delta_v u^{-1}$$

.

It is straightforward to verify that $\Delta_{u_{I\cap J}} = \Delta_{I\cap J}$ and $\Delta_v \subseteq \Delta_I \backslash \Delta_{I\cap J}$. From these two facts, it follows that $u'\Delta_{u_{I\cap J}}u'^{-1} = u\Delta_{I\cap J}u^{-1}$. Let $\mathcal{A}_1$ be the hyperplane arrangement corresponding to Δ_u. Let $\mathcal{A}_0$ be the hyperplane arrangement corresponding to $\Delta_{I\cap J}$. Let $\mathcal{A}_2$ be the hyperplane arrangement corresponding to Δ_v. We can study $\mathcal{A} := \mathcal{A}_1 \uplus \mathcal{A}_0 \uplus \mathcal{A}_2$ in place of $\mathcal{A}_w$.

Lemma 5.6. [8] Let c be a chamber inside $\mathcal{A}_1 \uplus \mathcal{A}_0$. Let c' be the chamber of $\mathcal{A}_0$ containing c. Then the chamber graph of c with respect to $\mathcal{A}$ is isomorphic to the chamber graph of c' with respect to $\mathcal{A}_0 \uplus \mathcal{A}_2$.

The proof of the following corollary is taken from [8] but listed below for convenience, with a minor typographical change to distinguish a left quotient from a right quotient.

Corollary 5.7. [8] In the above decomposition, if $\mathcal{A}_{w'}$ is uniform with respect to $\mathcal{A}_u$ and v is the longest element of ${}^{I\cap J}W_I$, then $R_u(q) = P_u(q)$ implies that $R_w(q) = P_w(q)$.

Proof. If v is the longest element of ${}^{I\cap J}W_I$, then $w' := u_{I\cap J}v$ is the longest element of W_I. Then it is obvious that $\mathcal{A}_{w'}$ is uniform with respect to $\mathcal{A}_u$. Now, it follows from the above lemma that $R_w(q)/R_u(q) = R_{u_{I\cap J}v}(q)/R_{u_{I\cap J}}(q)$. Since we also know that the right hand side equals $P_v^{^JW}(q)$, then $R_u(q) = P_u(q)$ implies that $R_w(q) = P_w(q)$. $\square$

5.3 The Necessary and Sufficient Condition for $R_w(q) = P_w(q)$

In this section, we prove the main theorem.

Theorem 5.8. Let (W, S) be a finite, crystallographic Coxeter system. Let $w \in W$. Then $R_w(q) = P_w(q)$ if and only if $[e, w]$ is rank-symmetric.

Proof. Recall that $R_w(q)$ is palindromic. If $[e, w]$ is not rank-symmetric, then $P_w(q)$ is not palindromic, whence $R_w(q) \neq P_w(q)$.

Suppose then, that $[e, w]$ is rank-symmetric. Let uv be a BP-decomposition of w (or w^{-1}), where $u \in W_J$ and $v \in {}^JW$. By Theorem 4.13, there are two cases: either $[e, v]_J =$

$[e,v] \cap {}^JW$ is a chain, or there is a subset $I \subseteq S$ such that $[e,v]_J$ is the embedded subposet ${}^{I\cap J}W_I$. In simply-laced Coxeter groups, the latter case is always true, but in non-simply-laced Coxeter groups, we sometimes get $[e,v]_J$ a chain without v being the top element of a quotient, so we must treat them separately.

Suppose first that v is the top element of a quotient ${}^{I\cap J}W_I$ We may decompose the inversion hyperplane arrangement $\mathcal{A}_w$ as per the last section and apply Corollary 5.7 to see that if $R_u(q) = P_u(q)$ then $R_w(q) = P_w(q)$. So, if we think to substitute w with some $u \in W_J$ such that $[e,u]$ is rank-symmetric, then we can formulate an inductive argument on the rank of our Coxeter group and thusly get the result.

Now, assume the first case is true, and that $v \in {}^JW$ is a chain element; that is, that $[e,v]_J = [e,v] \cap {}^JW$ is a chain. Suppose $v = s_{i(1)}s_{i(2)}\cdots$. Then

$$\ell(u) < \ell(us_{i(1)}) < \ell(us_{i(1)}s_{i(2)}) < \cdots < \ell(uv). \tag{29}$$

Hence the corresponding chanbers in $\mathcal{A}_w$ have different distance from the fundamental chamber. This shows that each chamber in $\mathcal{A}_u$ has different distance from the fundamental chamber as well, hence

$$\frac{R_w(q)}{R_u(q)} = 1 + q + \cdots + q^{\ell(v)}, \tag{30}$$

Proving that

$$P_u(q)P_v^J(q) = R_u(q)(1 + q + \cdots + q^{\ell(v)}) \tag{31}$$

through an induction argument.

□

Here is an example of a Billey-Postnikov decomposition in relation to a hyperplane arrangement.

Example 5.9. Let (W,S) be a type G_2 Coxeter system with simple reflections labelled as in Figure 9. It has $|T| = 6$ reflections corresponding to six hyperplanes in the Coxeter arrangement. Let $w = s_1s_2s_1s_2 \in W$. Let $J = S \setminus \{s_1\}$. $\mathcal{A}_w$ has four hyperplanes of all the original six, these being only the ones which separate w from the fundamental chamber e; so $P_w(q) = 1 + 2q + 2q^2 + 2q^3 + q^4 = R_w(q)$. To see how this factorizes, let $u = s_1$ and $v = s_2s_1s_2$. Then $u \in W_J$, $v \in {}^JW$, and $w = uv$ is a BP-decomposition. Furthermore, $P_u(q) = 1 + q$, $P^J_v(q) = 1 + q + q^2 + q^3$, and $P_w(q) = P_u(q)P^J_v(q)$.

5.4 Final Thoughts

Recall the property of Theorem 4.13. The reader might ask whether this property is true for non-crystallographic Coxeter groups. The answer is: mostly, but not quite. It is true in the case of $I_2(m)/A_1$ quotients since they are all chain posets. The exceptional cases H_3 and H_4 have been examined by computer [10, 11]. The property holds true for type $H_3/I_2(5)$ and H_4/H_3 quotients.

Example 5.10. Let (W, S) be a type H_3 Coxeter system with simple reflections labelled as in Figure 7. Let $J = S \setminus \{s_3\}$. The Hasse diagram of W^J is shown in Figure 16. Notice how the only proper, palindromic lower intervals are chains.

However, there are two obstructions to the property holding true for H_4/A_3, one of which occurs also in H_3/A_2. Let (W, S) be a type H_4 Coxeter system with simple reflections labelled as in Figure 8. Let $J = S \setminus \{s_1\}$. Let

$$v_1 = s_2s_1s_2s_3s_1s_2s_1 \quad \text{and} \quad v_2 = (s_1s_2s_3)^3s_4s_1s_2s_3s_1s_2s_1. \tag{32}$$

Then $[e, v_1]_J$ and $[e, w_2]_J$ are both rank-symmetric, non-chain intervals; yet neither v nor w is the top element of a $W_I^{I\cap J}$ for some $I \subset S$.

It is not known at the time of writing whether Conjecture 1.3 is true for all finite Coxeter groups, since the proof machinery in Chapter V relies on the property of leaf quotients. Should the conjecture be true, then the cases where we have a BP-factorization $w = uv_1$ or $w = uv_2$ for type H_3 or H_4 quotients would need to be handled in a different

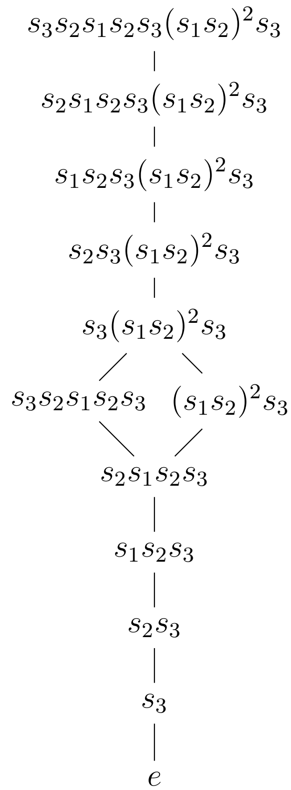

Figure 16: Type $H_3/I_2(5)$ quotient

way. It could, of course, be checked by brute force, which would close a gap in the proof machinery and entail the truth of the conjecture for all cases; however, this would not be terribly interesting. This is where we ask: could there be a more general geometric or bijective principle that would prove the conjecture true? Could it be done without utilizing the leaf-quotient property at all? These are questions worth exploring.

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