NUMERICAL APPROACH TO ENERGY MINIMIZATION OF FLUID CONFIGURATIONS USING PHASE-FIELD MODELS

by

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ABSTRACT

We consider a fluid, under isothermal conditions and confined to a bounded container of homogeneous makeup, whose Gibbs free energy, per unit volume, is a prescribed function of its density distribution. Based on the Van der Waals-Cahn-Hilliard Theory of phase transitions, we minimize our functional, whose phase field formulation is obtained by considering an energy of the type

$$E_\epsilon(u) = \int_\Omega \left(\epsilon|\nabla u|^2 + \tfrac{1}{\epsilon}a^2u^2(1-u)^2 + uG(x) + \lambda u\right) dx,$$

where u is the phase function, G is a potential energy, and λ represents volume constraint. We know that these minimizers, E_ϵ, as ϵ goes to 0, will $\Gamma-$converge to the minimizer of the capillary energy functional.

Although numerical approaches to this minimization exists, current approaches are unable to distinguish between local and global minimizers of the functional. I propose a mesh-grid-based optimization approach, with Dirichlet boundary conditions. Assuming convexity of our system, we utilize a logarithmic barrier optimization scheme in hopes to guarantee convergence to the global minimum of our energy functional.

I. INTRODUCTION

Capillarity describes the tendency of liquids under certain surface conditions to sink or rise as a result of surface tension. The theory of capillarity as an application to static equilibrium in fluids has been studied for centuries. Common problems in this field include the "liquid-drop problem," the study of Sessile drops, analysis of fluid-like bubbles, and the calculation of an equilibrium configuration for a fluid(s) on a surface or in a container. We will focus on the latter of these, in whose field recent discoveries have been made.

Classic capillary theory, as described by Young and Laplace, which can be read in [3], states that in a multiphase system (including a solid phase, liquid phase, and gaseous/vapor phase), where the solid surface in question is smooth and of homogenenous chemical makeup and the liquid is pure, there is a relationship between the surface tensions between the phases and the contact angle between the liquid and solid phase. In particular, Young's equation takes the form

$$cos(\theta_Y) = -\frac{\sigma_{SL} - \sigma_{SV}}{\sigma_{LV}}, \tag{1.1}$$

where σ_{AB} represents the surface tension between any two phases A, B, θ_Y represents the so-called Young's angle, and S, L, V represent the Solid, Liquid, and Vapor phases, respectively.

Classic capillary theory takes a geometric form, modeling the free energy of the system as a function of the geometric configuration of the liquid, and states that equilibrium can be gained from the solution which minimizes the free energy functional given in [6]:

$$E(\omega,t) = \int_{\partial_S\omega}\sigma_{SL}(x)dA_x + \int_{\partial_V\omega}\sigma_{LV}(x)dA_x + \int_{\partial_S/\partial_S\omega}\sigma_{SV}(x)dA_x + \int_{\omega}\rho_L(x)G(x,t)dV_x, \quad (1.2)$$

where $E(\omega,t)$ is the free energy of a liquid drop ω at a time t, $\partial_S\omega$ represents the interface between the liquid and solid, $\partial_V\omega$ is the interface between the liquid and vapor, and $\partial_S/\partial_S\omega$ is the interface between the solid and vapor, ρ_L represents the density of our liquid, and $G(x,t)$ represents some generic potential external force, usually gravity, at a point x in our domain.

Recent studies have shown that there are situations in which the multiphase system may achieve equilibrium at states which do not minimize the free energy. You can see examples of such phenomena in daily experiences, such as water droplets adhering to a window regardless of gravitational pull. The theory behind such a phenomenon describes the potential of a non-minimal equilibrium state as a result of the contact angle of the liquid with the surface, known as contact angle hysteresis, and can be predicted by the introduction of a dissipation mechanism into the classical theory. The need for an adaptive mechanism gave way to a new model for the classic capillary theory.

Phase Field Model

With the study of hysteresis, new methods for describing capillary theory have arisen, and a new model for the free energy has been created in order to deal with non-minimial equilibria. This new approach is called a phase field model. The concepts behind the phase field model can be attributed to van der Waals, but Cahn was the first to use a phase field model to study wetting phenomena. In fact, the Cahn-Hilliard model for free energy inspired the modern formulations of similar models. We often refer to the phase field formulation in capillary study as the van der Waals-Cahn-Hilliard theory of phase transitions.

The phase field model is quintessential to the study of hysteresis, as the classic geometric model does not allow for the natural merging/splitting of drops, and the new model alone allows introduction of boundary conditions which can model the frictional pinning forces involved in contact angle hysteresis [8]. Additionally, the phase field model handles phase transition differently than the classic, interchanging the sharp interfaces of the geometric theory with smoother ones by the introduction of narrow transitional layers between phases.

Consider a bounded container Ω in $\mathbb{R}^3$ under isothermic conditions whose solid domain $\partial\Omega$ is of homogeneous chemical makeup. A phase field model of the time-independent capillary problem can be expressed by the energy functional

$$E_\epsilon(u) = \int_\Omega \left[\epsilon|\nabla u|^2 + \frac{1}{\epsilon}W(u) + uG(x) + \lambda u \right] dx, \tag{1.3}$$

where u is the phase function, λ is a Lagrange multiplier for the purposes of modeling the volume constraint of the system, i.e. $\int_\Omega u dx = V_\Omega$, for a volume V_Ω prescribed to the system, and $W(u) = a^2u^2(1-u)^2$ is a double-well type function that models a strictly non-negative potential, with the constant $a > 0$ to be determined. The potential W vanishes at exactly two points, $u = 0$ and $u = 1$, where the phase function u is 0 in the vapor stage and 1 in the liquid stage. u, continuous, will also take on values in $[0, 1]$ during transitional phases.

Boundary conditions for the system have been recovered, as mentioned by Fedeli [2]. Theory fixes the parameter a from $W(u)$ such that

$$a = 3\sigma_{LV}, \tag{1.4}$$

and finds that the Dirichlet and Neumann boundaries can be solved as follows:

$$cos(\theta_Y) = -4u_{\partial S}^3 + 6u_{\partial S}^2 - 1,\ 0 \leq u_{\partial S} \leq 1, \tag{1.5}$$

where $u_{\partial S}$ denotes the Dirichlet value on the boundary of the solid domain, or

$$\begin{aligned} -2\epsilon\frac{\partial u}{\partial \mathbf{n}} &= N, & (1.6)\\ inf_{s\geq 0}\left\{Ns + 2a\left(\frac{s^3}{3} - \frac{s^2}{2} + \frac{1}{6}\right)\right\} &= \sigma_{SL} - \sigma_{SV} & (1.7) \end{aligned}$$

where finding N in (1.7) will yield the correct Neumann boundary $\frac{\partial u}{\partial \mathbf{n}}$ in (1.6). With the correct Dirichlet/Neumann boundary conditions added, we will find that (1.3) will $\Gamma-$ converge to the geometric capillary energy as described by Young. That is, as ϵ tends to zero, we will find solutions for (1.3) which serve as good approximations for the minimizers of the overall system.

For the purpose of simplifying the following proof, consider the functional (1.3) without the volume constraint or gravitational energy, and consider the possibility of a two-phase function for W. That is, temporarily consider the following definition for $E_\epsilon(u)$:

$$E_\epsilon(u) = \int_\Omega \epsilon|\nabla u|^2 + \frac{1}{\epsilon}W(u)dV_x, \tag{1.8}$$

$$W(u) = a^2u_1^2(1-u_1)^2 + b^2u_2^2, \tag{1.9}$$

$$a = 3\sigma_{LV} > 0, \tag{1.10}$$

$$b = \frac{1}{2}(\sigma_{SV} + \sigma_{SL} - \sigma_{LV}) > 0. \tag{1.11}$$

Theorem - Let $\overline{E}_0$ and $\overline{E}_\epsilon$ be given by:

$$\overline{E_\epsilon}(u) = \begin{cases} E_\epsilon(u)\ if & u \in H^1(\Omega, \mathbb{R}^2)\, and\, u|_{\partial\Omega} = g; \\ +\infty & otherwise\, in\, L^1, \end{cases}$$

$$\overline{E_0}(u) = \begin{cases} E(\{u \equiv L\})\ if & u \in BV(\Omega, \{V, L\}); \\ +\infty & otherwise\, in\, L^1. \end{cases}$$

Then $\overline{E_0}$ is the $\Gamma-$limit of $\overline{E_\epsilon}$ as ϵ tends to zero in the topology of L^1. Moreover, for every $\epsilon > 0$, let $u^*_\epsilon = argmin\ \{\overline{E_\epsilon}(u) : \int_\Omega u = V_\Omega\}$; then the sequence $(u^*_\epsilon$ is pre-compact in L^1 and every cluster point, say u^*, belongs to $BV(\Omega, \{V, L\})$, and we have $u^* = argmin\ \{\overline{E_0}(u) : \int_\Omega u = V_\Omega\}$.

proof: Let us define a metric on $\mathbb{R}^2$ (other possible definitions of this metric exist):

$$d(v_1, v_2) = min \left\{ \int_{-\infty}^{+\infty} (\rho^2 + W(\rho))dt, \quad \rho(-\infty) = v_1, \quad \rho(+\infty) = v_2 \right\}, \tag{1.12}$$

and a functional on $BV(\Omega, \{V, L\})$:

$$\begin{aligned} \tilde{E}_0(u) &= 2d(V, L)\mathcal{H}_{n-1}(\partial^*\{u \equiv L\} \cap \partial^*\{u \equiv V\}) \\ &\quad +2 \int_{\partial\Omega} d(u|_{\partial\Omega}(x), g(x))d\mathcal{H}_{n-1}, \end{aligned} \tag{1.13}$$

where $\partial^* A$ denotes the reduced boundary of the set A, $L \equiv (1, 0)$ and $V \equiv (0, 0)$ are the only zeros of $W(u)$, and $\mathcal{H}_{n-1}$ denotes the Hausdorff measure of dimension $n - 1$. Given that $\partial\Omega = \partial_S\Omega \cup \partial_V\Omega$, we choose $g : \partial\Omega \rightarrow\ R^2$ such that $g \equiv V$ on $\partial_V\Omega$ and $g \equiv (u_{\partial S}, 1) := S$ on $\partial_S\Omega$, where $u_{\partial S}$ is the unique solution to (1.5). We want to show that in our situation, $2\overline{E_0} = \tilde{E}_0$. In the wetting setting, we lablel the "liquid" as the set $\{u \equiv L\}$ and the "vapor" as the set $\{u \equiv V\}$.

Moreover, considering our choice of g, the functional (1.13) becomes

$$\tilde{E}_0(u) \;=\; 2d(V,L)|\Sigma_{LV}(u)| + 2d(V,S)|\Sigma_{SV}(u)| + 2d(S,L)|\Sigma_{SL}(u)|, \tag{1.14}$$

where Σ_{AB} is the interface between phases A, B, and $|A|$ denotes the measure of the set A. The integral in (1.13) decomposes into the sum of two terms, as g is constant and u can assume only the values V and L. We need only show that our chosen metric and the equation (1.5) for Young's angle give us the correct surface tensions when parameters a and b are chosen appropriately in the potential W. Substituting the appropriate values, we obtain

$$\begin{aligned}
\sigma_{LV} &= d(L,V) = min\int_{-\infty}^{+\infty}[\rho_1^2 + W(\rho_1,0)]dt & (1.15)\\
&= 2\int_0^1\sqrt{W(\tau,0)}d\tau = \frac{a}{3} \\
\sigma_{SV} &= d(S,V) = 2\int_0^{u|_{\partial\Omega}}\sqrt{W(\tau,0)}d\tau + 2\int_0^1\sqrt{W(\tau,0)}d\tau & (1.16)\\
&= -2a\left(-\frac{u|_{\partial\Omega}^2}{2} + \frac{u|_{\partial\Omega}^3}{3}\right) + b \\
\sigma_{SL} &= d(S,L) = 2\int_{u|_{\partial\Omega}}^1\sqrt{W(\tau,0)}d\tau + 2\int_0^1\sqrt{W(\tau,0)}d\tau & (1.17)\\
&= -2a\left(\frac{1}{6} - \frac{u|_{\partial\Omega}^2}{2} + \frac{u|_{\partial\Omega}^3}{3}\right) + b.
\end{aligned}$$

Hence, b is as described in (1.11) and (1.5) holds. Observe that in the form of W listed in (1.9), the minimal u_2 is always zero, and thus, we can continue to think of W only in terms of u as one input. With this proof, we can easily conclude that $\Gamma-$convergence holds in our gravity and volume-restricted model. By standard properties of $\Gamma-$ convergence, since G will be modeled by the gravitational potential, $u \to \int uGdV_x$ is continuous in the L^1 topology and is therefore admissible. Similarly, the subspace $\{u \in L^1, \int u = V_\Omega\}$ is closed in the L^1 topology and, in the recovering sequence of the $\Gamma - lim\,sup$, we can always assume that $\int u_\epsilon = \int u$.

This proof by Turco [8] ensures our ability to assume $\Gamma-$ convergence of our system upon a proper solution. $\square$

Numerical Implementations

Currently, numerical implementations of the phase field model in (1.3) exist for varying purposes. Most of these focus on acquiring data about contact angle hysteresis or focus on the equilibria obtained over heterogeneous surfaces rather than homogeneous. Instead, we wish to minimize our energy functional in accordance with the classical capillary theory, and we will take a unique approach in doing so.

There is some ambiguity with how to handle certain parts of our energy system (the boundary conditions, the gravitational potential, the volume constraints, and the general scheme of minimization).

While both Dirichlet and Neumann boundary conditions have proven sufficient for the solution of the system, in the numerical implementations seen in [2] and [8], Neumann boundary data is chosen. Use of Neumann data can be pertinent to the solution of systems modeling hysteresis. But for ease of calculation, and because we are not studying a system with dissipation effects, we will choose to use the Dirichlet boundary conditions instead.

In some schemes [8], the function $G(u)$ as a time-independent function is excluded, as gravity is considered to be negligent in the solution of the system. However, we will complete our energy minimization including the typical gravity potential as a function of height in the container.

Also, as in the numerical implementation by Turco [8], one can model the volume constraint in our system as is written in (1.3), by way of a Lagrange multiplier. However, there is also the option to use a penalty function in order to model the volume constraint instead, and as you will see by our choice of minimization scheme later on, we will handle the volume constraint in this way. We consider this equality constraint as we opt for an equality-constrained minimzation scheme.

Lastly, the numerical implementations so far revolve around turning the energy functional into a parabolic PDE and solving with some sort of gradient flow method. No direct optimization scheme on the functional has been used. The issue with this choice of scheme, however, is the absence of guarantee of convergence upon a global minimum for the energy functional versus a local minimum. The theory of $\Gamma-$ convergence only guarantees the accuracy of global minimizers, not critical points [2], and numerical schemes like Turco's in [8] do not distinguish between these. We will, instead, use a pure optimization scheme, assuming the convexity or near-convexity of our system, in order to guarantee convergence to the global minimum of the system.

We arrive with our phase field energy functional system, with our above choices implemented:

$$\begin{aligned} min\ E_\epsilon(u) &= \int_\Omega \epsilon|\nabla u|^2 dx + \frac{1}{\epsilon}\int_\Omega W(u)dx + \int_\Omega uG(x))dx, && (1.18)\\ with &\quad 0 \leq u \leq 1, && (1.19)\\ such\ that &\quad \int_\Omega udx = V_\Omega, && (1.20) \end{aligned}$$

and now we need only decide the best numerical approach for minimizing the energy functional problem.

II. NUMERICAL APPROACH

In this section, we discuss our particular numerical approach towards finding the global minimizer of our energy functional. We must find an efficient way to model our data, as well as find a sufficient optimization scheme to satisfy our particular type of problem.

Firstly, to model our domain for a numerical approach, we will utilize a Finite Element Method framework to appropriately discretize our domain and model our phase-field equation. Secondly, we must choose an appropriate method of optimization to handle the equality and inequality constraints of our system. We want to ensure that our method will find a global minimum in any situation. We will use a logarithmic barrier technique for this purpose.

Finite Element Method Framework

The Finite Element Method [7] is a technique used to numerically solve partial differential equations. Many physical phenomena can be modeled by PDE's, including conservation and equilibrium of energies. Given certain constraints for the PDE in question, one would find the optimal solution to the original problem by solving a simplified numerical approximation. For example, one could solve the system

$$\begin{aligned} \Delta u &= f, \quad \textit{on } \Omega \\ u &= g_D, \quad \textit{on } \Gamma_D \\ \frac{\partial u}{\partial n} &= g_N, \quad \textit{on } \Gamma_N, \end{aligned} \tag{2.1}$$

which is known as Poisson's Equation, where f, g_D, and g_N are prescribed functions, u is the variable function, Ω is the domain, Γ_D represents the portion of the boundary subject to Dirichlet boundary conditions, and Γ_N represents the portion of the boundary subject to Neumann boundary conditions. Although our problem is not a PDE, we will still use the basic FEM framework to allow great flexibility in applying our methods to non-trivial domains.

To solve a PDE using the Finite Element Method, firstly, one would discretize the domain into a collection of "elements." Although triangular and quadrilateral elements are the most popular, the FEM can be adapted to more difficult geometries. Sophisticated mesh generators exist for complex domains, but for our purposes, with a simple domain, we will use a simplistic uniform triangular mesh:

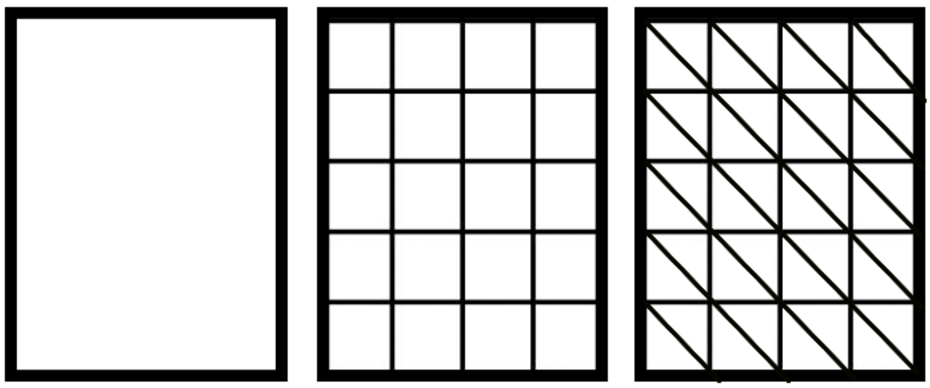

Figure 1: Domain Gridded and Triangulated

Referring to the above figure, each triangular element of the discretized domain can be handled individually. In actual FEM calculations, the approximate solution is represented by a series of "test functions." On each element, we model our variable function, u, by a series of "basis functions," which are usually linear functions to model the plane on an individual element. For our particular work, we will use basis functions to model density distribution as a plane over each triangular element. These basis functions model the linear edges of each triangular element, and the endpoints of each element edge are referred to as "nodes."

In the Finite Element Method, one would go on to model the PDE with a series of matrices known as the "Stiffness Matrix" and "Load Matrix," but we do not need to go so far for our particular optimization problem. In fact, the major contributions the FEM will make to our project are the mesh-grid and element-wise solution system concept and the linear basis functions.

Logarithmic Barrier Optimization

When considering which method to use in order to solve our optimization problem, we must consider the additional constraints in our system.

$$\min_u \quad \epsilon \int_\Omega |\nabla u|^2 dx + \frac{1}{\epsilon} \int_\Omega W(u) dx + \int_\Omega u G(x) dx \tag{2.2}$$

$$\text{with} \quad 0 \leq u \leq 1, \tag{2.3}$$

$$\text{subject to} \quad \int_\Omega u = V_\Omega \tag{2.4}$$

Because we have both equality and inequality constraints, we must carefully choose our method of optimization. Although Newton's Method seems to be a plausible option for our optimization problem, because we know our energy functional to be twice differentiable and infer that it is likely convex, Newton's Method cannot, on its own, handle inequality constraints like those present in our system. However, there is an adaptation, known as a barrier method, which we can use to bypass this issue by treating our inequality constraints as additional parts of our minimization term. More specifically, we will use the logarithmic barrier method [1] to optimize our problem.

The logarithmic barrier method for optimization trials a problem like

$$\min_{\mathrm{x}} \quad f(x) \tag{2.5}$$

$$\text{with} \quad h_i(x) \leq 0,\ i = 1, ..., m, \tag{2.6}$$

$$\text{subject to} \quad Ax = b. \tag{2.7}$$

The general barrier method uses an indicator function

$$I(x) = \begin{cases} 0 \text{ if } x \leq 0 \\ \infty \text{ if } x > 0 \end{cases}$$

to model inequality constraints, turning the minimization term of the former system into $min\ f(x) + \sum_{i=1}^{m} I_{(h_i(x) \leq 0)}$, where I is the indicator function. The logarithmic barrier method uses a series of minimization equations to serve as a smooth approximation for this indicator term. As t approaches ∞, we get that $\frac{1}{t}\phi(x) = \left(\frac{-1}{t}\right) \sum_{i=1}^{m} \ln\left(-h_i(x)\right)$ converges upon our indictator function, as the domain of $\phi(x)$ is precisely the points which satisfy our inequality constraints. We transform our system into the following:

$$\min\ tf(x) + \phi(x), \tag{2.8}$$

$$\text{with } Ax = b. \tag{2.9}$$

Then the solution to the minimization problem for any given t is found by the central path x_t^* as t approaches ∞ via Newton's Method, where this central path refers to the set of points x_t^* for $t > 0$ which strictly satisfy these Primal Feasibility and Stationarity conditions:

$$Ax_t^* = b, \quad h_i(x^*) \leq 0, \quad i = 1, ..., m \ \text{– Primal Feasibility} \tag{2.10}$$

$$t\nabla f(x_t^*) + \nabla\phi(x^*) + A^T\vec{w} = 0 \ \text{– Stationarity,} \tag{2.11}$$

where $\vec{w}$ exists in $\mathbb{R}^n$. Assuming these conditions hold, we may traverse the central path as t approaches ∞ and approach a more accurate solution to our initial problem. With these conditions, specifically the primal feasibility, we assume a strictly feasible starting value for our variable function, x above, that is, one which satisfies our equality and inequality constraints. An infeasible start for the barrier method does exist; however, as we will explore in our Numerical Results, an infeasible starting point may also be transformed into one which is strictly feasible for simplicity of our optimization.

Considering our particular capillary problem, after employing the logarithmic barrier method, we have the following system to solve:

$$\min_u \quad tE_\epsilon(u) + \phi(u) \tag{2.12}$$

$$\text{with} \quad \int_\Omega(u) = V_\Omega, \tag{2.13}$$

where t is our iteration from the barrier method, and ϵ is our desired accuracy level for each iteration of the $\Gamma-$convergence. We assign this minimization problem to an equality-constrained Newton's Method. Consider the general formula for Newton's Method, described in terms of feasible guesses $u^{(k)}, u^{(k+1)}$:

$$u^{(k+1)} - u^{(k)} = -t^{(k)}\nabla u^{(k)}. \tag{2.14}$$

Like the general Newton's method, we iterate through our minimization by calculating our Newton's step. We will do this similarly here, where we solve

$$\begin{bmatrix} \nabla_u^2 f(u) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta u_{nt} \\ w \end{bmatrix} = \begin{bmatrix} -\nabla_u f(u) \\ 0 \end{bmatrix} \tag{2.15}$$

for our Newton step Δu_{nt}, where $f(u)$ represents our adjusted equation $tE_\epsilon(u) + \phi(u)$. We find our Newton direction via backtracking line search. We iterate through our Newton steps, computing u_t^* for each t, and then increasing $t = \mu t$ for some barrier parameter μ, until we get that $t \geq \frac{m}{\epsilon}$, which guarantees an $\epsilon-$suboptimal solution for our original equation, by the duality of our logarithmic barrier system, as described by Boyd [1].

Varying choices for initial t value and μ affect the number of iterations required for our system to converge upon a solution. For our optimization scheme, we will leave the choices of initial $t = t_0$ and barrier parameter μ as options for input into our numerical solver.

III. IMPLEMENTATION

In this section, we shall go through steps necessary to compute each portion of our energy functional as needed for our optimization scheme. Our logarithmic barrier method requires us to solve the following system for our Newton step Δu_{nt}:

$$\nabla_u^2(tE_\epsilon(u) + \phi(u))\Delta u_{nt} + A^T w + \nabla_u(tE_\epsilon(u) + \phi(u)) = 0, \tag{3.1}$$

$$A\Delta u_{nt} = 0 \tag{3.2}$$

during our Newton's method centering steps, for feasible configurations u, where $\phi(u)$, our barrier function, represents our inequality constraints, and A our equality constraint. We will show calculations for the first term of this system and include steps taken to ensure ease of calculation. We must calculate appropriate values for each of the terms in this equation, beginning with those attributed to the inequality constraints.

Configuration of Barrier Function

Firstly, we will calculate the gradient and Hessian for our barrier function $\phi(u)$. Knowing that $\phi(u) = -\sum_{i=1}^{m} \ln(-f_i(u))$, we get that

$$\nabla_u \phi(u) = \sum_{i=1}^{m} \frac{1}{-f_i(u)} \nabla_u f_i(u) \tag{3.3}$$

$$\nabla_u^2 \phi(u) = \sum_{i=1}^{m} \frac{1}{f_i^2(u)} \nabla_u f_i(u) \nabla_u f_i(u)^T + \sum_{i=1}^{m} \frac{1}{-f_i(u)} \nabla_u^2 f_i(u). \tag{3.4}$$

The inequality constraints of our system, as represented by $\phi(u)$, are restrictions on the phase function of our fluid configuration as it corresponds with density change. In standard form, we have that

$$-u \leq 0, \tag{3.5}$$

$$u - 1 \leq 0. \tag{3.6}$$

Because we are evaluating our functional on a discretized domain, these two inequality constraints apply themselves to every node of our meshed domain. Letting n represent the number of nodes in any given mesh, our constraints become

$$-u_i \leq 0, \quad i = 1, ..., n \tag{3.7}$$

$$u_i - 1 \leq 0, \quad i = 1, ..., n, \tag{3.8}$$

yielding $2n$ total inequality constraints.

We get that

$$\begin{aligned} \nabla_u \phi(u) &= \sum_{i=1}^{n} \frac{1}{u_i} \nabla_u(-u_i) + \sum_{i=1}^{n} \frac{1}{1-u_i} \nabla_u(u_i - 1) \\ &= \sum_{i=1}^{n} \frac{2u_i - 1}{u_i(1-u_i)}, \end{aligned} \tag{3.9}$$

where we get the latter vector from the gradient of u_i being 1 at u_i and 0 everywhere else. Similarly, we get that

$$\begin{aligned} \nabla_u^2 \phi(u) &= \sum_{i=1}^{n} \frac{1}{u_i^2} \nabla_u(-u_i) \nabla(-u_i)^T + \sum_{i=1}^{n} \frac{1}{(u_i-1)^2} \nabla_u(u_i - 1) \nabla_u(u_i - 1)^T \\ &\quad + \sum_{i=1}^{n} \frac{1}{u_i} \nabla_u^2(-u_i) + \sum_{i=1}^{n} \frac{1}{1-u_i} \nabla_u^2(u_i - 1) \\ &= \sum_{i=1}^{n} \frac{1}{u_i^2} I_n + \sum_{i=1}^{n} \frac{1}{(u_i-1)^2} I_n + \sum_{i=1}^{n} 0 \\ &= \sum_{i=1}^{n} \frac{2u_i(u_i-1)+1}{u_i^2(1-u_i)^2} I_n, \end{aligned} \tag{3.10}$$

where I_n is the n-dimensional Identity matrix. With the gradient and Hessian of our barrier function having been calculated, we must now calculate the gradient and Hessian of our energy functional

$$\begin{aligned} & tE_\epsilon(u) \\ = \; & t\left(\epsilon\int_\Omega |\nabla u|^2 dx + \frac{1}{\epsilon}\int_\Omega W(u)dx + \int_\Omega uG(x)dx\right) \\ = \; & t\left(\epsilon\int_\Omega |\nabla u|^2 dx + \frac{1}{\epsilon}\int_\Omega a^2u^2(1-u)^2 dx + \int_\Omega u\cdot \rho g y\, dx\right). \end{aligned} \tag{3.11}$$

We will show calculations for each of these three gradient terms where the first is denoted as the Dirichlet Integral, the second as our Double-Well Term, and the third as our Gravity Term, and we will calculate the respective Hessian at each step. First, however, we must configure our domain to be more easily integrable.

Rotation of Elements for Ease of Integration

Because we may use any mesh, uniform or otherwise, to model our domain Ω, we will rotate our triangular elements into standard form to ensure ease of integration. We will do this by applying a rotation matrix to our (x, y, u) coordinate system on each element.

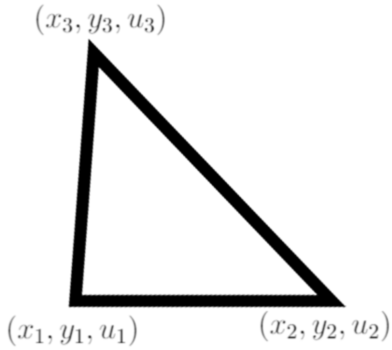

Figure 1: Element Triangle Before Rotation

In order to rotate our element into standard triangle position, we calculate the angle θ between the hypotenuse and the horizontal. Then, we apply the following rotation matrix, in the negative direction, keeping in mind to preserve the values of u_1, u_2, and u_3. We have

$$\mathrm{R} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplying this matrix by our element coordinate matrix:

$$\text{Element} = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_1 & u_2 & u_3 \end{bmatrix},$$

we get our rotated matrix and thereby our rotated element.

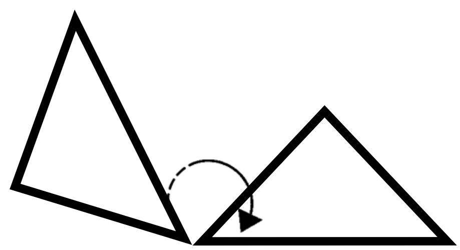

Figure 2: Element Rotated into Standard Position

In all future calculations over an element, note that $\langle x_1, y_1, u_1\rangle, \langle x_2, y_2, u_2\rangle, \langle x_3, y_3, u_3\rangle$ will refer to the coordinates post-rotation. The equation $u(x, y)$ will also be calculated with the rotated element.

In addition to rotating our element into standard position, we will also divide the element into two triangular regions, T_1, T_2, about the center, and label the nodes of each of these triangles by their relative position (left, middle, and right).

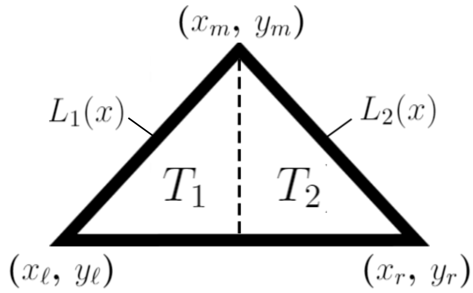

Figure 3: Division and Relabeling of Rotated Element

In the above figure, we label the left-most (x, y) coordinates as (x_ℓ, y_ℓ), the middle coordinates as (x_m, y_m), and the right-most coordinates as (x_r, y_r). We do not use new notation for the u-coordinates of our elements, as they did not change post-rotation. Because of this, we will note that calculations involving $u(x, y)$ will contain node values labeled position-wise, as well as those labeled numerically. The lines $L_1(x), L_2(x)$, which will be used as integration bounds, take the form

$$L_1(x) = \frac{y_m - y_\ell}{x_m - x_\ell}(x - x_\ell) + y_\ell \tag{3.12}$$

$$L_2(x) = \frac{y_r - y_m}{x_r - x_m}(x - x_r) + y_r. \tag{3.13}$$

where we may create $L_2(x)$ from $L_1(x)$ by simply replacing x_ℓ and y_ℓ with x_r and y_r, respectively. We will get that for the integral of any function $f(x, y)$ over an element T,

$$\int_T f(x,y)dydx = \int_{T_1} f(x,y)dydx - \int_{T_2} f(x,y)dydx \tag{3.14}$$

$$= \int_{x_\ell}^{x_m}\int_{y_\ell}^{y=L_1(x)} f(x,y)dydx - \int_{x_r}^{x_m}\int_{y_r}^{y=L_2(x)} f(x,y)dydx \tag{3.15}$$

as the bounds appropriated for the integration of T_2 produce the negative integral over T_2.

Calculation of Plane Equation

After rotating element triangles into standard position, consider three vertices of a single element triangle T: v_1, v_2, v_3. Then $v_1 = \langle x_1, y_1, u_1\rangle, v_2 = \langle x_2, y_2, u_2\rangle$, and $v_3 = \langle x_3, y_3, u_3\rangle$. To find the equation of the height $u(x, y)$ of a single element, we find the plane equation on that element. We begin by calculating two vectors in our plane:

$$\begin{aligned} \overrightarrow{v_1v_2} &= \langle x_2 - x_1, y_2 - y_1, u_2 - u_1\rangle, && (3.16) \\ \overrightarrow{v_3v_2} &= \langle x_3 - x_2, y_3 - y_2, u_3 - u_2\rangle. && (3.17) \end{aligned}$$

Then the determinant of our cross-product matrix
will yield the following equation for the plane $u(x, y)$:

$$\begin{aligned} 0 = {} & ((y_2 - y_1)(u_3 - u_2) - (y_3 - y_2)(u_2 - u_1))(x - x_1) \qquad (3.18) \\ & -((x_2 - x_1)(u_3 - u_2) - (x_3 - x_2)(u_2 - u_1))(y - y_1) \\ & +((x_2 - x_1)(y_3 - y_2) - (x_3 - x_2)(y_2 - y_1))(u - u_1). \end{aligned}$$

To simplify our final plane equation, assign the following quantities:

$$\begin{aligned} \Delta &= (x_2 - x_1)(y_3 - y_2) - (x_3 - x_2)(y_2 - y_1) && (3.19) \\ C_x &= (y_2 - y_1)(u_3 - u_2) - (y_3 - y_2)(u_2 - u_1) && (3.20) \\ C_y &= (x_2 - x_1)(u_3 - u_2) - (x_3 - x_2)(u_2 - u_1). && (3.21) \end{aligned}$$

Our final equation for element height is:

$$u(x, y) = \frac{C_y(y - y_1) - C_x(x - x_1)}{\Delta} + u_1. \qquad (3.22)$$

Also consider the following quantities:

$$u_x = \frac{-C_x}{\Delta}, \quad u_x^2 = \frac{C_x^2}{\Delta^2} \tag{3.23}$$
$$u_y = \frac{C_y}{\Delta}, \quad u_y^2 = \frac{C_y^2}{\Delta^2}. \tag{3.24}$$

Dirichlet Integral

The first of three terms needed for our phase-field equation is $\int_\Omega |\nabla u|^2 dydx$, to be calculated on each element T of our domain. Our standardized triangular element T takes the following form:

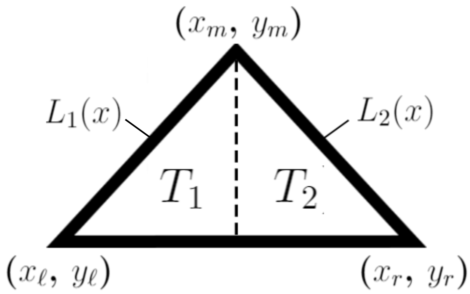

Figure 4: Rotated Element Divided into Easily Integrable Regions

Recall that we have defined

$$L_1(x) = \frac{y_m - y_\ell}{x_m - x_\ell}(x - x_\ell) + y_\ell \tag{3.25}$$
$$L_2(x) = \frac{y_r - y_m}{x_r - x_m}(x - x_r) + y_r. \tag{3.26}$$

The integral over T can be calculated by summing over T_1 and T_2. We have

$$\begin{aligned} \int_T |\nabla u|^2 dydx &= \int_{T_1} |\nabla u|^2 dydx + \int_{T_2} |\nabla u|^2 dydx \qquad (3.27) \\ &= \int_{x_\ell}^{x_m}\int_{y_\ell}^{y=L_1(x)} |\nabla u|^2 dydx - \int_{x_r}^{x_m}\int_{y_r}^{y=L_2(x)} |\nabla u|^2 dydx. \end{aligned}$$

Each of the integrals T_1, T_2 will be integrated similarly, so without loss of generality, we shall show only the integration of T_1. We have

$$\begin{aligned}
\int_{T_1} |\nabla u|^2 dydx &= \int_{x_\ell}^{x_m} \int_{y_\ell}^{y=L_1(x)} |\nabla u|^2 dydx \qquad (3.28)\\
&= \int_{x_\ell}^{x_m} \int_{y_\ell}^{y=L_1(x)} (u_x^2 + u_y^2) dydx \\
&= \int_{x_\ell}^{x_m} \int_{y_\ell}^{y=L_1(x)} \frac{C_x^2 + C_y^2}{\Delta^2} dydx \\
&= \frac{C_x^2 + C_y^2}{\Delta^2} \int_{x_\ell}^{x_m} \int_{y_\ell}^{y=L_1(x)} 1 dydx \\
&= \frac{C_x^2 + C_y^2}{\Delta^2} \int_{x_\ell}^{x_m} \frac{y_m - y_\ell}{x_m - x_\ell}(x - x_\ell) dx \\
&= \frac{C_x^2 + C_y^2}{\Delta^2} \frac{y_m - y_\ell}{x_m - x_\ell} \left[\frac{x_m^2 - x_\ell^2}{2} - x_\ell(x_m - x_\ell) \right] \\
&= \frac{C_x^2 + C_y^2}{2\Delta^2} (y_m - y_\ell)(x_m - x_\ell).
\end{aligned}$$

Define this to be the quantity I_1, the integral over the left-hand triangle. Then

$$I_1 = \frac{C_x^2 + C_y^2}{2\Delta^2} (y_m - y_\ell)(x_m - x_\ell). \qquad (3.29)$$

We may compute a similar result for the integral over the right-hand triangle, by replacing $x_\ell$and y_ℓ with x_r and y_r, respectively:

$$I_2 = -\frac{C_x^2 + C_y^2}{2\Delta^2} (y_m - y_r)(x_m - x_r). \qquad (3.30)$$

Now we must calculate the gradient of the integrated term with respect to each of the nodes, u_1, u_2, u_3, of our integrated element T. Considering C_x and C_y

from before, we get that

$$
\begin{aligned}
C_x^2 + C_y^2 &= (u_3^2 + u_2^2 - 2u_2u_3)((y_2 - y_1)^2 + (x_2 - x_1)^2) \qquad (3.31)\\
&+(u_2^2 + u_1^2 - 2u_2u_1)((y_3 - y_2)^2 + (x_3 - x_2)^2)\\
&-2\left\{(u_2u_3 - u_2^2 - u_1u_3 + u_2u_1)\right.\\
&\left.\cdot[(y_3 - y_2)(y_2 - y_1) + (x_3 - x_2)(x_2 - x_1)]\right\}.
\end{aligned}
$$

We gain our node-wise partial derivatives:

$$
\begin{aligned}
\frac{\partial}{\partial u_1}\int_T \mid \nabla u \mid^2 dydx &= \frac{1}{\Delta^2}[(y_m - y_\ell)(x_m - x_\ell) - (y_r - y_m)(x_r - x_m)]\\
&\cdot[C_x(y_3 - y_2) + C_y(x_3 - x_2)], \qquad (3.32)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial u_2}\int_T \mid \nabla u \mid^2 dydx &= \frac{1}{\Delta^2}[(y_m - y_\ell)(x_m - x_\ell) - (y_r - y_m)(x_r - x_m)]\\
&\cdot[C_x(y_1 - y_3) + C_y(x_1 - x_3)], \qquad (3.33)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial u_3}\int_T \mid \nabla u \mid^2 dydx &= \frac{1}{\Delta^2}[(y_m - y_\ell)(x_m - x_\ell) - (y_r - y_m)(x_r - x_m)]\\
&\cdot[C_x(y_2 - y_1) + C_y(x_2 - x_1)]. \qquad (3.34)
\end{aligned}
$$

For the purposes of our Newton's method for optimization. we will also need to calculate the Hessian for this term. That is, letting $d(u)$ represent the Dirichlet Integral term, we must calculate the partial derivatives $\frac{\partial d(u)}{\partial u_1^2}$, $\frac{\partial d(u)}{\partial u_1 \partial u_2}$, $\frac{\partial d(u)}{\partial u_1 \partial u_3}$, $\frac{\partial d(u)}{\partial u_2 \partial u_1}$, $\frac{\partial d(u)}{\partial u_2^2}$, $\frac{\partial d(u)}{\partial u_2 \partial u_3}$, $\frac{\partial d(u)}{\partial u_3 \partial u_1}$, $\frac{\partial d(u)}{\partial u_3 \partial u_2}$, and $\frac{\partial d(u)}{\partial u_3^2}$.

We get that

$$\frac{\partial d(u)}{\partial u_1^2} = \frac{1}{\Delta^2}[(y_m - y_\ell)(x_m - x_\ell) - (y_r - y_m)(x_r - x_m)] \cdot [(y_3 - y_2)^2 + (x_3 - x_2)^2], \tag{3.35}$$

$$\frac{\partial d(u)}{\partial u_1 \partial u_2} = \frac{1}{\Delta^2}[(y_m - y_\ell)(x_m - x_\ell) - (y_r - y_m)(x_r - x_m)] \cdot [(y_3 - y_2)(y_1 - y_3) + (x_3 - x_2)(x_1 - x_3)], \tag{3.36}$$

$$\frac{\partial d(u)}{\partial u_1 \partial u_3} = \frac{1}{\Delta^2}[(y_m - y_\ell)(x_m - x_\ell) - (y_r - y_m)(x_r - x_m)] \cdot [(y_3 - y_2)(y_2 - y_1) + (x_3 - x_2)(x_2 - x_1)], \tag{3.37}$$

$$\frac{\partial d(u)}{\partial u_2 \partial u_1} = \frac{1}{\Delta^2}[(y_m - y_\ell)(x_m - x_\ell) - (y_r - y_m)(x_r - x_m)] \cdot [(y_1 - y_3)(y_3 - y_2) + (x_1 - x_3)(x_3 - x_2)], \tag{3.38}$$

$$\frac{\partial d(u)}{\partial u_2^2} = \frac{1}{\Delta^2}[(y_m - y_\ell)(x_m - x_\ell) - (y_r - y_m)(x_r - x_m)] \cdot [(y_1 - y_3)^2 + (x_1 - x_3)^2], \tag{3.39}$$

$$\frac{\partial d(u)}{\partial u_2 \partial u_3} = \frac{1}{\Delta^2}[(y_m - y_\ell)(x_m - x_\ell) - (y_r - y_m)(x_r - x_m)] \cdot [(y_1 - y_3)(y_2 - y_1) + (x_1 - x_3)(x_2 - x_1)], \tag{3.40}$$

$$\frac{\partial d(u)}{\partial u_3 \partial u_1} = \frac{1}{\Delta^2}[(y_m - y_\ell)(x_m - x_\ell) - (y_r - y_m)(x_r - x_m)] \cdot [(y_2 - y_1)(y_3 - y_2) + (x_2 - x_1)(x_3 - x_2)], \tag{3.41}$$

$$\frac{\partial d(u)}{\partial u_3 \partial u_2} = \frac{1}{\Delta^2}[(y_m - y_\ell)(x_m - x_\ell) - (y_r - y_m)(x_r - x_m)] \cdot [(y_2 - y_1)(y_1 - y_3) + (x_2 - x_1)(x_1 - x_3)], \tag{3.42}$$

$$\frac{\partial d(u)}{\partial u_3^2} = \frac{1}{\Delta^2}[(y_m - y_\ell)(x_m - x_\ell) - (y_r - y_m)(x_r - x_m)] \cdot [(y_2 - y_1)^2 + (x_2 - x_1)^2]. \tag{3.43}$$

Double-Well Term

The second term to be integrated is our double-well term: $\int_\Omega a^2u^2(1-u)^2dydx$, where a is a constant dependent on the phase-plane model. We have that

$$\int_\Omega a^2u^2(1-u)^2\,dydx = a^2\int_\Omega (u^4-2u^3+u^2)\,dydx \tag{3.44}$$

We will integrate the first of these terms, $\int_\Omega u^4$, and will apply the integration to the remaining terms. Once again, without loss of generality, we will show only the integration of the left-hand triangle T_1. Recall that $u(x,y) = \frac{C_y(y-y_1)-C_x(x-x_1)}{\Delta} + u_1$. Also, recall that $L_1(x) = \frac{y_m-y_\ell}{x_m-x_\ell}(x-x_\ell)+y_\ell$, which gives

$$\begin{aligned}
\int_{T_1} u^4dydx &= \int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)} u^4\,dydx \qquad (3.45)\\
&= \int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)} \left[\frac{C_y(y-y_1)-C_x(x-x_1)}{\Delta}+u_1\right]^4 dydx\\
&= \int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)} \left[\frac{C_y(y-y_1)-C_x(x-x_1)}{\Delta}\right]^4 dydx\\
&\quad +4u_1\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)} \left[\frac{C_y(y-y_1)-C_x(x-x_1)}{\Delta}\right]^3 dydx\\
&\quad +6u_1^2\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)} \left[\frac{C_y(y-y_1)-C_x(x-x_1)}{\Delta}\right]^2 dydx\\
&\quad +4u_1^3\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)} \frac{C_y(y-y_1)-C_x(x-x_1)}{\Delta} dydx\\
&\quad +u_1^4\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)} 1dydx.
\end{aligned}$$

Notice that the majority of the terms above take the form of $\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)}\left[\frac{C_y(y-y_1)-C_x(x-x_1)}{\Delta}\right]^n dydx$ for some n. Therefore, by integrating the first term of this summation, $\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)}\left[\frac{C_y(y-y_1)-C_x(x-x_1)}{\Delta}\right]^4 dydx$, we can form the additional integral terms by adjusting this integral for the appropriate exponent n and related coefficients. We calculate

$$\begin{aligned}
& \int_{x_\ell}^{x_m} \int_{y_\ell}^{L_1(x)} \left[\frac{C_y(y-y_1) - C_x(x-x_1)}{\Delta} \right]^4 dydx \qquad (3.46) \\
= & \frac{1}{\Delta^4} \int_{x_\ell}^{x_m} \int_{y_\ell}^{L_1(x)} \left[C_y(y-y_1) - C_x(x-x_1) \right]^4 dydx \\
= & \frac{1}{5\Delta^4 C_y} \int_{x_\ell}^{x_m} \left[C_y(\frac{y_m - y_\ell}{x_m - x_\ell}(x - x_\ell) + y_\ell - y_1) - C_x(x-x_1) \right]^5 \\
& - \frac{1}{5\Delta^4 C_y} \int_{x_\ell}^{x_m} [C_y(y_\ell - y_1) - C_x(x-x_1)]^5 dx \\
= & \frac{1}{30\Delta^4 C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - 30\Delta^4 C_y C_x} \\
& \cdot \left[(C_y(y_m - y_1) - C_x(x_m - x_1))^6 - (C_y(y_\ell - y_1) - C_x(x_\ell - x_1))^6 \right] \\
& + \frac{1}{30\Delta^4 C_y C_x} \\
& \cdot \left[(C_y(y_\ell - y_1) - C_x(x_m - x_1))^6 - (C_y(y_\ell - y_1) - C_x(x_\ell - x_1))^6 \right].
\end{aligned}$$

With the remaining terms of $\int_{T_1} u^4$ taking similar form, we simply substitute coefficients and exponents based on the appropriate exponent n.

Then we get that

$$
\begin{aligned}
&\int_{T_1} u^4 dydx \qquad (3.47)\\
= \quad & \frac{1}{30\Delta^4 C_y^2 \frac{y_m-y_\ell}{x_m-x_\ell} - 30\Delta^4 C_y C_x}\\
&\cdot \left[(C_y(y_m-y_1)-C_x(x_m-x_1))^6-(C_y(y_\ell-y_1)-C_x(x_\ell-x_1))^6\right]\\
&+\frac{1}{30\Delta^4 C_y C_x}\left[(C_y(y_\ell-y_1)-C_x(x_m-x_1))^6-(C_y(y_\ell-y_1)-C_x(x_\ell-x_1))^6\right]\\
&+\frac{u_1}{5\Delta^3 C_y^2 \frac{y_m-y_\ell}{x_m-x_\ell} - 5\Delta^3 C_y C_x}\\
&\cdot \left[(C_y(y_m-y_1)-C_x(x_m-x_1))^5-(C_y(y_\ell-y_1)-C_x(x_\ell-x_1))^5\right]\\
&+\frac{u_1}{5\Delta^3 C_y C_x}\left[(C_y(y_\ell-y_1)-C_x(x_m-x_1))^5-(C_y(y_\ell-y_1)-C_x(x_\ell-x_1))^5\right]\\
&+\frac{u_1^2}{2\Delta^2 C_y^2 \frac{y_m-y_\ell}{x_m-x_\ell} - 2\Delta^2 C_y C_x}\\
&\cdot \left[(C_y(y_m-y_1)-C_x(x_m-x_1))^4-(C_y(y_\ell-y_1)-C_x(x_\ell-x_1))^4\right]\\
&+\frac{u_1^2}{2\Delta^2 C_y C_x}\left[(C_y(y_\ell-y_1)-C_x(x_m-x_1))^4-(C_y(y_\ell-y_1)-C_x(x_\ell-x_1))^4\right]\\
&+\frac{2u_1^3}{3\Delta C_y^2 \frac{y_m-y_\ell}{x_m-x_\ell} - 3\Delta C_y C_x}\\
&\cdot \left[(C_y(y_m-y_1)-C_x(x_m-x_1))^3-(C_y(y_\ell-y_1)-C_x(x_\ell-x_1))^3\right]\\
&+\frac{2u_1^3}{3\Delta C_y C_x}\left[(C_y(y_\ell-y_1)-C_x(x_m-x_1))^3-(C_y(y_\ell-y_1)-C_x(x_\ell-x_1))^3\right]\\
&+\frac{u_1^4 x_m(y_m-y_\ell)}{2}.
\end{aligned}
$$

Similarly, we have that

$$\begin{aligned}
& \int_{T_1} -2u^3 dydx && (3.48)\\
= \quad & \frac{-1}{10\Delta^3 C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - 10\Delta^3 C_y C_x} \\
& \cdot \left[(C_y(y_m - y_1) - C_x(x_m - x_1))^5 - (C_y(y_\ell - y_1) - C_x(x_\ell - x_1))^5 \right] \\
& + \frac{-1}{10\Delta^3 C_y C_x} \left[(C_y(y_\ell - y_1) - C_x(x_m - x_1))^5 - (C_y(y_\ell - y_1) - C_x(x_\ell - x_1))^5 \right] \\
& + \frac{-u_1}{2\Delta^2 C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - 2\Delta^2 C_y C_x} \\
& \cdot \left[(C_y(y_m - y_1) - C_x(x_m - x_1))^4 - (C_y(y_\ell - y_1) - C_x(x_\ell - x_1))^4 \right] \\
& + \frac{-u_1}{2\Delta^2 C_y C_x} \left[(C_y(y_\ell - y_1) - C_x(x_m - x_1))^4 - (C_y(y_\ell - y_1) - C_x(x_\ell - x_1))^4 \right] \\
& + \frac{-u_1^2}{\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x} \\
& \cdot \left[(C_y(y_m - y_1) - C_x(x_m - x_1))^3 - (C_y(y_\ell - y_1) - C_x(x_\ell - x_1))^3 \right] \\
& + \frac{-u_1^2}{\Delta C_y C_x} \left[(C_y(y_\ell - y_1) - C_x(x_m - x_1))^3 - (C_y(y_\ell - y_1) - C_x(x_\ell - x_1))^3 \right] \\
& - u_1^3 x_m (y_m - y_\ell).
\end{aligned}$$

Likewise, we have

$$\begin{aligned}
& \int_{T_1} u^2 dydx && (3.49)\\
= \quad & \frac{1}{12\Delta^2 C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - 12\Delta^2 C_y C_x} \\
& \cdot \left[(C_y(y_m - y_1) - C_x(x_m - x_1))^4 - (C_y(y_\ell - y_1) - C_x(x_\ell - x_1))^4 \right] \\
& + \frac{1}{12\Delta^2 C_y C_x} \left[(C_y(y_\ell - y_1) - C_x(x_m - x_1))^4 - (C_y(y_\ell - y_1) - C_x(x_\ell - x_1))^4 \right] \\
& + \frac{u_1}{3\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - 3\Delta C_y C_x} \\
& \cdot \left[(C_y(y_m - y_1) - C_x(x_m - x_1))^3 - (C_y(y_\ell - y_1) - C_x(x_\ell - x_1))^3 \right] \\
& + \frac{u_1}{3\Delta C_y C_x} \left[(C_y(y_\ell - y_1) - C_x(x_m - x_1))^3 - (C_y(y_\ell - y_1) - C_x(x_\ell - x_1))^3 \right] \\
& + \frac{u_1^2 x_m (y_m - y_\ell)}{2}.
\end{aligned}$$

Define the following quantities:

$$\alpha = C_y(y_m - y_1) - C_x(x_m - x_1), \tag{3.50}$$

$$\beta = C_y(y_\ell - y_1) - C_x(x_m - x_1), \tag{3.51}$$

$$\gamma = C_y(y_\ell - y_1) - C_x(x_\ell - x_1). \tag{3.52}$$

Then we have that

$$\begin{aligned}
&\int_{T_1} a^2u^2(1-u)^2dydx \qquad (3.53)\\
= \;& \frac{a^2}{\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x}\\
&\cdot \left[\frac{\alpha^6}{30\Delta^3} + \frac{\alpha^5(2u_1 - 1)}{10\Delta^2} + \frac{\alpha^4(6u_1^2 - 6u_1 + 1)}{12\Delta} + \frac{\alpha^3(2u_1^3 - 3u_1^2 + u_1)}{3}\right]\\
&+\frac{a^2}{\Delta C_y C_x} \cdot \left[\frac{\beta^6}{30\Delta^3} + \frac{\beta^5(2u_1 - 1)}{10\Delta^2} + \frac{\beta^4(6u_1^2 - 6u_1 + 1)}{12\Delta} + \frac{\beta^3(2u_1^3 - 3u_1^2 + u_1)}{3}\right]\\
&+\left(\frac{a^2}{\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x} - \frac{a^2}{\Delta C_y C_x}\right)\\
&\cdot\left[\frac{\gamma^6}{30\Delta^3} + \frac{\gamma^5(2u_1 - 1)}{10\Delta^2} + \frac{\gamma^4(6u_1^2 - 6u_1 + 1)}{12\Delta} + \frac{\gamma^3(2u_1^3 - 3u_1^2 + u_1)}{3}\right]\\
&+\frac{a^2x_m(y_m - y_\ell)(u_1^4 - 2u_1^3 + u_1^2)}{2}.
\end{aligned}$$

We can now take the partial derivatives of this quantity with respect to nodes u_1, u_2, and u_3. We calculate

$$
\frac{\partial}{\partial u_1}\int_{T_1} a^2u^2(1-u)^2 dydx \tag{3.54}
$$

$$
\begin{aligned}
= \; & \frac{2a^2\Delta C_y \frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2) - a^2\Delta(C_x(x_3-x_2)+C_y(y_3-y_2))}{-(\Delta C_y^2 \frac{y_m-y_\ell}{x_m-x_\ell} - \Delta C_y C_x)^2} \\
& \cdot\left[\frac{\alpha^6}{30\Delta^3} + \frac{\alpha^5(2u_1-1)}{10\Delta^2} + \frac{\alpha^4(6u_1^2-6u_1+1)}{12\Delta} + \frac{\alpha^3(2u_1^3-3u_1^2+u_1)}{3}\right] \\
& + \frac{a^2[(y_m-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]}{\Delta C_y^2 \frac{y_m-y_\ell}{x_m-x_\ell} - \Delta C_y C_x} \\
& \cdot\left[\frac{\alpha^5}{5\Delta^3} + \frac{\alpha^4(2u_1-1)}{2\Delta^2} + \frac{\alpha^3(6u_1^2-6u_1+1)}{3\Delta} + \frac{\alpha^2(2u_1^3-3u_1^2+u_1)}{1}\right] \\
& + \frac{a^2}{\Delta C_y^2 \frac{y_m-y_\ell}{x_m-x_\ell} - \Delta C_y C_x} \\
& \cdot\left[\frac{\alpha^5}{5\Delta^2} + \frac{\alpha^4(12u_1-6)}{12\Delta} + \frac{\alpha^3(6u_1^2-6u_1+1)}{3}\right] \\
& + \frac{a^2\Delta(C_x(x_3-x_2)+C_y(y_3-y_2))}{-(\Delta C_y C_x)^2} \\
& \cdot\left[\frac{\beta^6}{30\Delta^3} + \frac{\beta^5(2u_1-1)}{10\Delta^2} + \frac{\beta^4(6u_1^2-6u_1+1)}{12\Delta} + \frac{\beta^3(2u_1^3-3u_1^2+u_1)}{3}\right] \\
& + \frac{a^2[(y_\ell-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]}{\Delta C_y C_x} \\
& \cdot\left[\frac{\beta^5}{5\Delta^3} + \frac{\beta^4(2u_1-1)}{2\Delta^2} + \frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta} + \frac{\beta^2(2u_1^3-3u_1^2+u_1)}{1}\right] \\
& + \frac{a^2}{\Delta C_y C_x}\cdot\left[\frac{\beta^5}{5\Delta^2} + \frac{\beta^4(12u_1-6)}{12\Delta} + \frac{\beta^3(6u_1^2-6u_1+1)}{3}\right] \\
& + \left[\frac{2a^2\Delta C_y \frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2) - a^2\Delta(C_x(x_3-x_2)+C_y(y_3-y_2))}{-(\Delta C_y^2 \frac{y_m-y_\ell}{x_m-x_\ell} - \Delta C_y C_x)^2}\right. \\
& \left. + \frac{a^2\Delta(C_x(x_3-x_2)+C_y(y_3-y_2))}{-(\Delta C_y C_x)^2}\right] \\
& \cdot\left[\frac{\gamma^6}{30\Delta^3} + \frac{\gamma^5(2u_1-1)}{10\Delta^2} + \frac{\gamma^4(6u_1^2-6u_1+1)}{12\Delta} + \frac{\gamma^3(2u_1^3-3u_1^2+u_1)}{3}\right] \\
& + \left[\frac{a^2}{\Delta C_y C_x} - \frac{a^2}{\Delta C_y^2 \frac{y_m-y_\ell}{x_m-x_\ell} - \Delta C_y C_x}\right]\frac{(y_\ell-y_1)(x_3-x_2)-(x_\ell-x_1)(y_3-y_2)}{1} \\
& \cdot\left[\frac{\gamma^5}{5\Delta^3} + \frac{\gamma^4(2u_1-1)}{2\Delta^2} + \frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta} + \frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right] \\
& + \left[\frac{a^2}{\Delta C_y C_x} - \frac{a^2}{\Delta C_y^2 \frac{y_m-y_\ell}{x_m-x_\ell} - \Delta C_y C_x}\right] \\
& \cdot\left[\frac{\gamma^5}{5\Delta^2} + \frac{\gamma^4(12u_1-6)}{12\Delta} + \frac{\gamma^3(6u_1^2-6u_1+1)}{3}\right] \\
& + a^2x_m(y_m-y_\ell)(2u_1^3-3u_1^2+u_1);
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial u_2}\int_{T_1} a^2u^2(1-u)^2dydx \qquad (3.55)\\
=\;& \frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-a^2\Delta(C_x(x_1-x_3)+C_y(y_1-y_3))}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\\
&\cdot\left[\frac{\alpha^6}{30\Delta^3}+\frac{\alpha^5(2u_1-1)}{10\Delta^2}+\frac{\alpha^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\alpha^3(2u_1^3-3u_1^2+u_1)}{3}\right]\\
&+\frac{a^2[(y_m-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)]}{\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x}\\
&\cdot\left[\frac{\alpha^5}{5\Delta^3}+\frac{\alpha^4(2u_1-1)}{2\Delta^2}+\frac{\alpha^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\alpha^2(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&+\frac{a^2\Delta(C_x(x_1-x_3)+C_y(y_1-y_3))}{-(\Delta C_yC_x)^2}\\
&\cdot\left[\frac{\beta^6}{30\Delta^3}+\frac{\beta^5(2u_1-1)}{10\Delta^2}+\frac{\beta^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\beta^3(2u_1^3-3u_1^2+u_1)}{3}\right]\\
&+\frac{a^2[(y_\ell-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)]}{\Delta C_yC_x}\\
&\cdot\left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&+\left[\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-a^2\Delta(C_x(x_1-x_3)+C_y(y_1-y_3))}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\right.\\
&\left.+\frac{a^2\Delta(C_x(x_1-x_3)+C_y(y_1-y_3))}{-(\Delta C_yC_x)^2}\right]\\
&\cdot\left[\frac{\gamma^6}{30\Delta^3}+\frac{\gamma^5(2u_1-1)}{10\Delta^2}+\frac{\gamma^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\gamma^3(2u_1^3-3u_1^2+u_1)}{3}\right]\\
&+\left[\frac{a^2}{\Delta C_yC_x}-\frac{a^2}{\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x}\right]\frac{(y_\ell-y_1)(x_1-x_3)-(x_\ell-x_1)(y_1-y_3)}{1}\\
&\cdot\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right];
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial u_3}\int_{T_1} a^2u^2(1-u)^2 dydx \qquad (3.56)\\
= &\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-a^2\Delta(C_x(x_2-x_1)+C_y(y_2-y_1))}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\\
&\cdot\left[\frac{\alpha^6}{30\Delta^3}+\frac{\alpha^5(2u_1-1)}{10\Delta^2}+\frac{\alpha^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\alpha^3(2u_1^3-3u_1^2+u_1)}{3}\right]\\
&+\frac{a^2[(y_m-y_1)(x_2-x_1)-(x_m-x_1)(y_2-y_1)]}{\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x}\\
&\cdot\left[\frac{\alpha^5}{5\Delta^3}+\frac{\alpha^4(2u_1-1)}{2\Delta^2}+\frac{\alpha^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\alpha^2(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&+\frac{a^2\Delta(C_x(x_2-x_1)+C_y(y_2-y_1))}{-(\Delta C_yC_x)^2}\\
&\cdot\left[\frac{\beta^6}{30\Delta^3}+\frac{\beta^5(2u_1-1)}{10\Delta^2}+\frac{\beta^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\beta^3(2u_1^3-3u_1^2+u_1)}{3}\right]\\
&+\frac{a^2[(y_\ell-y_1)(x_2-x_1)-(x_m-x_1)(y_2-y_1)]}{\Delta C_yC_x}\\
&\cdot\left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&+\left[\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-a^2\Delta(C_x(x_2-x_1)+C_y(y_2-y_1))}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\right.\\
&\left.+\frac{a^2\Delta(C_x(x_2-x_1)+C_y(y_2-y_1))}{-(\Delta C_yC_x)^2}\right]\\
&\cdot\left[\frac{\gamma^6}{30\Delta^3}+\frac{\gamma^5(2u_1-1)}{10\Delta^2}+\frac{\gamma^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\gamma^3(2u_1^3-3u_1^2+u_1)}{3}\right]\\
&+\left[\frac{a^2}{\Delta C_yC_x}-\frac{a^2}{\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x}\right]\frac{(y_\ell-y_1)(x_2-x_1)-(x_\ell-x_1)(y_2-y_1)}{1}\\
&\cdot\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right].
\end{aligned}
$$

We can calculate this integral over T_2 by replacing x_ℓ with x_r and y_ℓ with y_r.

We must also calculate the Hessian for this Double-Well Term, which we will do over T_1.

$$
\begin{aligned}
&\frac{\partial}{\partial u_1^2}\int_{T_1} a^2u^2(1-u)^2dydx \qquad (3.57)\\
=\; & \frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)-a^2\Delta(C_x(x_3-x_2)+C_y(y_3-y_2))}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\\
&\cdot[[(y_m-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]\\
&\cdot\left[\frac{\alpha^5}{5\Delta^3}+\frac{\alpha^4(2u_1-1)}{2\Delta^2}+\frac{\alpha^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\alpha^2(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&+\left[\frac{\alpha^5}{5\Delta^2}+\frac{\alpha^4(2u_1-1)}{2\Delta}+\frac{\alpha^3(6u_1^2-6u_1+1)}{3}\right]]\\
&+\left[\frac{\alpha^6}{30\Delta^3}+\frac{\alpha^5(2u_1-1)}{10\Delta^2}+\frac{\alpha^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\alpha^3(2u_1^3-3u_1^2+u_1)}{3}\right]\\
&\cdot\left[\frac{2a^2\Delta[\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)^2-(x_3-x_2)(y_3-y_2)]}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\right.\\
&-\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)-a^2\Delta(C_x(x_3-x_2)+C_y(y_3-y_2))}{(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^3}\\
&\left.\cdot[-4\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)-\Delta(C_y(y_3-y_2)+C_x(x_3-x_2))]\right]\\
&+\frac{a^2[(y_m-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]^2}{\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x}\\
&\cdot\left[\frac{\alpha^4}{\Delta^3}+\frac{2\alpha^3(2u_1-1)}{\Delta^2}+\frac{\alpha^2(6u_1^2-6u_1+1)}{1\Delta}+\frac{2\alpha(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&+\frac{a^2[(y_m-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]}{\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x}\\
&\cdot\left[\frac{\alpha^4}{\Delta^2}+\frac{\alpha^3(4u_1-2)}{\Delta}+\frac{\alpha^2(6u_1^2-6u_1+1)}{1}\right]\\
&+\left[\frac{\alpha^5}{5\Delta^3}+\frac{\alpha^4(2u_1-1)}{2\Delta^2}+\frac{\alpha^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\alpha^2(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&\cdot\frac{-a^2[(y_m-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]}{(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\\
&\cdot[2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)-\Delta(C_y(y_3-y_2)+C_x(x_3-x_2))]\\
&+\frac{a^2}{\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x}\cdot\{[(y_m-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]\\
&\cdot\left[\frac{\alpha^4}{\Delta^2}+\frac{\alpha^3(12u_1-6)}{3\Delta}+\frac{\alpha^2(6u_1^2-6u_1+1)}{1}\right]+\left[\frac{\alpha^4}{\Delta}+\frac{\alpha^3(4u_1-2)}{1}\right]\}
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{\alpha^5}{5\Delta^2}+\frac{\alpha^4(12u_1-6)}{12\Delta}+\frac{\alpha^3(6u_1^2-6u_1+1)}{3}\right]\\
&\cdot\frac{-a^2\{2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)-\Delta[C_y(y_3-y_2)+C_x(x_3-x_2)]\}}{(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\\
&+\frac{a^2\Delta[C_x(x_3-x_2)+C_y(y_3-y_2)]}{-(\Delta C_yC_x)^2}\\
&\cdot\{[(y_\ell-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]\\
&\cdot\left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+1)}{1}\right]\\
&+\left[\frac{\beta^5}{5\Delta^2}+\frac{\beta^4(2u_1-1)}{2\Delta}+\frac{\beta^3(6u_1^2-6u_1+1)}{3}\right]\Bigg\}\\
&+\left[\frac{\beta^6}{30\Delta^3}+\frac{\beta^5(2u_1-1)}{10\Delta^2}+\frac{\beta^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\beta^3(2u_1^3-3u_1^2+1)}{3}\right]\\
&\cdot\frac{-2a^2\Delta^3(C_yC_x)^2(x_3-x_2)(y_3-y_2)+2a^2\Delta^3(C_yC_x)[C_y(y_3-y_2)+C_x(x_3-x_2)]^2}{(\Delta C_yC_x)^4}\\
&+\frac{a^2[(y_\ell-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]}{\Delta C_yC_x}\\
&\cdot\{[(y_\ell-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]\\
&\cdot\left[\frac{\beta^4}{\Delta^3}+\frac{2\beta^3(2u_1-1)}{\Delta^2}+\frac{\beta^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\beta(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&+\left[\frac{\beta^4}{\Delta^2}+\frac{\beta^3(4u_1-2)}{\Delta}+\frac{\beta^2(6u_1^2-6u_1+1)}{1}\right]\Bigg\}\\
&+\left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&\cdot\frac{-a^2\Delta[(y_\ell-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)][C_y(y_3-y_2)+C_x(x_3-x_2)]}{(\Delta C_yC_x)^2}\\
&+\frac{a^2}{\Delta C_yC_x}\cdot\{[(y_\ell-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]\\
&\cdot\left[\frac{\beta^4}{\Delta^2}+\frac{\beta^3(4u_1-2)}{\Delta}+\frac{\beta^2(6u_1^2-6u_1+1)}{1}\right]+\left[\frac{\beta^4}{\Delta}+\frac{\beta^3(4u_1-2)}{1}\right]\Bigg\}\\
&+\left[\frac{\beta^5}{5\Delta^2}+\frac{\beta^4(2u_1-1)}{2\Delta}+\frac{\beta^3(6u_1^2-6u_1+1)}{3}\right]\\
&\cdot\left[\frac{-a^2\Delta[C_y(y_3-y_2)+C_x(x_3-x_2)]}{(\Delta C_yC_x)^2}\right.\\
&+\left[\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)-a^2\Delta(C_x(x_3-x_2)+C_y(y_3-y_2))}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\right.\\
&\left.+\frac{a^2\Delta(C_x(x_3-x_2)+C_y(y_3-y_2))}{-(\Delta C_yC_x)^2}\right]
\end{aligned}
$$

$$\cdot \{[(y_\ell - y_1)(x_3 - x_2) - (x_\ell - x_1)(y_3 - y_2)]$$

$$\cdot \left[\frac{\gamma^5}{5\Delta^3} + \frac{\gamma^4(2u_1-1)}{2\Delta^2} + \frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta} + \frac{\gamma^2(2u_1^3-3u_1^2+1)}{1}\right]$$

$$+ \left[\frac{\gamma^5}{5\Delta^2} + \frac{\gamma^4(2u_1-1)}{2\Delta} + \frac{\gamma^3(6u_1^2-6u_1+1)}{3}\right]\Bigg\}$$

$$+ \left[\frac{\gamma^6}{30\Delta^3} + \frac{\gamma^5(2u_1-1)}{10\Delta^2} + \frac{\gamma^4(6u_1^2-6u_1+1)}{12\Delta} + \frac{\gamma^3(2u_1^3-3u_1^2+1)}{3}\right]$$

$$\cdot \left[\left\{(2a^2)\left\{2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2) - [C_y(y_3-y_2)+C_x(x_3-x_2)]\right\}\right.\right.$$

$$\cdot\ (2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2) - [C_y(y_3-y_2)+C_x(x_3-x_2)])$$

$$-(a^2)(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell} - C_yC_x)\left[2\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)^2\right.$$

$$\left.\left. -[(x_3-x_2)^2+(y_3-y_2)^2]\right]\right\} \cdot \frac{1}{\Delta(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell} - C_yC_x)^3}$$

$$+ \left\{2a^2[C_y(y_3-y_2)+C_x(x_3-x_2)]^2 - a^2(C_yC_x)[(x_3-x_2)^2+(y_3-y_2)^2]\right\}$$

$$\left.\cdot\frac{1}{\Delta(C_yC_x)^3}\right]$$

$$+\frac{a^2\left[\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell} - 2\Delta C_yC_x\right]}{\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell} - (\Delta C_yC_x)^2} \cdot \left\{[(y_\ell-y_1)(x_3-x_2) - (x_\ell-x_1)(y_3-y_2)]^2\right.$$

$$\cdot \left[\frac{\gamma^4}{\Delta^3} + \frac{2\gamma^3(2u_1-1)}{\Delta^2} + \frac{\gamma^2(6u_1^2-6u_1+1)}{\Delta} + \frac{2\gamma(2u_1^3-3u_1^2+u_1)}{1}\right]$$

$$+[(y_\ell-y_1)(x_3-x_2) - (x_\ell-x_1)(y_3-y_2)]$$

$$\left.\cdot \left[\frac{2\gamma^4}{\Delta^2} + \frac{2\gamma^3(4u_1-2)}{\Delta} + \frac{2\gamma^2(6u_1^2-6u_1+1)}{1}\right] + \left[\frac{\gamma^4}{\Delta} + \frac{\gamma^3(4u_1-2)}{1}\right]\right\}$$

$$+ \left[\frac{\gamma^5}{5\Delta^3} + \frac{\gamma^4(2u_1-1)}{2\Delta^2} + \frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta} + \frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right]$$

$$\cdot\left\{\frac{a^2\Delta^3}{\left[\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell} - (\Delta C_yC_x)^2\right]^2} \cdot \left\{\left[C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell} - (C_yC_x)^2\right]\right.\right.$$

$$\left.\cdot\ 2\left[C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2) - (C_y(y_3-y_2)+C_x(x_3-x_2))\right]\right\}$$

$$-\left(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell} - 2C_yC_x\right)$$

$$\left.\cdot\left[(3C_x-C_y)C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2) - 2[C_y(y_3-y_2)+C_x(x_3-x_2)]\right]\right\}$$

$$+(a^2x_m)(y_m-y_\ell)(6u_1^2-6u_1+1)$$

$$
\begin{aligned}
&\frac{\partial}{\partial u_1 \partial u_2}\int_{T_1} a^2u^2(1-u)^2 dydx \qquad (3.58)\\
= \quad &\frac{2a^2\Delta C_y \frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2) - a^2\Delta(C_x(x_3-x_2)+C_y(y_3-y_2))}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell} - \Delta C_y C_x)^2}\\
&\cdot[(y_m-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)]\\
&\cdot\left[\frac{\alpha^5}{5\Delta^3}+\frac{\alpha^4(2u_1-1)}{2\Delta^2}+\frac{\alpha^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\alpha^2(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&+\left[\frac{\alpha^6}{30\Delta^3}+\frac{\alpha^5(2u_1-1)}{10\Delta^2}+\frac{\alpha^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\alpha^3(2u_1^3-3u_1^2+u_1)}{3}\right]\\
&\cdot\Bigg[\frac{2a^2\Delta\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)(x_1-x_3)}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell} - \Delta C_y C_x)^2}\\
&-\frac{a^2\Delta[(x_3-x_2)(y_1-y_3)+(y_3-y_2)(x_1-x_3)]}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell} - \Delta C_y C_x)^2}\\
&-\frac{2a^2\Delta C_y \frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2) - a^2\Delta(C_x(x_3-x_2)+C_y(y_3-y_2))}{(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell} - \Delta C_y C_x)^3}\\
&\cdot[-4\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-\Delta(C_y(y_1-y_3)+C_x(x_1-x_3))]\Bigg]\\
&+\frac{a^2[(y_m-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]}{\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell} - \Delta C_y C_x}\\
&\cdot[(y_m-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)]\\
&\cdot\left[\frac{\alpha^4}{\Delta^3}+\frac{2\alpha^3(2u_1-1)}{\Delta^2}+\frac{\alpha^2(6u_1^2-6u_1+1)}{1\Delta}+\frac{2\alpha(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&+\left[\frac{\alpha^5}{5\Delta^3}+\frac{\alpha^4(2u_1-1)}{2\Delta^2}+\frac{\alpha^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\alpha^2(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&\cdot\frac{-a^2[(y_m-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]}{(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell} - \Delta C_y C_x)^2}\\
&\cdot[2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-\Delta(C_y(y_1-y_3)+C_x(x_1-x_3))]\\
&+\frac{a^2}{\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell} - \Delta C_y C_x}\cdot[(y_m-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)]\\
&\cdot\left[\frac{\alpha^4}{\Delta^2}+\frac{\alpha^3(12u_1-6)}{3\Delta}+\frac{\alpha^2(6u_1^2-6u_1+1)}{1}\right]\\
&+\left[\frac{\alpha^5}{5\Delta^2}+\frac{\alpha^4(12u_1-6)}{12\Delta}+\frac{\alpha^3(6u_1^2-6u_1+1)}{3}\right]\\
&\cdot\frac{-a^2\{2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-\Delta[C_y(y_1-y_3)+C_x(x_1-x_3)]\}}{(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell} - \Delta C_y C_x)^2}
\end{aligned}
$$

$$+\frac{a^2\Delta[C_x(x_3-x_2)+C_y(y_3-y_2)]}{-(\Delta C_y C_x)^2}$$

$$\cdot\{[(y_\ell-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)]$$

$$\cdot\left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+1)}{1}\right]\right\}$$

$$+\left[\frac{\beta^6}{30\Delta^3}+\frac{\beta^5(2u_1-1)}{10\Delta^2}+\frac{\beta^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\beta^3(2u_1^3-3u_1^2+1)}{3}\right]$$

$$\cdot\left\{\frac{2a^2[C_y(y_3-y_2)+C_x(x_3-x_2)][C_y(y_1-y_3)+C_x(x_1-x_3)]}{\Delta(C_yC_x)^3}\right.$$

$$\left.-\frac{a^2C_yC_x[(x_3-x_2)(y_1-y_3)+(y_3-y_2)(x_1-x_3)]}{\Delta(C_yC_x)^3}\right\}$$

$$+\frac{a^2[(y_\ell-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]}{\Delta C_y C_x}$$

$$\cdot\{[(y_\ell-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)]$$

$$\cdot\left[\frac{\beta^4}{\Delta^3}+\frac{2\beta^3(2u_1-1)}{\Delta^2}+\frac{\beta^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\beta(2u_1^3-3u_1^2+u_1)}{1}\right]\right\}$$

$$+\left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+u_1)}{1}\right]$$

$$\cdot\frac{-a^2\Delta[(y_\ell-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)][C_y(y_1-y_3)+C_x(x_1-x_3)]}{(\Delta C_y C_x)^2}$$

$$+\frac{a^2}{\Delta C_y C_x}\cdot\{[(y_\ell-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)]$$

$$\cdot\left[\frac{\beta^4}{\Delta^2}+\frac{\beta^3(4u_1-2)}{\Delta}+\frac{\beta^2(6u_1^2-6u_1+1)}{1}\right]\right\}$$

$$+\left[\frac{\beta^5}{5\Delta^2}+\frac{\beta^4(2u_1-1)}{2\Delta}+\frac{\beta^3(6u_1^2-6u_1+1)}{3}\right]$$

$$\cdot\left[\frac{-a^2\Delta[C_y(y_1-y_3)+C_x(x_1-x_3)]}{(\Delta C_y C_x)^2}\right]$$

$$+\left[\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)-a^2\Delta(C_x(x_3-x_2)+C_y(y_3-y_2))}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_y C_x)^2}\right.$$

$$\left.+\frac{a^2\Delta(C_x(x_3-x_2)+C_y(y_3-y_2))}{-(\Delta C_y C_x)^2}\right]$$

$$\cdot\{[(y_\ell-y_1)(x_1-x_3)-(x_\ell-x_1)(y_1-y_3)]$$

$$\cdot\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+1)}{1}\right]\right\}$$

$$+\left[\frac{\gamma^6}{30\Delta^3}+\frac{\gamma^5(2u_1-1)}{10\Delta^2}+\frac{\gamma^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\gamma^3(2u_1^3-3u_1^2+1)}{3}\right]$$

$$\cdot\left[\left\{(2a^2)\left\{2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)-[C_y(y_3-y_2)+C_x(x_3-x_2)]\right\}\right.\right.$$

$$\cdot\ (2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-[C_y(y_1-y_3)+C_x(x_1-x_3)])$$

$$-(a^2)(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-C_yC_x)\left[2\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)(x_1-x_3)\right.$$

$$-[(x_3-x_2)(x_1-x_3)+(y_3-y_2)(y_1-y_3)]]\}$$

$$\cdot\frac{1}{\Delta(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-C_yC_x)^3}$$

$$+\left\{2a^2[C_y(y_3-y_2)+C_x(x_3-x_2)][C_y(y_1-y_3)+C_x(x_1-x_3)]\right.$$

$$\left.-\ a^2(C_yC_x)[(x_3-x_2)(x_1-x_3)+(y_3-y_2)(y_1-y_3)]\right\}$$

$$\left.\cdot\frac{1}{\Delta(C_yC_x)^3}\right]$$

$$+\frac{a^2\left[\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-2\Delta C_yC_x\right]}{\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(\Delta C_yC_x)^2}\cdot\{[(y_\ell-y_1)(x_1-x_3)-(x_\ell-x_1)(y_1-y_3)]$$

$$\cdot[(y_\ell-y_1)(x_3-x_2)-(x_\ell-x_1)(y_3-y_2)]$$

$$\cdot\left[\frac{\gamma^4}{\Delta^3}+\frac{2\gamma^3(2u_1-1)}{\Delta^2}+\frac{\gamma^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\gamma(2u_1^3-3u_1^2+u_1)}{1}\right]$$

$$+[(y_\ell-y_1)(x_1-x_3)-(x_\ell-x_1)(y_1-y_3)]\cdot\left[\frac{\gamma^4}{\Delta}+\frac{\gamma^3(4u_1-2)}{1}\right]\Bigg\}$$

$$+\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right]$$

$$\cdot\left\{\frac{a^2\Delta^3}{\left[\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(\Delta C_yC_x)^2\right]^2}\cdot\left\{\left[C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(C_yC_x)^2\right]\right.\right.$$

$$\left.\cdot\ 2\left[C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-(C_y(y_1-y_3)+C_x(x_1-x_3))\right]\right\}$$

$$-\left(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-2C_yC_x\right)$$

$$\left.\cdot\left[(3C_x-C_y)C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-2[C_y(y_1-y_3)+C_x(x_1-x_3)]\right]\right\}$$

$$
\begin{aligned}
&\frac{\partial}{\partial u_1 \partial u_3} \int_{T_1} a^2 u^2 (1-u)^2 dy dx \qquad (3.59)\\
= \quad & \frac{2a^2 \Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_3 - x_2) - a^2 \Delta (C_x (x_3 - x_2) + C_y (y_3 - y_2))}{-(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2}\\
&\cdot [(y_m - y_1)(x_2 - x_1) - (x_m - x_1)(y_2 - y_1)]\\
&\cdot \left[\frac{\alpha^5}{5\Delta^3} + \frac{\alpha^4 (2u_1 - 1)}{2\Delta^2} + \frac{\alpha^3 (6u_1^2 - 6u_1 + 1)}{3\Delta} + \frac{\alpha^2 (2u_1^3 - 3u_1^2 + u_1)}{1} \right]\\
&+ \left[\frac{\alpha^6}{30\Delta^3} + \frac{\alpha^5 (2u_1 - 1)}{10\Delta^2} + \frac{\alpha^4 (6u_1^2 - 6u_1 + 1)}{12\Delta} + \frac{\alpha^3 (2u_1^3 - 3u_1^2 + u_1)}{3} \right]\\
&\cdot \Bigg[\frac{2a^2 \Delta \frac{y_m - y_\ell}{x_m - x_\ell}(x_3 - x_2)(x_2 - x_1)}{-(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2}\\
&- \frac{a^2 \Delta [(x_3 - x_2)(y_2 - y_1) + (y_3 - y_2)(x_2 - x_1)]}{-(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2}\\
&- \frac{2a^2 \Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_3 - x_2) - a^2 \Delta (C_x (x_3 - x_2) + C_y (y_3 - y_2))}{(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^3}\\
&\cdot [-4\Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_2 - x_1) - \Delta (C_y (y_2 - y_1) + C_x (x_2 - x_1))] \Bigg]\\
&+ \frac{a^2 [(y_m - y_1)(x_3 - x_2) - (x_m - x_1)(y_3 - y_2)]}{\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x}\\
&\cdot [(y_m - y_1)(x_2 - x_1) - (x_m - x_1)(y_2 - y_1)]\\
&\cdot \left[\frac{\alpha^4}{\Delta^3} + \frac{2\alpha^3 (2u_1 - 1)}{\Delta^2} + \frac{\alpha^2 (6u_1^2 - 6u_1 + 1)}{1\Delta} + \frac{2\alpha (2u_1^3 - 3u_1^2 + u_1)}{1} \right]\\
&+ \left[\frac{\alpha^5}{5\Delta^3} + \frac{\alpha^4 (2u_1 - 1)}{2\Delta^2} + \frac{\alpha^3 (6u_1^2 - 6u_1 + 1)}{3\Delta} + \frac{\alpha^2 (2u_1^3 - 3u_1^2 + u_1)}{1} \right]\\
&\cdot \frac{-a^2 [(y_m - y_1)(x_3 - x_2) - (x_m - x_1)(y_3 - y_2)]}{(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2}\\
&\cdot [2\Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_2 - x_1) - \Delta (C_y (y_2 - y_1) + C_x (x_2 - x_1))]\\
&+ \frac{a^2}{\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x} \cdot [(y_m - y_1)(x_2 - x_1) - (x_m - x_1)(y_2 - y_1)]\\
&\cdot \left[\frac{\alpha^4}{\Delta^2} + \frac{\alpha^3 (12u_1 - 6)}{3\Delta} + \frac{\alpha^2 (6u_1^2 - 6u_1 + 1)}{1} \right]\\
&+ \left[\frac{\alpha^5}{5\Delta^2} + \frac{\alpha^4 (12u_1 - 6)}{12\Delta} + \frac{\alpha^3 (6u_1^2 - 6u_1 + 1)}{3} \right]\\
&\cdot \frac{-a^2 \{2\Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_2 - x_1) - \Delta [C_y (y_2 - y_1) + C_x (x_2 - x_1)]\}}{(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2}
\end{aligned}
$$

$$+\frac{a^2\Delta[C_x(x_3-x_2)+C_y(y_3-y_2)]}{-(\Delta C_y C_x)^2}$$

$$\cdot\{[(y_\ell-y_1)(x_2-x_1)-(x_m-x_1)(y_2-y_1)]$$

$$\cdot\left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+1)}{1}\right]\right\}$$

$$+\left[\frac{\beta^6}{30\Delta^3}+\frac{\beta^5(2u_1-1)}{10\Delta^2}+\frac{\beta^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\beta^3(2u_1^3-3u_1^2+1)}{3}\right]$$

$$\cdot\left\{\frac{2a^2[C_y(y_3-y_2)+C_x(x_3-x_2)][C_y(y_2-y_1)+C_x(x_2-x_1)]}{\Delta(C_yC_x)^3}\right.$$

$$\left.-\frac{a^2C_yC_x[(x_3-x_2)(y_2-y_1)+(y_3-y_2)(x_2-x_1)]}{\Delta(C_yC_x)^3}\right\}$$

$$+\frac{a^2[(y_\ell-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]}{\Delta C_yC_x}$$

$$\cdot\{[(y_\ell-y_1)(x_2-x_1)-(x_m-x_1)(y_2-y_1)]$$

$$\cdot\left[\frac{\beta^4}{\Delta^3}+\frac{2\beta^3(2u_1-1)}{\Delta^2}+\frac{\beta^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\beta(2u_1^3-3u_1^2+u_1)}{1}\right]\right\}$$

$$+\left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+u_1)}{1}\right]$$

$$\cdot\frac{-a^2\Delta[(y_\ell-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)][C_y(y_2-y_1)+C_x(x_2-x_1)]}{(\Delta C_yC_x)^2}$$

$$+\frac{a^2}{\Delta C_yC_x}\cdot\{[(y_\ell-y_1)(x_2-x_1)-(x_m-x_1)(y_2-y_1)]$$

$$\cdot\left[\frac{\beta^4}{\Delta^2}+\frac{\beta^3(4u_1-2)}{\Delta}+\frac{\beta^2(6u_1^2-6u_1+1)}{1}\right]\right\}$$

$$+\left[\frac{\beta^5}{5\Delta^2}+\frac{\beta^4(2u_1-1)}{2\Delta}+\frac{\beta^3(6u_1^2-6u_1+1)}{3}\right]$$

$$\cdot\left[\frac{-a^2\Delta[C_y(y_2-y_1)+C_x(x_2-x_1)]}{(\Delta C_yC_x)^2}\right]$$

$$+\left[\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)-a^2\Delta(C_x(x_3-x_2)+C_y(y_3-y_2))}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\right.$$

$$\left.+\frac{a^2\Delta(C_x(x_3-x_2)+C_y(y_3-y_2))}{-(\Delta C_yC_x)^2}\right]$$

$$\cdot\{[(y_\ell-y_1)(x_2-x_1)-(x_\ell-x_1)(y_2-y_1)]$$

$$\cdot\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+1)}{1}\right]\right\}$$

$$+\left[\frac{\gamma^6}{30\Delta^3}+\frac{\gamma^5(2u_1-1)}{10\Delta^2}+\frac{\gamma^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\gamma^3(2u_1^3-3u_1^2+1)}{3}\right]$$

$$\cdot\left[\left\{(2a^2)\left\{2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)-[C_y(y_3-y_2)+C_x(x_3-x_2)]\right\}\right.\right.$$

$$\cdot\ (2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-[C_y(y_2-y_1)+C_x(x_2-x_1)])$$

$$-(a^2)(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-C_yC_x)\left[2\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)(x_2-x_1)\right.$$

$$-[(x_3-x_2)(x_2-x_1)+(y_3-y_2)(y_2-y_1)]]\}$$

$$\cdot\frac{1}{\Delta(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-C_yC_x)^3}$$

$$+\left\{2a^2[C_y(y_3-y_2)+C_x(x_3-x_2)][C_y(y_2-y_1)+C_x(x_2-x_1)]\right.$$

$$\left.-\ a^2(C_yC_x)[(x_3-x_2)(x_2-x_1)+(y_3-y_2)(y_2-y_1)]\right\}$$

$$\left.\cdot\frac{1}{\Delta(C_yC_x)^3}\right]$$

$$+\frac{a^2\left[\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-2\Delta C_yC_x\right]}{\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(\Delta C_yC_x)^2}\cdot\{[(y_\ell-y_1)(x_2-x_1)-(x_\ell-x_1)(y_2-y_1)]$$

$$\cdot[(y_\ell-y_1)(x_3-x_2)-(x_\ell-x_1)(y_3-y_2)]$$

$$\cdot\left[\frac{\gamma^4}{\Delta^3}+\frac{2\gamma^3(2u_1-1)}{\Delta^2}+\frac{\gamma^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\gamma(2u_1^3-3u_1^2+u_1)}{1}\right]$$

$$\left.+[(y_\ell-y_1)(x_2-x_1)-(x_\ell-x_1)(y_2-y_1)]\cdot\left[\frac{\gamma^4}{\Delta}+\frac{\gamma^3(4u_1-2)}{1}\right]\right\}$$

$$+\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right]$$

$$\cdot\left\{\frac{a^2\Delta^3}{\left[\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(\Delta C_yC_x)^2\right]^2}\cdot\left\{\left[C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(C_yC_x)^2\right]\right.\right.$$

$$\left.\cdot\ 2\left[C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-(C_y(y_2-y_1)+C_x(x_2-x_1))\right]\right\}$$

$$-\left(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-2C_yC_x\right)$$

$$\left.\cdot\left[(3C_x-C_y)C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-2[C_y(y_2-y_1)+C_x(x_2-x_1)]\right]\right\}$$

$$
\begin{aligned}
&\frac{\partial}{\partial u_2 \partial u_1} \int_{T_1} a^2 u^2 (1-u)^2 dy dx \qquad (3.60)\\
= \quad & \left[\frac{2a^2 \Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_1 - x_3) - a^2 \Delta [C_y(y_1 - y_3) + C_x(x_1 - x_3)]}{-(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2} \right] \\
& \cdot \Big(\Big\{ [(y_m - y_1)(x_3 - x_2) - (x_m - x_1)(y_3 - y_2)] \\
& \cdot \left[\frac{\alpha^5}{5\Delta^3} + \frac{\alpha^4(2u_1 - 1)}{2\Delta^2} + \frac{\alpha^3(6u_1^2 - 6u_1 + 1)}{3\Delta} + \frac{\alpha^2(2u_1^3 - 3u_1^2 + u_1)}{1} \right] \Big\} \\
& + \left[\frac{\alpha^5}{5\Delta^2} + \frac{\alpha^4(2u_1 - 1)}{2\Delta} + \frac{\alpha^3(6u_1^2 - 6u_1 + 1)}{3} \right] \Big) \\
& + \left[\frac{\alpha^6}{30\Delta^3} + \frac{\alpha^5(2u_1 - 1)}{10\Delta^2} + \frac{\alpha^4(6u_1^2 - 6u_1 + 1)}{12\Delta} + \frac{\alpha^3(2u_1^3 - 3u_1^2 + u_1)}{3} \right] \\
& \cdot \Bigg[\frac{2a^2 \Delta \frac{y_m - y_\ell}{x_m - x_\ell}(x_3 - x_2)(x_1 - x_3)}{-(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2} \\
& - \frac{a^2 \Delta [(x_1 - x_3)(y_3 - y_2) + (y_1 - y_3)(x_3 - x_2)]}{-(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2} \\
& - \frac{2a^2 \Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_1 - x_3) - a^2 \Delta (C_x(x_1 - x_3) + C_y(y_1 - y_3))}{(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^3} \\
& \cdot [-4\Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_3 - x_2) - \Delta(C_y(y_3 - y_2) + C_x(x_3 - x_2))] \Bigg] \\
& + \frac{a^2 [(y_m - y_1)(x_1 - x_3) - (x_m - x_1)(y_1 - y_3)]}{\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x} \\
& \cdot \Big(\Big\{ [(y_m - y_1)(x_3 - x_2) - (x_m - x_1)(y_3 - y_2)] \\
& \cdot \left[\frac{\alpha^4}{\Delta^3} + \frac{2\alpha^3(2u_1 - 1)}{\Delta^2} + \frac{\alpha^2(6u_1^2 - 6u_1 + 1)}{\Delta} + \frac{2\alpha(2u_1^3 - 3u_1^2 + u_1)}{1} \right] \Big\} \\
& + \left[\frac{\alpha^4}{\Delta^2} + \frac{\alpha^3(4u_1 - 2)}{\Delta} + \frac{\alpha^2(6u_1^2 - 6u_1 + 1)}{1} \right] \Big) \\
& + \left[\frac{\alpha^5}{5\Delta^3} + \frac{\alpha^4(2u_1 - 1)}{2\Delta^2} + \frac{\alpha^3(6u_1^2 - 6u_1 + 1)}{3\Delta} + \frac{\alpha^2(2u_1^3 - 3u_1^2 + u_1)}{1} \right] \\
& \cdot \Bigg\{ \frac{-a^2 [(y_m - y_1)(x_1 - x_3) - (x_m - x_1)(y_1 - y_3)]}{(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2} \\
& \cdot \left[2\Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_3 - x_2) - \Delta(C_y(y_3 - y_2) + C_x(x_3 - x_2)) \right] \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left[\frac{a^2\Delta[C_y(y_1-y_3)+C_x(x_1-x_3)]}{-(\Delta C_yC_x)^2}\right] \\
&\cdot (\{[(y_\ell-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)] \\
&\cdot \left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+u_1)}{1}\right]\Bigg\} \\
&+ \left[\frac{\beta^5}{5\Delta^2}+\frac{\beta^4(2u_1-1)}{2\Delta}+\frac{\beta^3(6u_1^2-6u_1+1)}{3}\right]\Bigg) \\
&+ \left[\frac{\beta^6}{30\Delta^3}+\frac{\beta^5(2u_1-1)}{10\Delta^2}+\frac{\beta^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\beta^3(2u_1^3-3u_1^2+u_1)}{3}\right] \\
&\cdot \Bigg\{\frac{2a^2[C_y(y_3-y_2)+C_x(x_3-x_2)][C_y(y_1-y_3)+C_x(x_1-x_3)]}{\Delta(C_yC_x)^3} \\
&- \frac{a^2C_yC_x[(x_3-x_2)(y_1-y_3)+(y_3-y_2)(x_1-x_3)]}{\Delta(C_yC_x)^3}\Bigg\} \\
&+ \left[\frac{a^2[(y_\ell-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)]}{\Delta C_yC_x}\right] \\
&\cdot (\{[(y_\ell-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)] \\
&\cdot \left[\frac{\beta^4}{\Delta^3}+\frac{2\beta^3(2u_1-1)}{\Delta^2}+\frac{\beta^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\beta(2u_1^3-3u_1^2+u_1)}{1}\right]\Bigg\} \\
&+ \left[\frac{\beta^4}{\Delta^2}+\frac{\beta^3(4u_1-2)}{\Delta}+\frac{\beta^2(6u_1^2-6u_1+1)}{1}\right]\Bigg) \\
&+ \left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+u_1)}{1}\right] \\
&\cdot \frac{-a^2\Delta[(y_\ell-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)][C_y(y_3-y_2)+C_x(x_3-x_2)]}{(\Delta C_yC_x)^2} \\
&+ \left[\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-a^2\Delta(C_x(x_1-x_3)+C_y(y_1-y_3))}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\right. \\
&\left.+\frac{a^2\Delta(C_x(x_1-x_3)+C_y(y_1-y_3))}{-(\Delta C_yC_x)^2}\right] \\
&\cdot (\{[(y_\ell-y_1)(x_3-x_2)-(x_\ell-x_1)(y_3-y_2)] \\
&\cdot \left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right]\Bigg\} \\
&+ \left[\frac{\gamma^5}{5\Delta^2}+\frac{\gamma^4(2u_1-1)}{2\Delta}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3}\right]\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{\gamma^6}{30\Delta^3}+\frac{\gamma^5(2u_1-1)}{10\Delta^2}+\frac{\gamma^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\gamma^3(2u_1^3-3u_1^2+u_1)}{3}\right]\\
&\cdot\left[\left\{(2a^2)\left\{2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-[C_y(y_1-y_3)+C_x(x_1-x_3)]\right\}\right.\right.\\
&\cdot\ (2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)-[C_y(y_3-y_2)+C_x(x_3-x_2)])\\
&-(a^2)(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-C_yC_x)\left[2\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)(x_3-x_2)\right.\\
&-[(x_1-x_3)(x_3-x_2)+(y_1-y_3)(y_3-y_2)]]\}\\
&\cdot\frac{1}{\Delta(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-C_yC_x)^3}\\
&+\left\{2a^2[C_y(y_1-y_3)+C_x(x_1-x_3)][C_y(y_3-y_2)+C_x(x_3-x_2)]\right.\\
&\left.-\ a^2(C_yC_x)[(x_1-x_3)(x_3-x_2)+(y_1-y_3)(y_3-y_2)]\right\}\\
&\left.\cdot\frac{1}{\Delta(C_yC_x)^3}\right]\\
&+\frac{a^2\Delta[C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-2C_yC_x][(y_\ell-y_1)(x_1-x_3)-(x_\ell-x_1)(y_1-y_3)]}{\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(\Delta C_yC_x)^2}\\
&\cdot(\{[(y_\ell-y_1)(x_3-x_2)-(x_\ell-x_1)(y_3-y_2)]\\
&\left.\cdot\left[\frac{\gamma^4}{\Delta^3}+\frac{2\gamma^3(2u_1-1)}{\Delta^2}+\frac{\gamma^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\gamma(2u_1^3-3u_1^2+u_1)}{1}\right]\right\}\\
&\left.+\left[\frac{\gamma^4}{\Delta^2}+\frac{\gamma^3(4u_1-2)}{\Delta}+\frac{\gamma^2(6u_1^2-6u_1+u_1)}{1}\right]\right)\\
&+\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&\cdot\left\{\frac{a^2\Delta^3}{\left[\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(\Delta C_yC_x)^2\right]^2}\cdot\left\{\left[C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(C_yC_x)^2\right]\right.\right.\\
&\left.\cdot\ 2\left[C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)-(C_y(y_3-y_2)+C_x(x_3-x_2))\right]\right\}\\
&-\left(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-2C_yC_x\right)\\
&\left.\cdot\left[(3C_x-C_y)C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)-2[C_y(y_3-y_2)+C_x(x_3-x_2)]\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial u_2^2}\int_{T_1} a^2u^2(1-u)^2dydx \qquad (3.61)\\
= \;& \left[\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-a^2\Delta[C_y(y_1-y_3)+C_x(x_1-x_3)]}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\right]\\
&\cdot\{[(y_m-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)]\\
&\cdot\left[\frac{\alpha^5}{5\Delta^3}+\frac{\alpha^4(2u_1-1)}{2\Delta^2}+\frac{\alpha^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\alpha^2(2u_1^3-3u_1^2+u_1)}{1}\right]\Bigg\}\\
&+\left[\frac{\alpha^6}{30\Delta^3}+\frac{\alpha^5(2u_1-1)}{10\Delta^2}+\frac{\alpha^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\alpha^3(2u_1^3-3u_1^2+u_1)}{3}\right]\\
&\cdot\Bigg[\frac{2a^2\Delta\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)^2-a^2\Delta[2(x_1-x_3)(y_1-y_3)]}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\\
&-\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-a^2\Delta(C_x(x_1-x_3)+C_y(y_1-y_3))}{(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^3}\\
&\cdot[-4\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-\Delta(C_y(y_1-y_3)+C_x(x_1-x_3))]\Bigg]\\
&+\frac{a^2[(y_m-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)]^2}{\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x}\\
&\cdot\left[\frac{\alpha^4}{\Delta^3}+\frac{2\alpha^3(2u_1-1)}{\Delta^2}+\frac{\alpha^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\alpha(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&+\left[\frac{\alpha^5}{5\Delta^3}+\frac{\alpha^4(2u_1-1)}{2\Delta^2}+\frac{\alpha^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\alpha^2(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&\cdot\Bigg\{\frac{-a^2[(y_m-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)]}{(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\\
&\cdot\left[2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-\Delta(C_y(y_1-y_3)+C_x(x_1-x_3))\right]\Bigg\}\\
&+\left[\frac{a^2\Delta[C_y(y_1-y_3)+C_x(x_1-x_3)]}{-(\Delta C_yC_x)^2}\right]\\
&\cdot\{[(y_\ell-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)]\\
&\cdot\left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+u_1)}{1}\right]\Bigg\}\\
&+\left[\frac{\beta^6}{30\Delta^3}+\frac{\beta^5(2u_1-1)}{10\Delta^2}+\frac{\beta^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\beta^3(2u_1^3-3u_1^2+u_1)}{3}\right]\\
&\cdot\left\{\frac{2a^2[C_y(y_1-y_3)+C_x(x_1-x_3)]^2-a^2C_yC_x[2(x_1-x_3)(y_1-y_3)]}{\Delta(C_yC_x)^3}\right\}\\
&+\left[\frac{a^2[(y_\ell-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)]^2}{\Delta C_yC_x}\right]\\
&\cdot\left[\frac{\beta^4}{\Delta^3}+\frac{2\beta^3(2u_1-1)}{\Delta^2}+\frac{\beta^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\beta(2u_1^3-3u_1^2+u_1)}{1}\right]
\end{aligned}
$$

$$+\left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+u_1)}{1}\right]$$

$$\cdot\frac{-a^2\Delta[(y_\ell-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)][C_y(y_1-y_3)+C_x(x_1-x_3)]}{(\Delta C_yC_x)^2}$$

$$+\left[\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-a^2\Delta(C_x(x_1-x_3)+C_y(y_1-y_3))}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\right.$$

$$\left.+\frac{a^2\Delta(C_x(x_1-x_3)+C_y(y_1-y_3))}{-(\Delta C_yC_x)^2}\right]$$

$$\cdot\{[(y_\ell-y_1)(x_1-x_3)-(x_\ell-x_1)(y_1-y_3)]$$

$$\left.\cdot\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right]\right\}$$

$$+\left[\frac{\gamma^6}{30\Delta^3}+\frac{\gamma^5(2u_1-1)}{10\Delta^2}+\frac{\gamma^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\gamma^3(2u_1^3-3u_1^2+u_1)}{3}\right]$$

$$\cdot\left[\left\{(2a^2)\left\{2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-[C_y(y_1-y_3)+C_x(x_1-x_3)]\right\}\right.\right.$$

$$\cdot\ (2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-[C_y(y_1-y_3)+C_x(x_1-x_3)])$$

$$\left.-(a^2)(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-C_yC_x)\left[2\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)^2-[(x_1-x_3)^2+(y_1-y_3)^2]\right]\right\}$$

$$\cdot\frac{1}{\Delta(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-C_yC_x)^3}$$

$$\left.+\left\{\frac{2a^2[C_y(y_1-y_3)+C_x(x_1-x_3)]^2-a^2(C_yC_x)[(x_1-x_3)^2+(y_1-y_3)^2]}{\Delta(C_yC_x)^3}\right\}\right]$$

$$+\frac{a^2\Delta[C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-2C_yC_x][(y_\ell-y_1)(x_1-x_3)-(x_\ell-x_1)(y_1-y_3)]^2}{\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(\Delta C_yC_x)^2}$$

$$\cdot\left[\frac{\gamma^4}{\Delta^3}+\frac{2\gamma^3(2u_1-1)}{\Delta^2}+\frac{\gamma^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\gamma(2u_1^3-3u_1^2+u_1)}{1}\right]$$

$$+\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right]$$

$$\cdot\left\{\frac{a^2\Delta^3}{\left[\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(\Delta C_yC_x)^2\right]^2}\cdot\left\{\left[C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(C_yC_x)^2\right]\right.\right.$$

$$\left.\cdot\ 2\left[C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-(C_y(y_1-y_3)+C_x(x_1-x_3))\right]\right\}$$

$$-\left(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-2C_yC_x\right)$$

$$\left.\cdot\left[(3C_x-C_y)C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-2[C_y(y_1-y_3)+C_x(x_1-x_3)]\right]\right\}$$

$$
\begin{aligned}
& \frac{\partial}{\partial u_2 \partial u_3} \int_{T_1} a^2 u^2 (1-u)^2 dy dx \qquad (3.62) \\
= \quad & \left[\frac{2a^2 \Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_1 - x_3) - a^2 \Delta [C_y(y_1 - y_3) + C_x(x_1 - x_3)]}{-(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2} \right] \\
& \cdot \{[(y_m - y_1)(x_2 - x_1) - (x_m - x_1)(y_2 - y_1)] \\
& \cdot \left[\frac{\alpha^5}{5\Delta^3} + \frac{\alpha^4(2u_1 - 1)}{2\Delta^2} + \frac{\alpha^3(6u_1^2 - 6u_1 + 1)}{3\Delta} + \frac{\alpha^2(2u_1^3 - 3u_1^2 + u_1)}{1} \right] \} \\
& + \left[\frac{\alpha^6}{30\Delta^3} + \frac{\alpha^5(2u_1 - 1)}{10\Delta^2} + \frac{\alpha^4(6u_1^2 - 6u_1 + 1)}{12\Delta} + \frac{\alpha^3(2u_1^3 - 3u_1^2 + u_1)}{3} \right] \\
& \cdot \left[\frac{2a^2 \Delta \frac{y_m - y_\ell}{x_m - x_\ell}(x_2 - x_1)(x_1 - x_3)}{-(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2} \right. \\
& - \frac{a^2 \Delta [(x_1 - x_3)(y_2 - y_1) + (y_1 - y_3)(x_2 - x_1)]}{-(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2} \\
& - \frac{2a^2 \Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_1 - x_3) - a^2 \Delta (C_x(x_1 - x_3) + C_y(y_1 - y_3))}{(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^3} \\
& \left. \cdot [-4\Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_2 - x_1) - \Delta(C_y(y_2 - y_1) + C_x(x_2 - x_1))] \right] \\
& + \frac{a^2[(y_m - y_1)(x_1 - x_3) - (x_m - x_1)(y_1 - y_3)]}{\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x} \\
& \cdot \{[(y_m - y_1)(x_2 - x_1) - (x_m - x_1)(y_2 - y_1)] \\
& \cdot \left[\frac{\alpha^4}{\Delta^3} + \frac{2\alpha^3(2u_1 - 1)}{\Delta^2} + \frac{\alpha^2(6u_1^2 - 6u_1 + 1)}{\Delta} + \frac{2\alpha(2u_1^3 - 3u_1^2 + u_1)}{1} \right] \} \\
& + \left[\frac{\alpha^5}{5\Delta^3} + \frac{\alpha^4(2u_1 - 1)}{2\Delta^2} + \frac{\alpha^3(6u_1^2 - 6u_1 + 1)}{3\Delta} + \frac{\alpha^2(2u_1^3 - 3u_1^2 + u_1)}{1} \right] \\
& \cdot \left\{ \frac{-a^2[(y_m - y_1)(x_1 - x_3) - (x_m - x_1)(y_1 - y_3)]}{(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2} \right. \\
& \left. \cdot \left[2\Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_2 - x_1) - \Delta(C_y(y_2 - y_1) + C_x(x_2 - x_1)) \right] \right\} \\
& + \left[\frac{a^2 \Delta [C_y(y_1 - y_3) + C_x(x_1 - x_3)]}{-(\Delta C_y C_x)^2} \right] \\
& \cdot \{[(y_\ell - y_1)(x_2 - x_1) - (x_m - x_1)(y_2 - y_1)] \\
& \cdot \left[\frac{\beta^5}{5\Delta^3} + \frac{\beta^4(2u_1 - 1)}{2\Delta^2} + \frac{\beta^3(6u_1^2 - 6u_1 + 1)}{3\Delta} + \frac{\beta^2(2u_1^3 - 3u_1^2 + u_1)}{1} \right] \}
\end{aligned}
$$

$$+\left[\frac{\beta^6}{30\Delta^3}+\frac{\beta^5(2u_1-1)}{10\Delta^2}+\frac{\beta^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\beta^3(2u_1^3-3u_1^2+u_1)}{3}\right]$$

$$\cdot\left\{\frac{2a^2[C_y(y_2-y_1)+C_x(x_2-x_1)][C_y(y_1-y_3)+C_x(x_1-x_3)]}{\Delta(C_yC_x)^3}\right.$$

$$\left.-\ \frac{a^2C_yC_x[(x_2-x_1)(y_1-y_3)+(y_2-y_1)(x_1-x_3)]}{\Delta(C_yC_x)^3}\right\}$$

$$+\left[\frac{a^2[(y_\ell-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)]}{\Delta C_yC_x}\right]$$

$$\cdot\{[(y_\ell-y_1)(x_2-x_1)-(x_m-x_1)(y_2-y_1)]$$

$$\left.\cdot\left[\frac{\beta^4}{\Delta^3}+\frac{2\beta^3(2u_1-1)}{\Delta^2}+\frac{\beta^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\beta(2u_1^3-3u_1^2+u_1)}{1}\right]\right\}$$

$$+\left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+u_1)}{1}\right]$$

$$\cdot\frac{-a^2\Delta[(y_\ell-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)][C_y(y_2-y_1)+C_x(x_2-x_1)]}{(\Delta C_yC_x)^2}$$

$$+\left[\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-a^2\Delta(C_x(x_1-x_3)+C_y(y_1-y_3))}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\right.$$

$$\left.+\frac{a^2\Delta(C_x(x_1-x_3)+C_y(y_1-y_3))}{-(\Delta C_yC_x)^2}\right]$$

$$\cdot\{[(y_\ell-y_1)(x_2-x_1)-(x_\ell-x_1)(y_2-y_1)]$$

$$\left.\cdot\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right]\right\}$$

$$+\left[\frac{\gamma^6}{30\Delta^3}+\frac{\gamma^5(2u_1-1)}{10\Delta^2}+\frac{\gamma^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\gamma^3(2u_1^3-3u_1^2+u_1)}{3}\right]$$

$$\cdot\left[\left\{(2a^2)\left\{2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-[C_y(y_1-y_3)+C_x(x_1-x_3)]\right\}\right.\right.$$

$$\cdot\ (2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-[C_y(y_2-y_1)+C_x(x_2-x_1)])$$

$$-(a^2)(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-C_yC_x)\left[2\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)(x_2-x_1)\right.$$

$$-[(x_1-x_3)(x_2-x_1)+(y_1-y_3)(y_2-y_1)]]\}$$

$$\cdot\frac{1}{\Delta(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-C_yC_x)^3}$$

$$+\left\{2a^2[C_y(y_1-y_3)+C_x(x_1-x_3)][C_y(y_2-y_1)+C_x(x_2-x_1)]\right.$$

$$\left.-\ a^2(C_yC_x)[(x_1-x_3)(x_2-x_1)+(y_1-y_3)(y_2-y_1)]\right\}$$

$$\left.\cdot\frac{1}{\Delta(C_yC_x)^3}\right]$$

$$
\begin{aligned}
&+\frac{a^2\Delta[C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-2C_yC_x][(y_\ell-y_1)(x_1-x_3)-(x_\ell-x_1)(y_1-y_3)]}{\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(\Delta C_yC_x)^2} \\
&\cdot\{[(y_\ell-y_1)(x_2-x_1)-(x_\ell-x_1)(y_2-y_1)] \\
&\left.\cdot\left[\frac{\gamma^4}{\Delta^3}+\frac{2\gamma^3(2u_1-1)}{\Delta^2}+\frac{\gamma^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\gamma(2u_1^3-3u_1^2+u_1)}{1}\right]\right\} \\
&+\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right] \\
&\cdot\left\{\frac{a^2\Delta^3}{\left[\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(\Delta C_yC_x)^2\right]^2}\cdot\left\{\left[C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(C_yC_x)^2\right]\right.\right. \\
&\left.\cdot\ 2\left[C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-(C_y(y_2-y_1)+C_x(x_2-x_1))\right]\right\} \\
&-\left(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-2C_yC_x\right) \\
&\left.\cdot\left[(3C_x-C_y)C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-2[C_y(y_2-y_1)+C_x(x_2-x_1)]\right]\right\}
\end{aligned}
$$

$$\frac{\partial}{\partial u_3 \partial u_1} \int_{T_1} a^2 u^2 (1-u)^2 dydx \tag{3.63}$$

$$= \left[\frac{2a^2 \Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_2 - x_1) - a^2 \Delta [C_y(y_2 - y_1) + C_x(x_2 - x_1)]}{-(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2} \right]$$

$$\cdot (\{[(y_m - y_1)(x_3 - x_2) - (x_m - x_1)(y_3 - y_2)]$$

$$\cdot \left[\frac{\alpha^5}{5\Delta^3} + \frac{\alpha^4(2u_1 - 1)}{2\Delta^2} + \frac{\alpha^3(6u_1^2 - 6u_1 + 1)}{3\Delta} + \frac{\alpha^2(2u_1^3 - 3u_1^2 + u_1)}{1} \right] \}$$

$$+ \left[\frac{\alpha^5}{5\Delta^2} + \frac{\alpha^4(2u_1 - 1)}{2\Delta} + \frac{\alpha^3(6u_1^2 - 6u_1 + 1)}{3} \right])$$

$$+ \left[\frac{\alpha^6}{30\Delta^3} + \frac{\alpha^5(2u_1 - 1)}{10\Delta^2} + \frac{\alpha^4(6u_1^2 - 6u_1 + 1)}{12\Delta} + \frac{\alpha^3(2u_1^3 - 3u_1^2 + u_1)}{3} \right]$$

$$\cdot \left[\frac{2a^2 \Delta \frac{y_m - y_\ell}{x_m - x_\ell}(x_3 - x_2)(x_2 - x_1)}{-(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2} \right.$$

$$- \frac{a^2 \Delta [(x_2 - x_1)(y_3 - y_2) + (y_2 - y_1)(x_3 - x_2)]}{-(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2}$$

$$- \frac{2a^2 \Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_2 - x_1) - a^2 \Delta (C_x(x_2 - x_1) + C_y(y_2 - y_1))}{(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^3}$$

$$\left. \cdot [-4\Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_3 - x_2) - \Delta(C_y(y_3 - y_2) + C_x(x_3 - x_2))] \right]$$

$$+ \frac{a^2[(y_m - y_1)(x_2 - x_1) - (x_m - x_1)(y_2 - y_1)]}{\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x}$$

$$\cdot (\{[(y_m - y_1)(x_3 - x_2) - (x_m - x_1)(y_3 - y_2)]$$

$$\cdot \left[\frac{\alpha^4}{\Delta^3} + \frac{2\alpha^3(2u_1 - 1)}{\Delta^2} + \frac{\alpha^2(6u_1^2 - 6u_1 + 1)}{\Delta} + \frac{2\alpha(2u_1^3 - 3u_1^2 + u_1)}{1} \right] \}$$

$$+ \left[\frac{\alpha^4}{\Delta^2} + \frac{\alpha^3(4u_1 - 2)}{\Delta} + \frac{\alpha^2(6u_1^2 - 6u_1 + 1)}{1} \right])$$

$$+ \left[\frac{\alpha^5}{5\Delta^3} + \frac{\alpha^4(2u_1 - 1)}{2\Delta^2} + \frac{\alpha^3(6u_1^2 - 6u_1 + 1)}{3\Delta} + \frac{\alpha^2(2u_1^3 - 3u_1^2 + u_1)}{1} \right]$$

$$\cdot \left\{ \frac{-a^2[(y_m - y_1)(x_2 - x_1) - (x_m - x_1)(y_2 - y_1)]}{(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2} \right.$$

$$\left. \cdot \left[2\Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_3 - x_2) - \Delta(C_y(y_3 - y_2) + C_x(x_3 - x_2)) \right] \right\}$$

$$
\begin{aligned}
&+\left[\frac{a^2\Delta[C_y(y_2-y_1)+C_x(x_2-x_1)]}{-(\Delta C_yC_x)^2}\right]\\
&\cdot(\{[(y_\ell-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]\\
&\cdot\left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+u_1)}{1}\right]\Bigg\}\\
&+\left[\frac{\beta^5}{5\Delta^2}+\frac{\beta^4(2u_1-1)}{2\Delta}+\frac{\beta^3(6u_1^2-6u_1+1)}{3}\right]\Bigg)\\
&+\left[\frac{\beta^6}{30\Delta^3}+\frac{\beta^5(2u_1-1)}{10\Delta^2}+\frac{\beta^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\beta^3(2u_1^3-3u_1^2+u_1)}{3}\right]\\
&\cdot\Bigg\{\frac{2a^2[C_y(y_3-y_2)+C_x(x_3-x_2)][C_y(y_2-y_1)+C_x(x_2-x_1)]}{\Delta(C_yC_x)^3}\\
&-\frac{a^2C_yC_x[(x_3-x_2)(y_2-y_1)+(y_3-y_2)(x_2-x_1)]}{\Delta(C_yC_x)^3}\Bigg\}\\
&+\left[\frac{a^2[(y_\ell-y_1)(x_2-x_1)-(x_m-x_1)(y_2-y_1)]}{\Delta C_yC_x}\right]\\
&\cdot(\{[(y_\ell-y_1)(x_3-x_2)-(x_m-x_1)(y_3-y_2)]\\
&\cdot\left[\frac{\beta^4}{\Delta^3}+\frac{2\beta^3(2u_1-1)}{\Delta^2}+\frac{\beta^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\beta(2u_1^3-3u_1^2+u_1)}{1}\right]\Bigg\}\\
&+\left[\frac{\beta^4}{\Delta^2}+\frac{\beta^3(4u_1-2)}{\Delta}+\frac{\beta^2(6u_1^2-6u_1+1)}{1}\right]\Bigg)\\
&+\left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&\cdot\frac{-a^2\Delta[(y_\ell-y_1)(x_2-x_1)-(x_m-x_1)(y_2-y_1)][C_y(y_3-y_2)+C_x(x_3-x_2)]}{(\Delta C_yC_x)^2}\\
&+\left[\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-a^2\Delta(C_x(x_2-x_1)+C_y(y_2-y_1))}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\right.\\
&\left.+\frac{a^2\Delta(C_x(x_2-x_1)+C_y(y_2-y_1))}{-(\Delta C_yC_x)^2}\right]\\
&\cdot(\{[(y_\ell-y_1)(x_3-x_2)-(x_\ell-x_1)(y_3-y_2)]\\
&\cdot\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right]\Bigg\}\\
&+\left[\frac{\gamma^5}{5\Delta^2}+\frac{\gamma^4(2u_1-1)}{2\Delta}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3}\right]\Bigg)
\end{aligned}
$$

$$+\left[\frac{\gamma^6}{30\Delta^3}+\frac{\gamma^5(2u_1-1)}{10\Delta^2}+\frac{\gamma^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\gamma^3(2u_1^3-3u_1^2+u_1)}{3}\right]$$

$$\cdot\left[\left\{(2a^2)\left\{2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-[C_y(y_2-y_1)+C_x(x_2-x_1)]\right\}\right.\right.$$

$$\cdot\ (2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)-[C_y(y_3-y_2)+C_x(x_3-x_2)])$$

$$-(a^2)(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-C_yC_x)\left[2\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)(x_3-x_2)\right.$$

$$-[(x_2-x_1)(x_3-x_2)+(y_2-y_1)(y_3-y_2)]]\}$$

$$\cdot\frac{1}{\Delta(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-C_yC_x)^3}$$

$$+\left\{2a^2[C_y(y_2-y_1)+C_x(x_2-x_1)][C_y(y_3-y_2)+C_x(x_3-x_2)]\right.$$

$$\left.-\ a^2(C_yC_x)[(x_2-x_1)(x_3-x_2)+(y_2-y_1)(y_3-y_2)]\right\}$$

$$\left.\cdot\frac{1}{\Delta(C_yC_x)^3}\right]$$

$$+\frac{a^2\Delta[C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-2C_yC_x][(y_\ell-y_1)(x_2-x_1)-(x_\ell-x_1)(y_2-y_1)]}{\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(\Delta C_yC_x)^2}$$

$$\cdot(\{[(y_\ell-y_1)(x_3-x_2)-(x_\ell-x_1)(y_3-y_2)]$$

$$\left.\cdot\left[\frac{\gamma^4}{\Delta^3}+\frac{2\gamma^3(2u_1-1)}{\Delta^2}+\frac{\gamma^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\gamma(2u_1^3-3u_1^2+u_1)}{1}\right]\right\}$$

$$\left.+\left[\frac{\gamma^4}{\Delta^2}+\frac{\gamma^3(4u_1-2)}{\Delta}+\frac{\gamma^2(6u_1^2-6u_1+u_1)}{1}\right]\right)$$

$$+\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right]$$

$$\cdot\left\{\frac{a^2\Delta^3}{\left[\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(\Delta C_yC_x)^2\right]^2}\cdot\left\{\left[C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(C_yC_x)^2\right]\right.\right.$$

$$\left.\cdot\ 2\left[C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)-(C_y(y_3-y_2)+C_x(x_3-x_2))\right]\right\}$$

$$-\left(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-2C_yC_x\right)$$

$$\left.\cdot\left[(3C_x-C_y)C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}(x_3-x_2)-2[C_y(y_3-y_2)+C_x(x_3-x_2)]\right]\right\}$$

$$
\begin{aligned}
&\frac{\partial}{\partial u_3 \partial u_2} \int_{T_1} a^2 u^2 (1-u)^2 dy dx \qquad (3.64)\\
= \quad & \left[\frac{2a^2 \Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_2 - x_1) - a^2 \Delta [C_y (y_2 - y_1) + C_x (x_2 - x_1)]}{-(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2} \right]\\
& \cdot \{[(y_m - y_1)(x_1 - x_3) - (x_m - x_1)(y_1 - y_3)]\\
& \cdot \left[\frac{\alpha^5}{5\Delta^3} + \frac{\alpha^4 (2u_1 - 1)}{2\Delta^2} + \frac{\alpha^3 (6u_1^2 - 6u_1 + 1)}{3\Delta} + \frac{\alpha^2 (2u_1^3 - 3u_1^2 + u_1)}{1} \right] \Bigg\}\\
& + \left[\frac{\alpha^6}{30\Delta^3} + \frac{\alpha^5 (2u_1 - 1)}{10\Delta^2} + \frac{\alpha^4 (6u_1^2 - 6u_1 + 1)}{12\Delta} + \frac{\alpha^3 (2u_1^3 - 3u_1^2 + u_1)}{3} \right]\\
& \cdot \left[\frac{2a^2 \Delta \frac{y_m - y_\ell}{x_m - x_\ell}(x_1 - x_3)(x_2 - x_1)}{-(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2} \right.\\
& - \frac{a^2 \Delta [(x_2 - x_1)(y_1 - y_3) + (y_2 - y_1)(x_1 - x_3)]}{-(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2}\\
& - \frac{2a^2 \Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_2 - x_1) - a^2 \Delta (C_x (x_2 - x_1) + C_y (y_2 - y_1))}{(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^3}\\
& \left. \cdot [-4\Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_1 - x_3) - \Delta (C_y (y_1 - y_3) + C_x (x_1 - x_3))] \right]\\
& + \frac{a^2 [(y_m - y_1)(x_2 - x_1) - (x_m - x_1)(y_2 - y_1)]}{\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x}\\
& \cdot \{[(y_m - y_1)(x_1 - x_3) - (x_m - x_1)(y_1 - y_3)]\\
& \cdot \left[\frac{\alpha^4}{\Delta^3} + \frac{2\alpha^3 (2u_1 - 1)}{\Delta^2} + \frac{\alpha^2 (6u_1^2 - 6u_1 + 1)}{\Delta} + \frac{2\alpha (2u_1^3 - 3u_1^2 + u_1)}{1} \right] \Bigg\}\\
& + \left[\frac{\alpha^5}{5\Delta^3} + \frac{\alpha^4 (2u_1 - 1)}{2\Delta^2} + \frac{\alpha^3 (6u_1^2 - 6u_1 + 1)}{3\Delta} + \frac{\alpha^2 (2u_1^3 - 3u_1^2 + u_1)}{1} \right]\\
& \cdot \left\{ \frac{-a^2 [(y_m - y_1)(x_2 - x_1) - (x_m - x_1)(y_2 - y_1)]}{(\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - \Delta C_y C_x)^2} \right.\\
& \left. \cdot \left[2\Delta C_y \frac{y_m - y_\ell}{x_m - x_\ell}(x_1 - x_3) - \Delta (C_y (y_1 - y_3) + C_x (x_1 - x_3)) \right] \right\}\\
& + \left[\frac{a^2 \Delta [C_y (y_2 - y_1) + C_x (x_2 - x_1)]}{-(\Delta C_y C_x)^2} \right]\\
& \cdot \{[(y_\ell - y_1)(x_1 - x_3) - (x_m - x_1)(y_1 - y_3)]\\
& \cdot \left[\frac{\beta^5}{5\Delta^3} + \frac{\beta^4 (2u_1 - 1)}{2\Delta^2} + \frac{\beta^3 (6u_1^2 - 6u_1 + 1)}{3\Delta} + \frac{\beta^2 (2u_1^3 - 3u_1^2 + u_1)}{1} \right] \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&+\left[\frac{\beta^6}{30\Delta^3}+\frac{\beta^5(2u_1-1)}{10\Delta^2}+\frac{\beta^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\beta^3(2u_1^3-3u_1^2+u_1)}{3}\right]\\
&\cdot\left\{\frac{2a^2[C_y(y_1-y_3)+C_x(x_1-x_3)][C_y(y_2-y_1)+C_x(x_2-x_1)]}{\Delta(C_yC_x)^3}\right.\\
&\left.-\ \frac{a^2C_yC_x[(x_1-x_3)(y_2-y_1)+(y_1-y_3)(x_2-x_1)]}{\Delta(C_yC_x)^3}\right\}\\
&+\left[\frac{a^2[(y_\ell-y_1)(x_2-x_1)-(x_m-x_1)(y_2-y_1)]}{\Delta C_yC_x}\right]\\
&\cdot\{[(y_\ell-y_1)(x_1-x_3)-(x_m-x_1)(y_1-y_3)]\\
&\cdot\left.\left[\frac{\beta^4}{\Delta^3}+\frac{2\beta^3(2u_1-1)}{\Delta^2}+\frac{\beta^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\beta(2u_1^3-3u_1^2+u_1)}{1}\right]\right\}\\
&+\left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&\cdot\frac{-a^2\Delta[(y_\ell-y_1)(x_2-x_1)-(x_m-x_1)(y_2-y_1)][C_y(y_1-y_3)+C_x(x_1-x_3)]}{(\Delta C_yC_x)^2}\\
&+\left[\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-a^2\Delta(C_x(x_2-x_1)+C_y(y_2-y_1))}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\right.\\
&\left.+\frac{a^2\Delta(C_x(x_2-x_1)+C_y(y_2-y_1))}{-(\Delta C_yC_x)^2}\right]\\
&\cdot\{[(y_\ell-y_1)(x_1-x_3)-(x_\ell-x_1)(y_1-y_3)]\\
&\cdot\left.\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right]\right\}\\
&+\left[\frac{\gamma^6}{30\Delta^3}+\frac{\gamma^5(2u_1-1)}{10\Delta^2}+\frac{\gamma^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\gamma^3(2u_1^3-3u_1^2+u_1)}{3}\right]\\
&\cdot\left[\left\{(2a^2)\left\{2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-[C_y(y_2-y_1)+C_x(x_2-x_1)]\right\}\right.\right.\\
&\cdot\ (2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-[C_y(y_1-y_3)+C_x(x_1-x_3)])\\
&-(a^2)(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-C_yC_x)\left[2\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)(x_1-x_3)\right.\\
&-[(x_1-x_3)(x_2-x_1)+(y_1-y_3)(y_2-y_1)]]\}\\
&\cdot\frac{1}{\Delta(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-C_yC_x)^3}\\
&+\left\{2a^2[C_y(y_2-y_1)+C_x(x_2-x_1)][C_y(y_1-y_3)+C_x(x_1-x_3)]\right.\\
&\left.-\ a^2(C_yC_x)[(x_1-x_3)(x_2-x_1)+(y_1-y_3)(y_2-y_1)]\right\}\\
&\left.\cdot\frac{1}{\Delta(C_yC_x)^3}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{a^2\Delta[C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-2C_yC_x][(y_\ell-y_1)(x_2-x_1)-(x_\ell-x_1)(y_2-y_1)]}{\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(\Delta C_yC_x)^2} \\
&\cdot\{[(y_\ell-y_1)(x_1-x_3)-(x_\ell-x_1)(y_1-y_3)] \\
&\left.\cdot\left[\frac{\gamma^4}{\Delta^3}+\frac{2\gamma^3(2u_1-1)}{\Delta^2}+\frac{\gamma^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\gamma(2u_1^3-3u_1^2+u_1)}{1}\right]\right\} \\
&+\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right] \\
&\cdot\left\{\frac{a^2\Delta^3}{\left[\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(\Delta C_yC_x)^2\right]^2}\cdot\left\{\left[C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(C_yC_x)^2\right]\right.\right. \\
&\left.\cdot\ 2\left[C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-(C_y(y_1-y_3)+C_x(x_1-x_3))\right]\right\} \\
&-\left(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-2C_yC_x\right) \\
&\left.\cdot\left[(3C_x-C_y)C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}(x_1-x_3)-2[C_y(y_1-y_3)+C_x(x_1-x_3)]\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial u_3^2}\int_{T_1} a^2u^2(1-u)^2dydx \qquad (3.65)\\
= \quad &\left[\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-a^2\Delta[C_y(y_2-y_1)+C_x(x_2-x_1)]}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\right]\\
&\cdot\{[(y_m-y_1)(x_2-x_1)-(x_m-x_1)(y_2-y_1)]\\
&\cdot\left[\frac{\alpha^5}{5\Delta^3}+\frac{\alpha^4(2u_1-1)}{2\Delta^2}+\frac{\alpha^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\alpha^2(2u_1^3-3u_1^2+u_1)}{1}\right]\Bigg\}\\
&+\left[\frac{\alpha^6}{30\Delta^3}+\frac{\alpha^5(2u_1-1)}{10\Delta^2}+\frac{\alpha^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\alpha^3(2u_1^3-3u_1^2+u_1)}{3}\right]\\
&\cdot\left[\frac{2a^2\Delta\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)^2-a^2\Delta[2(x_2-x_1)(y_2-y_1)]}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\right.\\
&-\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-a^2\Delta(C_x(x_2-x_1)+C_y(y_2-y_1))}{(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^3}\\
&\left.\cdot[-4\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-\Delta(C_y(y_2-y_1)+C_x(x_2-x_1))]\right]\\
&+\frac{a^2[(y_m-y_1)(x_2-x_1)-(x_m-x_1)(y_2-y_1)]^2}{\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x}\\
&\cdot\left[\frac{\alpha^4}{\Delta^3}+\frac{2\alpha^3(2u_1-1)}{\Delta^2}+\frac{\alpha^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\alpha(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&+\left[\frac{\alpha^5}{5\Delta^3}+\frac{\alpha^4(2u_1-1)}{2\Delta^2}+\frac{\alpha^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\alpha^2(2u_1^3-3u_1^2+u_1)}{1}\right]\\
&\cdot\left\{\frac{-a^2[(y_m-y_1)(x_2-x_1)-(x_m-x_1)(y_2-y_1)]}{(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\right.\\
&\left.\cdot\left[2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-\Delta(C_y(y_2-y_1)+C_x(x_2-x_1))\right]\right\}\\
&+\left[\frac{a^2\Delta[C_y(y_2-y_1)+C_x(x_2-x_1)]}{-(\Delta C_yC_x)^2}\right]\\
&\cdot\{[(y_\ell-y_1)(x_2-x_1)-(x_m-x_1)(y_2-y_1)]\\
&\cdot\left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+u_1)}{1}\right]\Bigg\}\\
&+\left[\frac{\beta^6}{30\Delta^3}+\frac{\beta^5(2u_1-1)}{10\Delta^2}+\frac{\beta^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\beta^3(2u_1^3-3u_1^2+u_1)}{3}\right]\\
&\cdot\left\{\frac{2a^2[C_y(y_2-y_1)+C_x(x_2-x_1)]^2-a^2C_yC_x[2(x_2-x_1)(y_2-y_1)]}{\Delta(C_yC_x)^3}\right\}\\
&+\left[\frac{a^2[(y_\ell-y_1)(x_2-x_1)-(x_m-x_1)(y_2-y_1)]^2}{\Delta C_yC_x}\right]\\
&\cdot\left[\frac{\beta^4}{\Delta^3}+\frac{2\beta^3(2u_1-1)}{\Delta^2}+\frac{\beta^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\beta(2u_1^3-3u_1^2+u_1)}{1}\right]
\end{aligned}
$$

$$
+\left[\frac{\beta^5}{5\Delta^3}+\frac{\beta^4(2u_1-1)}{2\Delta^2}+\frac{\beta^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\beta^2(2u_1^3-3u_1^2+u_1)}{1}\right]
$$

$$
\cdot\frac{-a^2\Delta[(y_\ell-y_1)(x_2-x_1)-(x_m-x_1)(y_2-y_1)][C_y(y_2-y_1)+C_x(x_2-x_1)]}{(\Delta C_yC_x)^2}
$$

$$
+\left[\frac{2a^2\Delta C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-a^2\Delta(C_x(x_2-x_1)+C_y(y_2-y_1))}{-(\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-\Delta C_yC_x)^2}\right.
$$

$$
\left.+\frac{a^2\Delta(C_x(x_2-x_1)+C_y(y_2-y_1))}{-(\Delta C_yC_x)^2}\right]
$$

$$
\cdot\{[(y_\ell-y_1)(x_2-x_1)-(x_\ell-x_1)(y_2-y_1)]
$$

$$
\left.\cdot\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right]\right\}
$$

$$
+\left[\frac{\gamma^6}{30\Delta^3}+\frac{\gamma^5(2u_1-1)}{10\Delta^2}+\frac{\gamma^4(6u_1^2-6u_1+1)}{12\Delta}+\frac{\gamma^3(2u_1^3-3u_1^2+u_1)}{3}\right]
$$

$$
\cdot\left[\left\{(2a^2)\left\{2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-[C_y(y_2-y_1)+C_x(x_2-x_1)]\right\}\right.\right.
$$

$$
\cdot\left(2C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-[C_y(y_2-y_1)+C_x(x_2-x_1)]\right)
$$

$$
-(a^2)(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-C_yC_x)\left[2\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)^2\right.
$$

$$
\left.\left.\left.-[(x_2-x_1)^2+(y_2-y_1)^2]\right]\right\}
$$

$$
\cdot\frac{1}{\Delta(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-C_yC_x)^3}
$$

$$
+\left\{\frac{2a^2[C_y(y_2-y_1)+C_x(x_2-x_1)]^2-a^2(C_yC_x)[(x_2-x_1)^2+(y_2-y_1)^2]}{\Delta(C_yC_x)^3}\right\}
$$

$$
+\frac{a^2\Delta[C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-2C_yC_x][(y_\ell-y_1)(x_2-x_1)-(x_\ell-x_1)(y_2-y_1)]^2}{\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(\Delta C_yC_x)^2}
$$

$$
\cdot\left[\frac{\gamma^4}{\Delta^3}+\frac{2\gamma^3(2u_1-1)}{\Delta^2}+\frac{\gamma^2(6u_1^2-6u_1+1)}{\Delta}+\frac{2\gamma(2u_1^3-3u_1^2+u_1)}{1}\right]
$$

$$
+\left[\frac{\gamma^5}{5\Delta^3}+\frac{\gamma^4(2u_1-1)}{2\Delta^2}+\frac{\gamma^3(6u_1^2-6u_1+1)}{3\Delta}+\frac{\gamma^2(2u_1^3-3u_1^2+u_1)}{1}\right]
$$

$$
\cdot\left\{\frac{a^2\Delta^3}{\left[\Delta^2C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(\Delta C_yC_x)^2\right]^2}\cdot\left\{\left[C_y^3C_x\frac{y_m-y_\ell}{x_m-x_\ell}-(C_yC_x)^2\right]\right.\right.
$$

$$
\left.\cdot\,2\left[C_y\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-(C_y(y_2-y_1)+C_x(x_2-x_1))\right]\right\}
$$

$$
-\left(C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-2C_yC_x\right)
$$

$$
\left.\cdot\left[(3C_x-C_y)C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}(x_2-x_1)-2[C_y(y_2-y_1)+C_x(x_2-x_1)]\right]\right\}
$$

Gravity Term

We must now consider the gravity term of our energy distribution: $\int_\Omega(\rho g u y)\,dydx$, where ρ is our liquid density, g is the gravitational constant, and the variable y models the height within our container Ω. Looking at our left triangle, T_1, we break up our term into parts:

$$
\begin{aligned}
\int_{T_1}(\rho g u y)\,dydx &= \int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)}(\rho g u y)\,dydx \\
&= \rho g\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)}(yu)\,dydx \\
&= \rho g\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)} y[\frac{C_y(y-y_1)-C_x(x-x_1)}{\Delta}+u_1]dydx \\
&= \rho g\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)}\frac{yC_y(y-y_1)}{\Delta}dydx \\
&\quad -\rho g\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)}\frac{yC_x(x-x_1)}{\Delta}dydx \\
&\quad +\rho g\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)}(u_1 y)dydx.
\end{aligned}
\tag{3.66}
$$

Recall that $L_1(x) = \frac{y_m-y_\ell}{x_m-x_\ell}(x-x_\ell)+y_\ell$.

Integrating the first of these three terms gives us

$$
\begin{aligned}
&\rho g\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)}\frac{yC_y(y-y_1)}{\Delta}dydx \\
=\ &\frac{\rho g C_y}{\Delta}\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)}(y^2-y_1y)\,dydx \\
=\ &\frac{(\rho g C_y)}{3\Delta}\int_{x_\ell}^{x_m}\left[\frac{y_m-y_\ell}{x_m-x_\ell}(x-x_\ell)+y_\ell\right]^3-y_\ell^3\,dx \\
&-\frac{(\rho g C_y)y_1}{2\Delta}\int_{x_\ell}^{x_m}\left[\frac{y_m-y_\ell}{x_m-x_\ell}(x-x_\ell)+y_\ell\right]^2-y_\ell^2\,dx \\
=\ &\frac{(\rho g C_y)}{\Delta}\cdot\left[\frac{(y_m^4-y_\ell^4)-2(y_m^3-y_\ell^3)}{12\frac{y_m-y_\ell}{x_m-x_\ell}}-\frac{y_\ell^3x_m-y_\ell^3x_\ell}{3}+\frac{y_1(y_\ell^2x_m-y_\ell^2x_\ell}{2}\right].
\end{aligned}
\tag{3.67}
$$

Integrating the second term gives us

$$\begin{aligned}
&\rho g \int_{x_\ell}^{x_m} \int_{y_\ell}^{L_1(x)} \frac{yC_x(x-x_1)}{\Delta} dydx \qquad (3.68)\\
=\ & \frac{\rho g C_x}{\Delta} \int_{x_\ell}^{x_m} \int_{y_\ell}^{L_1(x)} y(x-x_1)dydx \\
=\ & \frac{\rho g C_x}{2\Delta} \int_{x_\ell}^{x_m} (x-x_1)\left[\frac{(y_m-y_\ell)^2}{(x_m-x_\ell)^2}(x-x_\ell)^2 + 2y_\ell \frac{y_m-y_\ell}{x_m-x_\ell}(x-x_\ell)\right] dx \\
=\ & \frac{\rho g C_x}{\Delta} \cdot \left[(x_m^4 - x_\ell^4)\left[\frac{(y_m-y_\ell)^2}{8(x_m-x_\ell)^2}\right]\right. \\
& +(x_m^3 - x_\ell^3)\left[\frac{y_\ell(y_m-y_\ell)}{3(x_m-x_\ell)} - \frac{(-x_1-2x_\ell)(y_m-y_\ell)^2}{6(x_m-x_\ell)^2}\right] \\
& +(x_m^2 - x_\ell^2)\left[\frac{(x_\ell^2+2x_1x_\ell)(y_m-y_\ell)^2}{4(x_m-x_\ell)^2} + \frac{y_\ell(-x_1-x_\ell)(y_m-y_\ell)}{2(x_m-x_\ell)}\right] \\
& \left. +(x_m - x_\ell)\left[\frac{y_\ell x_\ell x_1(y_m-y_\ell)}{(x_m-x_\ell)} - \frac{x_\ell^2 x_1(y_m-y_\ell)^2}{2(x_m-x_\ell)^2}\right]\right].
\end{aligned}$$

Finally, integrating the third term gives us

$$\begin{aligned}
&\rho g \int_{x_\ell}^{x_m} \int_{y_\ell}^{L_1(x)} (u_1 y) dydx \qquad (3.69)\\
=\ & \rho g u_1 \int_{x_\ell}^{x_m} \int_{y_\ell}^{L_1(x)} (y)dydx \\
=\ & \frac{(\rho g u_1)}{2} \int_{x_\ell}^{x_m} \left[\frac{(y_m-y_\ell)^2}{(x_m-x_\ell)^2}(x-x_\ell)^2 + 2y_\ell \frac{y_m-y_\ell}{x_m-x_\ell}(x-x_\ell)\right] dx \\
=\ & (\rho g u_1) \cdot \left[\frac{(y_m-y_\ell)^2}{2(x_m-x_\ell)^2}\left[\frac{1}{3}(x_m^3 - x_\ell^3) + x_\ell^2(x_m-x_\ell) - x_\ell(x_m^2 - x_\ell^2)\right]\right. \\
& \left. + \frac{y_\ell(y_m-y_\ell)}{(x_m-x_\ell)}\left[\frac{1}{2}(x_m^2 - x_\ell^2) - x_\ell(x_m-x_\ell)\right]\right].
\end{aligned}$$

We get that our final gravity term for the triangle T_1

$$
\begin{aligned}
&\int_{T_1} (\rho g u y)\, dydx \qquad (3.70)\\
= \quad & \int_{x_\ell}^{x_m} \int_{y_\ell}^{L_1(x)} (\rho g u y) dydx \\
= \quad & \frac{(\rho g C_y)}{\Delta} \cdot \left[\frac{(y_m^4 - y_\ell^4) - 2(y_m^3 - y_\ell^3)}{12 \frac{y_m - y_\ell}{x_m - x_\ell}} - \frac{y_\ell^3 x_m - y_\ell^3 x_\ell}{3} + \frac{y_1(y_\ell^2 x_m - y_\ell^2 x_\ell)}{2} \right] \\
& - \frac{\rho g C_x}{\Delta} \cdot \left[(x_m^4 - x_\ell^4) \left[\frac{(y_m - y_\ell)^2}{8(x_m - x_\ell)^2} \right] \right. \\
& + (x_m^3 - x_\ell^3) \left[\frac{y_\ell (y_m - y_\ell)}{3(x_m - x_\ell)} - \frac{(-x_1 - 2x_\ell)(y_m - y_\ell)^2}{6(x_m - x_\ell)^2} \right] \\
& + (x_m^2 - x_\ell^2) \left[\frac{(x_\ell^2 + 2x_1 x_\ell)(y_m - y_\ell)^2}{4(x_m - x_\ell)^2} + \frac{y_\ell(-x_1 - x_\ell)(y_m - y_\ell)}{2(x_m - x_\ell)} \right] \\
& \left. + (x_m - x_\ell) \left[\frac{y_\ell x_\ell x_1 (y_m - y_\ell)}{(x_m - x_\ell)} - \frac{x_\ell^2 x_1 (y_m - y_\ell)^2}{2(x_m - x_\ell)^2} \right] \right] \\
& + (\rho g u_1) \cdot \left\{ \frac{(y_m - y_\ell)^2}{2(x_m - x_\ell)^2} \left[\frac{1}{3}(x_m^3 - x_\ell^3) + x_\ell^2 (x_m - x_\ell) - x_\ell (x_m^2 - x_\ell^2) \right] \right. \\
& \left. + \frac{y_\ell (y_m - y_\ell)}{(x_m - x_\ell)} \left[\frac{1}{2}(x_m^2 - x_\ell^2) - x_\ell (x_m - x_\ell) \right] \right\}.
\end{aligned}
$$

We differentiate to yield our node-wise partial derivatives:

$$
\begin{aligned}
& \frac{\partial}{\partial u_1} \int_{T_1} (\rho g u y) dy dx \qquad (3.71) \\
= \; & \frac{\rho g (y_3 - y_2)}{\Delta} \\
& \cdot \left[\frac{(y_m^4 - y_\ell^4) - 2(y_m^3 - y_\ell^3)}{12 \frac{y_m - y_\ell}{x_m - x_\ell}} - \frac{y_\ell^3 x_m - y_\ell^3 x_\ell}{3} + \frac{y_1(y_\ell^2 x_m - y_\ell^2 x_\ell)}{2} \right] \\
& - \frac{\rho g (x_3 - x_2)}{\Delta} \\
& \cdot \left[(x_m^4 - x_\ell^4) \left[\frac{(y_m - y_\ell)^2}{8(x_m - x_\ell)^2} \right] \right. \\
& + (x_m^3 - x_\ell^3) \left[\frac{y_\ell(y_m - y_\ell)}{3(x_m - x_\ell)} - \frac{(-x_1 - 2x_\ell)(y_m - y_\ell)^2}{6(x_m - x_\ell)^2} \right] \\
& + (x_m^2 - x_\ell^2) \left[\frac{(x_\ell^2 + 2x_1 x_\ell)(y_m - y_\ell)^2}{4(x_m - x_\ell)^2} + \frac{y_\ell(-x_1 - x_\ell)(y_m - y_\ell)}{2(x_m - x_\ell} \right] \\
& \left. + (x_m - x_\ell) \left[\frac{y_\ell x_\ell x_1 (y_m - y_\ell)}{(x_m - x_\ell)} - \frac{x_\ell^2 x_1 (y_m - y_\ell)^2}{2(x_m - x_\ell)^2} \right] \right] \\
& + (\rho g) \\
& \cdot \left[\frac{(y_m - y_\ell)^2}{2(x_m - x_\ell)^2} \left[\frac{1}{3}(x_m^3 - x_\ell^3) + x_\ell^2 (x_m - x_\ell) - x_\ell (x_m^2 - x_\ell^2) \right] \right. \\
& \left. + \frac{y_\ell (y_m - y_\ell)}{(x_m - x_\ell)} \left[\frac{1}{2}(x_m^2 - x_\ell^2) - x_\ell (x_m - x_\ell) \right] \right],
\end{aligned}
$$

$$\begin{aligned}
& \frac{\partial}{\partial u_2} \int_{T_1} (\rho g u y) dy dx \qquad (3.72) \\
= \; & \frac{\rho g (y_1 - y_3)}{\Delta} \\
& \cdot \left[\frac{(y_m^4 - y_\ell^4) - 2(y_m^3 - y_\ell^3)}{12 \frac{y_m - y_\ell}{x_m - x_\ell}} - \frac{y_\ell^3 x_m - y_\ell^3 x_\ell}{3} + \frac{y_1 (y_\ell^2 x_m - y_\ell^2 x_\ell)}{2} \right] \\
& - \frac{\rho g (x_1 - x_3)}{\Delta} \\
& \cdot \left[(x_m^4 - x_\ell^4) \left[\frac{(y_m - y_\ell)^2}{8(x_m - x_\ell)^2} \right] \right. \\
& + (x_m^3 - x_\ell^3) \left[\frac{y_\ell (y_m - y_\ell)}{3(x_m - x_\ell)} - \frac{(-x_1 - 2x_\ell)(y_m - y_\ell)^2}{6(x_m - x_\ell)^2} \right] \\
& + (x_m^2 - x_\ell^2) \left[\frac{(x_\ell^2 + 2x_1 x_\ell)(y_m - y_\ell)^2}{4(x_m - x_\ell)^2} + \frac{y_\ell (-x_1 - x_\ell)(y_m - y_\ell)}{2(x_m - x_\ell} \right] \\
& \left. + (x_m - x_\ell) \left[\frac{y_\ell x_\ell x_1 (y_m - y_\ell)}{(x_m - x_\ell)} - \frac{x_\ell^2 x_1 (y_m - y_\ell)^2}{2(x_m - x_\ell)^2} \right] \right],
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\partial}{\partial u_3} \int_{T_1} (\rho g u y) dy dx \qquad (3.73) \\
= \; & \frac{\rho g (y_2 - y_1)}{\Delta} \\
& \cdot \left[\frac{(y_m^4 - y_\ell^4) - 2(y_m^3 - y_\ell^3)}{12 \frac{y_m - y_\ell}{x_m - x_\ell}} - \frac{y_\ell^3 x_m - y_\ell^3 x_\ell}{3} + \frac{y_1 (y_\ell^2 x_m - y_\ell^2 x_\ell)}{2} \right] \\
& - \frac{\rho g (x_2 - x_1)}{\Delta} \\
& \cdot \left[(x_m^4 - x_\ell^4) \left[\frac{(y_m - y_\ell)^2}{8(x_m - x_\ell)^2} \right] \right. \\
& + (x_m^3 - x_\ell^3) \left[\frac{y_\ell (y_m - y_\ell)}{3(x_m - x_\ell)} - \frac{(-x_1 - 2x_\ell)(y_m - y_\ell)^2}{6(x_m - x_\ell)^2} \right] \\
& + (x_m^2 - x_\ell^2) \left[\frac{(x_\ell^2 + 2x_1 x_\ell)(y_m - y_\ell)^2}{4(x_m - x_\ell)^2} + \frac{y_\ell (-x_1 - x_\ell)(y_m - y_\ell)}{2(x_m - x_\ell} \right] \\
& \left. + (x_m - x_\ell) \left[\frac{y_\ell x_\ell x_1 (y_m - y_\ell)}{(x_m - x_\ell)} - \frac{x_\ell^2 x_1 (y_m - y_\ell)^2}{2(x_m - x_\ell)^2} \right] \right]
\end{aligned}$$

Without loss of generality, we get our integral and node-wise partial derivative values for T_2 by replacing x_ℓ and y_ℓ with x_r and y_r respectively.

We must also calculate the Hessian for this term, but as all of the node-wise partial derivatives are constants with respect to the nodes, the second partial derivatives are all 0, and this term does not contribute to the overall Hessian of our energy functional.

Volume Constraint and Boundary Conditions

Along with the minimization of our energy functional, our problem has equality constraints which we must satisfy. Namely, that $\int_\Omega u(x,y)dydx = V_\Omega$, where our volume is a prescribed quantity depending on the fluid. In order to input this into our logarithmic barrier scheme, we need this equality constraint in terms of $Au = b$, so here, our volume equates to the right-hand side, and we need only devise A such that it represents the integral over the entire domain.

Consider how we would calculate the integral $\int_\Omega u(x,y)dydx$ on our descretized domain. The integral over the whole space is the sum of the integral over each element of our domain. However, in order to put this calculation in terms of the nodes of our system, the vector u, we must find the contribution each node has to the element integral. Consider a singular element T, rotated and divided into $T_1 \cup T_2$ as before. Then,

$$\begin{aligned}\int_T u(x,y)dydx &= \int_{T_1} u(x,y)dydx + \int_{T_2} u(x,y)dydx \\ &= \int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)} u(x,y)dydx - \int_{x_r}^{x_m}\int_{y_r}^{L_2(x)} u(x,y)dydx.\end{aligned} \tag{3.74}$$

Once again, as these integrals are computed similarly, we will focus only on the integral over T_1. We have

$$\begin{aligned}
&\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)} u(x,y)dydx \qquad (3.75)\\
= &\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)} \frac{C_y(y-y_1)-C_x(x-x_1)}{\Delta}dydx\\
&+\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)} u_1 dydx\\
= &\frac{1}{\Delta}\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)} C_y(y-y_1)-C_x(x-x_1)dydx\\
&+u_1\int_{x_\ell}^{x_m}\int_{y_\ell}^{L_1(x)} 1\, dydx\\
= &\frac{1}{2\Delta C_y}\int_{x_\ell}^{x_m}\left\{C_y\left[\frac{y_m-y_\ell}{x_m-x_\ell}(x-x_\ell)+y_\ell-y_1\right]-C_x(x-x_1)\right\}^2 dx\\
&-\frac{1}{2\Delta C_y}\int_{x_\ell}^{x_m}[C_y(y_\ell-y_1)-C_x(x-x_1)]^2 dx\\
&+u_1\int_{x_\ell}^{x_m}\frac{y_m-y_\ell}{x_m-x_\ell}(x-x_\ell)dx\\
= &\frac{[C_y(y_m-y_1)-C_x(x_m-x_1)]^3-[C_y(y_\ell-y_1)-C_x(x_\ell-x_1)]^3}{6\Delta C_y^2\frac{y_m-y_\ell}{x_m-x_\ell}-6\Delta C_y C_x}\\
&+\frac{[C_y(y_\ell-y_1)-C_x(x_m-x_1)]^3-[C_y(y_\ell-y_1)-C_x(x_\ell-x_1)]^3}{6\Delta C_y C_x}\\
&+\frac{u_1(y_m-y_\ell)(x_m-x_\ell)}{2}.
\end{aligned}$$

Now, in order to best factor this node-wise, we will look at how to do this for the numerators of these three terms. Noticing that the numerators of the first two terms take similar forms, we will just look at the factorization of $[C_y(y_m-y_1)-C_x(x_m-x_1)]^3$.

We have

$$
\begin{aligned}
& [C_y(y_m - y_1) - C_x(x_m - x_1)]^3 \qquad (3.76) \\
= \; & C_y^3(y_m - y_1)^3 - 3C_y^2 C_x (y_m - y_1)^2 (x_m - x_1) \\
& +3C_y C_x^2 (y_m - y_1)(x_m - x_1)^2 - C_x^3 (x_m - x_1)^3 \\
= \; & C_y^2 (y_m - y_1)^3 \{(u_3(x_2 - x_1) - u_2[(x_2 - x_1) + (x_3 - x_2)] + u_1(x_3 - x_2)\} \\
& -3C_y^2 (y_m - y_1)^2 (x_m - x_1) \\
& \cdot \{(u_3(y_2 - y_1) - u_2[(y_2 - y_1) + (y_3 - y_2)] + u_1(y_3 - y_2)\} \\
& +3C_x^2 (y_m - y_1)(x_m - x_1)^2 \\
& \cdot \{(u_3(x_2 - x_1) - u_2[(x_2 - x_1) + (x_3 - x_2)] + u_1(x_3 - x_2)\} \\
& -C_x^2 (x_m - x_1)^3 \{(u_3(y_2 - y_1) - u_2[(y_2 - y_1) + (y_3 - y_2)] + u_1(y_3 - y_2)\} \\
= \; & u_3(x_2 - x_1)[C_y^2 (y_m - y_1)^3 + 3C_x^2 (y_m - y_1)(x_m - x_1)^2] \\
& -u_3(y_2 - y_1)[3C_y^2 (y_m - y_1)^2 (x_m - x_1) + C_x^2 (x_m - x_1)^3] \\
& -u_2[(x_2 - x_1) + (x_3 - x_2)][C_y^2 (y_m - y_1)^3 + 3C_x^2 (y_m - y_1)(x_m - x_1)^2] \\
& +u_2[(y_2 - y_1) - (y_3 - y_2)][3C_y^2 (y_m - y_1)^2 (x_m - x_1) + C_x^2 (x_m - x_1)^3] \\
& +u_1(x_3 - x_2)[C_y^2 (y_m - y_1)^3 + 3C_x^2 (y_m - y_1)(x_m - x_1)^2] \\
& -u_1(y_3 - y_2)[3C_y^2 (y_m - y_1)^2 (x_m - x_1) + C_x^2 (x_m - x_1)^3].
\end{aligned}
$$

Repeating this process for all of the similar terms, we can get the distributions of the T_1 integral over our node values. That is, if $\int_{T_1} u(x,y)dydx = A_1u_1 + A_2u_2 + A_3u_3$, we get that

$$
\begin{aligned}
A_1 \quad = \quad & \frac{(x_3 - x_2)}{6\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - 6\Delta C_y C_x} \\
& \cdot \left\{ C_y^2[(y_m - y_1)^3 - (y_\ell) - y_1)^3] \right. \\
& \left. + 3C_x^2[(y_m - y_1)(x_m - x_1)^2 - (y_\ell - y_1)(x_\ell - x_1)^2] \right\} \\
& - \frac{(y_3 - y_2)}{6\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - 6\Delta C_y C_x} \\
& \cdot \left\{ C_x^2[(x_m - x_1)^3 - (x_\ell) - x_1)^3] \right. \\
& \left. + 3C_y^2[(y_m - y_1)^2(x_m - x_1) - (y_\ell - y_1)^2(x_\ell - x_1)] \right\} \\
& + \frac{(x_3 - x_2)}{6\Delta C_y C_x} \\
& \cdot \left\{ 3C_x^2[(y_\ell - y_1)(x_m - x_1)^2 - (y_\ell - y_1)(x_\ell - x_1)^2] \right\} \\
& - \frac{(y_3 - y_2)}{6\Delta C_y C_x} \cdot \left\{ C_x^2[(x_m - x_1)^3 - (x_\ell - x_1)^3 \right\} \\
& + \frac{(y_m - y_\ell)(x_m - x_\ell)}{2}.
\end{aligned}
\tag{3.77}
$$

We have that

$$
\begin{aligned}
A_2 \quad = \quad & \frac{-[(x_2 - x_1) + (x_3 - x_2)]}{6\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - 6\Delta C_y C_x} \\
& \cdot \left\{ C_y^2 [(y_m - y_1)^3 - (y_\ell) - y_1)^3] \right. \\
& \left. + 3C_x^2 [(y_m - y_1)(x_m - x_1)^2 - (y_\ell - y_1)(x_\ell - x_1)^2] \right\} \\
& - \frac{-[(y_2 - y_2) + (y_3 - y_2)]}{6\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - 6\Delta C_y C_x} \\
& \cdot \left\{ C_x^2 [(x_m - x_1)^3 - (x_\ell) - x_1)^3] \right. \\
& \left. + 3C_y^2 [(y_m - y_1)^2 (x_m - x_1) - (y_\ell - y_1)^2 (x_\ell - x_1)] \right\} \\
& + \frac{(x_3 - x_2)}{6\Delta C_y C_x} \\
& \cdot \left\{ 3C_x^2 [(y_\ell - y_1)(x_m - x_1)^2 - (y_\ell - y_1)(x_\ell - x_1)^2] \right\} \\
& - \frac{(y_3 - y_2)}{6\Delta C_y C_x} \cdot \left\{ C_x^2 [(x_m - x_1)^3 - (x_\ell - x_1)^3 \right\} .
\end{aligned}
\tag{3.78}
$$

Finally, we get that

$$
\begin{aligned}
A_3 \quad = \quad & \frac{(x_3 - x_2)}{6\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - 6\Delta C_y C_x} \\
& \cdot \left\{ C_y^2 [(y_m - y_1)^3 - (y_\ell) - y_1)^3] \right. \\
& \left. + 3C_x^2 [(y_m - y_1)(x_m - x_1)^2 - (y_\ell - y_1)(x_\ell - x_1)^2] \right\} \\
& - \frac{(y_3 - y_2)}{6\Delta C_y^2 \frac{y_m - y_\ell}{x_m - x_\ell} - 6\Delta C_y C_x} \\
& \cdot \left\{ C_x^2 [(x_m - x_1)^3 - (x_\ell) - x_1)^3] \right. \\
& \left. + 3C_y^2 [(y_m - y_1)^2 (x_m - x_1) - (y_\ell - y_1)^2 (x_\ell - x_1)] \right\} \\
& + \frac{(x_3 - x_2)}{6\Delta C_y C_x} \\
& \cdot \left\{ 3C_x^2 [(y_\ell - y_1)(x_m - x_1)^2 - (y_\ell - y_1)(x_\ell - x_1)^2] \right\} \\
& - \frac{(y_3 - y_2)}{6\Delta C_y C_x} \cdot \left\{ C_x^2 [(x_m - x_1)^3 - (x_\ell - x_1)^3 \right\} .
\end{aligned}
\tag{3.79}
$$

We calculate these coefficients similarly for the right-hand triangle of our element T_2 once again by replacing x_ℓ and y_ℓ with x_r and y_r, respectively.

We will cycle through all of the elements in our discretized domain, adding each of these terms to the vector A, so that we may achieve an appropriate vector such that $Au = V_\Omega$.

We must also consider our boundary conditions for the liquid-solid interface of our bounded container. From the study of the phase field model, we yielded a mechanism by which to recover the appropriate Dirichlet boundary conditions. Recall that

$$cos(\theta_Y) = -4u_{\partial S}^3 + 6u_{\partial S}^2 + 1. \tag{3.80}$$

To implement this, we will simply force the boundary nodes to assume the appropriate root of this equation ($0 \leq u_{\partial S} \leq 1$). As we allow for free motion of the fluid within our entire container until it reaches equilibrium, no boundary data is required for the vapor boundary.

In observing the density distribution of the system in equilibrium, we expect only to find the configuration which is most in line with classical capillary theory - that our data should take on the phases u=0 and u=1 in most of the container, with $|\{u \in (0,1)\}|$ being negligible. That is, we expect an infinitely thin layer of the transition phase between the liquid and the container, and likewise for the phase between the liquid and the vapor.

IV. FUTURE WORK

Future work will include the completion of a numerical solver based on our approach and yielding numerical results. Taking this work even further, we wish to try to adjust the scheme to optimize energy functionals for a bounded container containing three immiscible fluids. We hope our discretized approach will properly model asymmetric configurations under these circumstances.

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