A SOLID OBJECT FLOATING IN A BATH OF THREE FLUIDS

by

Henry E. Ickes

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Committee Members:

Raymond Treinen, Chair

Ricardo Torrejon

Stewart Welsh

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ABSTRACT

We consider a bounded container filled by three immiscible fluids and a solid object, presumably floating at one or more of the interfaces. Fluids are modeled by Caccioppoli sets. The energy in our model is assumed to come from gravity, adhesion energy, and surface tension. We shall use the theory of functions of bounded variation to show that the energy functional attains a minimum for some configurations of the fluids and solid.

I. INTRODUCTION

We consider a bounded cylindrical container $G := \Omega \times [0, p]$, where $\Omega \subset \mathbb{R}^2$ is bounded and simply connected. Suppose that G is filled with three fluids, which we denote by the sets $E_i, i = 1, 2, 3$, and a rigid object $\mathcal{B}$.

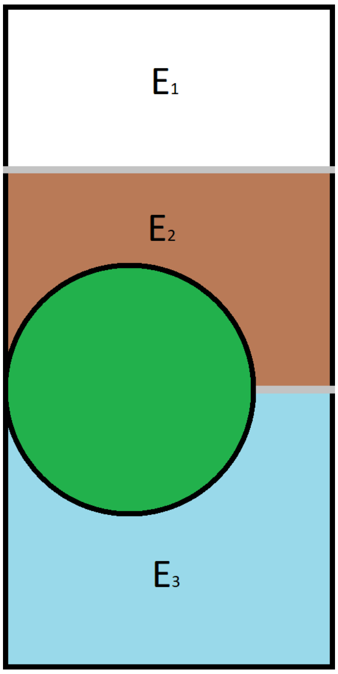

Figure I.1: A typical example of the problem we consider.

The density of each fluid is given by ρ_i and the density of the solid by ρ_0. The interface between each fluid is assumed to be governed by surface tension. If we denote the surface tension between fluids i, j by α_{ij}, then we will consider the "surface tension coefficient" of each fluid by

$$\alpha_i := \frac{1}{2}(\alpha_{ij} + \alpha_{ik} - \alpha_{jk}), \tag{I.1}$$

for $i, j, k = 1, 2, 3$ mutually distinct. The energy of the system is assumed to come from the following:

- g, the gravitational constant,
- the adhesion energy between fluid i and the boundary ∂G of the container, given by β_i,
- the adhesion energy between fluid i and the surface $\partial\mathcal{B}$ of the solid, given by τ_i.

We study the energy functional in the case where each fluid region is represented by a Caccioppoli set: each fluid's characteristic function is of bounded variation. Let $\mathcal{B}(c, R) := \{y = c + Rx : x \in \mathcal{B}\}$, where $c \in \mathbb{R}^3$ denotes a translation, and $R = R(d, \theta) \in SO(3)$ describes a rotation with respect to some axis, with unit vector d, about some angle θ. The quantities c, R are restricted by requiring that the floating body is contained in G. Throughout it shall be assumed that $\mathcal{B}(c, R)$ is a closed set. Then $E_i \subset G\backslash\mathcal{B}(c, R)$ is measurable, so we denote $V_i = \mathcal{L}^3(E_i)$ to be the volume of each fluid. We also define $E = \{E_1, E_2, E_3\}$. Lastly, we shall use the notation (x', x_n) for $x \in \mathbb{R}^3$. The energy functional is then

$$\begin{aligned}\mathcal{F}(c, R, E) := \sum_{i=1}^{3} \Bigg(\alpha_i \int_{\Omega\backslash\mathcal{B}(c,R)} |D\phi_{E_i}| + g\rho_i \int_{\Omega\backslash\mathcal{B}(c,R)} x_n\phi_{E_i}\, dx \\ +\beta_i \int_{\partial\Omega} \phi_{E_i}\, d\mathcal{H}^2 + \tau_i \int_{\partial\mathcal{B}} \phi_{E_i}\, d\mathcal{H}^2\Bigg) + g\rho_0 \int_{\mathcal{B}(c,R)} x_n\, dx,\end{aligned} \tag{I.2}$$

where

$$\int_{\Omega\backslash\mathcal{B}(c,R)} |D\phi_{E_i}| := sup\left\{\int_{\Omega\backslash\mathcal{B}(c,R)} \phi_{E_i}\, div(g)\, dx : g \in C_c^1(\Omega\backslash\mathcal{B}(c, R); \mathbb{R}^3), \|g\|_{C^0} \le 1\right\}$$

denotes the total variation of ϕ_{E_i}. The integrals over $\partial\Omega$ and $\partial\mathcal{B}$ denote the area of the wetted part of the container and the solid respectively. We prove that there is a minimizing configuration $(\mathcal{B}(c,R),E)$ to $\mathcal{F}$ in the class

$$\begin{aligned}\mathcal{C} :=&\{(c,R,E) : c \in \mathbb{R}^3, R \in SO(3) \text{ such that } \mathcal{B}(c,R) \subset \Omega, \\ &E_i \subset \Omega\backslash\mathcal{B}(c,R) \text{ measurable with } \mathcal{L}^3(E) = V_0\}.\end{aligned} \tag{I.3}$$

Chapter two is dedicated to listing definitions used throughout the text. In the following three chapters, we dedicate a chapter each to three papers that studied previous versions of our problem, giving an occasional exposition on the details of their proofs. Chapter three considers Emmer's lemma [2], which is fundamental to every case we study here. We will then proceed to study Emmer's proof in the case where G is filled with two fluids and no solids. Chapter four considers Massari's proof [3] where G is filled with three fluids. Finally, chapter five addresses the proof given by Bemelmans, Galdi, and Kyed [1] for the case when G is filled with two fluids and one solid.

In the final chapter, we will consider our problem. The proof given will use a combination of all three previous proofs. The first step is to establish a lower bound on $\mathcal{F}(c,R,E)$, then form a minimizing sequence $(c_j,R_j,E_j) \subset \mathcal{C}$. We then show that (c_j,R_j,E_j) is bounded in the sense that $|c_j| + |R_j| + \|E_j\|_{BV(G)} \leq c \in \mathbb{R}$, and thus form a convergent subsequence. To complete the proof, we show that $\mathcal{F}$ is lower semicontinuous with respect to our subsequence, ensuring that it converges in $\mathcal{C}$.

II. DEFINITIONS

Before we begin, we first list some definitions that are important in approaching these problems. The main definition here is the following:

Definition II.0.1 *Let $f : \mathbb{R} \to \mathbb{R}$. Then f is* ***lower semicontinuous*** *at $p \in \mathbb{R}$ if for any $\epsilon > 0$ there exists $\delta > 0$ so that if $|x-p| < \delta$, then $f(x) \geq f(p) - \epsilon$. Alternatively,*

$$\liminf_{x \to p} f(x) \geq f(p). \tag{II.1}$$

When dealing with our boundary, we need to place some restrictions on it. The two we use are the interior sphere condition and the Lipschitz condition. The latter is the stronger of the two.

Definition II.0.2 *Let $\Omega \subset \mathbb{R}^n$ be open. Then Ω has the* ***interior sphere condition*** *if for all $y \in \partial\Omega$ there exists $x \in \Omega, r > 0$ such that the open ball $B(x, \rho)$ satisfies $B(x, \rho) \subset \Omega, y \in \partial B(x, \rho)$.*

Definition II.0.3 *A function $f : \mathbb{R} \to \mathbb{R}$ is* ***Lipschitz*** *if there exists a positive L such that for all $x, y \in \mathbb{R}$,*

$$|f(x) - f(y)| \leq L|x - y|. \tag{II.2}$$

We call f ***locally Lipschitz*** *if for every $x \in \mathbb{R}$ there exists a neighborhood U of x so that $f\big|_U$ is Lipschitz.*

We also need to place restrictions on the functions we integrate. We require them to be of bounded variation:

Definition II.0.4 *Let $\Omega \subset \mathbb{R}^n$ be open, $f \in L^1(\Omega)$. The* ***total variation*** *of f is*

$$V(f,\Omega) := sup\left\{\int_\Omega f(x)\, div(\phi(x))dx : \phi \in C_c^1(\Omega,\mathbb{R}^n), \|\phi\|_{L^1(\Omega)} \leq 1\right\}, \qquad \text{(II.3)}$$

where C_c^1 is the set of continuously differentiable vector functions of compact support.

Definition II.0.5 *Let $\Omega \subset \mathbb{R}^n$ be open, $u \in L^1(\Omega)$. Then u has* ***bounded variation*** *if its total variation is finite. We write $BV(\Omega)$ for the space of functions of bounded variation.*

Lastly, we occasionally use the Hausdorff integral on our functions of bounded variation:

Definition II.0.6 *Let $n \in \mathbb{N}$, $A \subset \mathbb{R}^{n-1}$ be open, $f : A \to \mathbb{R}$, $\alpha : \mathbb{R}^n \to \mathbb{R}$, and $S = graph(f)$. The* ***Hausdorff integral*** *of α is defined by*

$$\int_S \alpha\, d\mathcal{H}^n := \int_A \alpha(y, f(y))\sqrt{1 + |Df(y)|^2}\, dy, \qquad \text{(II.4)}$$

where $y \in \mathbb{R}^{n-1}$, and Df is the gradient of f.

III. EMMER

Emmer[2] considered a two-fluid problem in a bounded container. His results are presented here. We begin with the energy given by

$$\sigma_1 \int_\Omega \sqrt{1+|Du|^2}\,dx + g\rho \int_\Omega u^2\,dx + \sigma_2 \int_{\partial\Omega} u\,d\mathcal{H}^{n-1}, \tag{III.1}$$

with σ_1 being the surface tension of the fluid, g the gravitational constant, ρ the density of the fluid, and σ_2 the adhesion energy. Note that the first integral gives the surface area of the fluid. We normalize the equation, dividing by σ_1, to obtain our desired functional

$$\mathcal{F}(u) = \int_\Omega \sqrt{1+|Du|^2}\,dx + \kappa \int_\Omega u^2\,dx + a \int_{\partial\Omega} u\,d\mathcal{H}^{n-1}, \tag{III.2}$$

with $u : \Omega \to \mathbb{R}$ representing the surface height, $\kappa \in \mathbb{R}^+$ the capillary constant, $a \in \mathbb{R}$ the surface tension. We shall prove that there is a minimizing element u for $\mathcal{F}$, under the restriction that u is measurable with $\mathcal{L}^n(u) = V$, with V being the volume of the fluid.

Emmer's original paper [2] is written in Italian. An English-translated version of the proof is given in Massari and Miranda's book [4]. We will use the proof given there as our reference.

Lemma III.0.1 *Let $\{\phi_j\}$ be a partition of unity and take $u_j = \phi_j u$. Let $A \subset \mathbb{R}^{n-1}$ be open, and $\psi_j : A \to \mathbb{R}$ be Lipschitz such that $\partial\Omega \cap spt(\phi_j) \subset graph(\psi_j)$. Let $\delta > 0$ such that $\{x \in \Omega | x' \in A, |x_n - \psi_j(x)| < \delta\}$ $= \{x \in \Omega | x' \in A, -\delta < x_n - \psi_j(x) < 0\}$. Then for $t \in (0,\delta)$ we have*

$$\int_{\partial\Omega} |u_j|\,d\mathcal{H}^{n-1} \leq \sqrt{1+L^2} \int_{\Omega_t} |Du_j| + \int_{S_t} |u_j|\,d\mathcal{H}^{n-1} \tag{III.3}$$

where L is the Lipschitz constant of ψ_j,

$$\begin{aligned} \Omega_t &:= \{x \in \Omega | x' \in A, -t < x_n - \psi_j(x) < 0\}, \\ S_t &:= \{x \in \Omega | x' \in A, -t = x_n - \psi_j(x)\}. \end{aligned} \tag{III.4}$$

Proof. Let $y : \Omega \to \partial\Omega$ by $y(x) = proj_{\partial\Omega}(x)$, and $z : \Omega \to S_t$ by $z(x) = proj_{S_t}(x)$. Let $f(y) = |u_j|\big|_{\partial\Omega}$, $f(z) = |u_j|\big|_{S_t}$, and interpolate between $f(y)$ and $f(z)$ by $g(t) = f((1-t)z + ty)$, so that $g(t)$ describes the line segment connecting $y(x)$ and $z(x)$ for any given x. Applying the Fundamental Theorem of Calculus (using g(0) and g(1)), we obtain

$$f(y) - f(z) = \int_0^1 Df(z + t(y-z))\, dt. \tag{III.5}$$

Now integrate (III.5) over A:

$$\int_A f(y) - f(z) dx = \int_A \left[\int_0^1 Df(z + t(y-z))\, dt \right] dx. \tag{III.6}$$

Using the definition of Ω_t, and the fact that $\partial\Omega$ is Lipschitz, we obtain

$$\begin{aligned} \int_A \left[\int_0^1 Df(z + t(y-z))\, dt \right] dx &\le \int_{\Omega_t} |Du_j| \sqrt{1 + |Dy|^2}\, dx \\ &\le \sqrt{1 + L^2} \int_{\Omega_t} |Du_j|\, dx. \end{aligned} \tag{III.7}$$

On the other hand, we use the definition of the Hausdorff integral on the left hand side of (III.6) to obtain

$$\int_A f(y) - f(z)\, dx = \int_A f(y)\, dx - \int_A f(z)\, dx = \int_{\partial\Omega} |u_j|\, d\mathcal{H}^{n-1} - \int_{S_t} |u_j|\, d\mathcal{H}^{n-1}. \tag{III.8}$$

We now have

$$\int_{\partial\Omega} |u_j|\, d\mathcal{H}^{n-1} - \int_{S_t} |u_j|\, d\mathcal{H}^{n-1} \leq \sqrt{1+L^2} \int_{\Omega_t} |Du_j|, \tag{III.9}$$

from which we obtain (III.3). ■

Integrate (III.3) with respect to $t \in (0, \delta)$ to obtain

$$\delta \int_{\partial\Omega} |u_j|\, d\mathcal{H}^{n-1} \leq \delta \sqrt{1+L^2} \int_{\Omega_\delta} |Du_j| + \int_{\Omega_\delta} |u_j|\, dx, \tag{III.10}$$

then adding with respect to j and using the Product Rule, we get

$$\int_{\partial\Omega} |u|\, d\mathcal{H}^{n-1} \leq \sqrt{1+L^2} \int_{\Omega_\delta} |Du| + c \int_{\Omega_\delta} |u|\, dx, \tag{III.11}$$

with $c = \left(\sum_j max|D\phi_j|\right) + 2/\delta$. Since ϕ_j is assumed to be a partition of unity, then for any $x \in \Omega$ we can form a neighborhood U of x were $\phi_j(x) = 0$ for all but finite j. Additionally, $\delta > 0$. Therefore, $c < \infty$.

Lemma III.0.2 *If* $1 - |a|\sqrt{1+L^2} \geq 0$, *then* $\mathcal{F}$ *is bounded below in* $BV(\Omega)$.

Proof. Notice (III.11) is of the form $|x| \leq p$, which is equivalent to $-p \leq x \leq p$. Therefore, (III.11) is equivalent to

$$\begin{aligned} &- |a|\sqrt{1+L^2} \int_{\Omega} |Du| - |a|c \int_{\Omega} |u|\, dx \leq a \int_{\partial\Omega} u\, d\mathcal{H}^{n-1} \\ &\leq |a|\sqrt{1+L^2} \int_{\Omega} |Du| + |a|c \int_{\Omega} |u|\, dx \end{aligned} \tag{III.12}$$

Using the left inequality of (III.12), together with the fact that

$|Du| \leq \sqrt{1+|Du|^2}$, we obtain

$$
\begin{aligned}
\mathcal{F}(u) &= \int_\Omega \sqrt{1+|Du|^2}\,dx + \kappa \int_\Omega u^2\,dx + a\int_{\partial\Omega} u\,d\mathcal{H}^{n-1} \\
&\geq \left(1 - |a|\sqrt{1+L^2}\right)\int_\Omega \sqrt{1+|Du|^2}\,dx + \kappa\int_\Omega u^2\,dx - |a|c\int_\Omega |u|\,dx \\
&\geq \left(1 - |a|\sqrt{1+L^2}\right)\int_\Omega \sqrt{1+|Du|^2}\,dx - \frac{a^2c^2}{2\kappa} meas(\Omega) + \frac{\kappa}{2}\int_\Omega u^2\,dx \\
&\geq -\frac{a^2c^2}{2\kappa} meas(\Omega) > -\infty.
\end{aligned}
\tag{III.13}
$$

■

Lemma III.0.3 *If* $1 - |a|\sqrt{1+L^2} \geq 0$, $\{u_j\} \subset BV(\Omega)$, $u \in BV(\Omega)$ *with*

$$
\lim_{j\to\infty}\int_\Omega |u_j - u|\,dx = 0,
$$

and $u_j \to u$ *a.e, then* $\mathcal{F}$ *is lower semicontinuous.*

Proof. In order to show our desired result, we subtract $\liminf_j \mathcal{F}(u)$ from both sides to obtain

$$
\mathcal{F}(u) - \left[\liminf_j \mathcal{F}(u_j)\right] \leq 0.
$$

Using the fact that $\liminf_k(a_k) = -\limsup_k(-a_k)$, we get

$$
\limsup_j\,[\mathcal{F}(u) - \mathcal{F}(u_j)] \leq 0, \tag{III.14}
$$

which we will use to show our conclusion. From (III.11), we have

$$
\begin{aligned}
\left| a\int_{\partial\Omega} u\,d\mathcal{H}^{n-1} - a\int_{\partial\Omega} u_j\,d\mathcal{H}^{n-1}\right| &\leq |a|\sqrt{1+L^2}\left[\int_{\Omega_\epsilon}|Du|\,dx + \int_{\Omega_\epsilon}|Du_j|\,dx\right] \\
&\quad + |a|c\int_\Omega |u-u_j|\,dx,
\end{aligned}
\tag{III.15}
$$

where Ω_ϵ is given in (III.4). Therefore,

$$\begin{aligned}
\mathcal{F}(u)-\mathcal{F}(u_j) &\leq \int_\Omega \sqrt{1+|Du|^2}\,dx - \int_\Omega \sqrt{1+|Du_j|^2}\,dx + \kappa \int_\Omega \left(u^2-u_j^2\right)\,dx \\
&\quad + |a|\sqrt{1+L^2}\left[\int_{\Omega_\epsilon} |Du|\,dx - \int_{\Omega_\epsilon} |Du_j|\,dx\right] + |a|c\int_\Omega |u-u_j|\,dx \\
&\leq \int_{\Omega-\Omega_\epsilon} \sqrt{1+|Du|^2}\,dx - \int_{\Omega-\Omega_\epsilon} \sqrt{1+|Du_j|^2}\,dx + \kappa \int_\Omega \left(u^2-u_j^2\right)\,dx \\
&\quad + \left(1+|a|\sqrt{1+L^2}\right)\int_{\Omega_\epsilon} \sqrt{1+|Du|^2}\,dx + |a|c\int_{\Omega_\epsilon} |u-u_j|\,dx.
\end{aligned} \tag{III.16}$$

Sending $j\to\infty$, since $u_j\to u$ a.e, we have the following:

$$\begin{aligned}
&\int_{\Omega-\Omega_\epsilon} \sqrt{1+|Du|^2}\,dx - \int_{\Omega-\Omega_\epsilon} \sqrt{1+|Du_j|^2}\,dx \to 0, \\
&\kappa\int_\Omega \left(u^2-u_j^2\right)\,dx \to 0, \\
&|a|c\int_\Omega |u-u_j|\,dx \to 0.
\end{aligned} \tag{III.17}$$

Therefore we obtain

$$\limsup_{j\to\infty}\left[\mathcal{F}(u)-\mathcal{F}(u_j)\right] \leq \left(1+|a|\sqrt{1+L^2}\right)\int_{\Omega_\epsilon} \sqrt{1+|Du|^2}\,dx. \tag{III.18}$$

As $\epsilon>0$ arbitrarily, the integral vanishes as $\epsilon\to 0$ and we obtain (III.14), as desired. ■

Theorem III.0.4 *If $1-|a|\sqrt{1+L^2}>0$, then there exists a $u_0\in BV(\Omega)$ so that for any $u\in BV(\Omega)$, $\mathcal{F}(u_0)\leq\mathcal{F}(u)$.*

Proof. By Lemma III.0.2, we see that $\mathcal{F}$ is bounded below, so we can form a minimizing sequence $\{u_j\}\subset BV(\Omega)$ and a subsequence $\{u_{j_n}\}$ which converges to some $u\in BV(\Omega)$. By Lemma III.0.3, $\mathcal{F}$ is lower semicontinuous with respect to this minimizing sequence. Therefore our u is a minimal element for $\mathcal{F}$, as desired.

■

IV. MASSARI

Massari's paper [3] addresses the free boundary problem with three fluids, but no solid. We define Ω as before, and define $E = (E_1, E_2, E_3)$ to be the set of fluids E_i. In this case the energy functional is

$$\mathcal{F}(E) = \sum_{i=1}^{3} \left(\gamma_i \int_\Omega |D\phi_{E_i}| + g\rho_i \int_\Omega x_n \phi_{E_i}\, dx + \beta_i \int_{\partial\Omega} \phi_{E_i}\, d\mathcal{H}^{n-1} \right) \tag{IV.1}$$

where g is the gravitational constant, ρ_i is the density, and β_i is the adhesion energy of liquid i to $\partial\Omega$. Additionally, define γ_{ij} to be the surface tension between fluids i and j, and denote γ_i to be the solutions to the linear equation

$$\gamma_i + \gamma_j = \gamma_{ij}, \qquad i, j = 1, 2, 3;\ i \neq j. \tag{IV.2}$$

Solving the system of three equations gives

$$\begin{aligned} \gamma_1 &= \frac{1}{2}(\gamma_{12} + \gamma_{13} - \gamma_{23}) \\ \gamma_2 &= \frac{1}{2}(\gamma_{12} + \gamma_{23} - \gamma_{13}) \\ \gamma_3 &= \frac{1}{2}(\gamma_{13} + \gamma_{23} - \gamma_{12}). \end{aligned} \tag{IV.3}$$

Lemma IV.0.1 *Suppose the following:*

(i) $\{E^h\}$ *is a sequence converging to* E

(ii) ϕ_{E_i} *is the characteristic function of* E_i*; in particular* $\sum_{i=1}^{3} \phi_{E_i} = 1$

(iii) The index i *in* $\{E_i\}_{i=1}^{3}$ *is labelled so that* $\beta_1 \leq \beta_2 \leq \beta_3$.

Then

$$\sum_{i=1}^{3} \beta_i \int_{\partial\Omega} (\phi_{E_i} - \phi_{E_i^h})\, d\mathcal{H}^{n-1} \le \sum_{j=1,3} |\beta_j - \beta_2| \int_{\partial\Omega} (\phi_{E_j} - \phi_{E_j^h})\, dx. \tag{IV.4}$$

Proof. Use (ii) to obtain

$$\begin{aligned} 2\int_{\partial\Omega} (\phi_{E_1} - \phi_{E_1^h})\, d\mathcal{H}^{n-1} &= \int_{\partial\Omega} [(\phi_{E_1} + \phi_{E_1} - 1) - (\phi_{E_1^h} + \phi_{E_1^h} - 1)]\, d\mathcal{H}^{n-1} \\ &= \int_{\partial\Omega} [(\phi_{E_1} - \phi_{E_2} - \phi_{E_3}) - (\phi_{E_1^h} - \phi_{E_2^h} - \phi_{E_3^h})]\, d\mathcal{H}^{n-1} \\ &= \int_{\partial\Omega} [(\phi_{E_1} - \phi_{E_1^h}) - (\phi_{E_2} - \phi_{E_2^h}) - (\phi_{E_3} - \phi_{E_3^h})]\, d\mathcal{H}^{n-1}. \end{aligned} \tag{IV.5}$$

Multiplying both sides by β_1 and using (iii), we obtain

$$2\beta_1 \int_{\partial\Omega} (\phi_{E_1} - \phi_{E_1^h})\, d\mathcal{H}^{n-1} \le \beta_2 \int_{\partial\Omega} [(\phi_{E_1} - \phi_{E_1^h}) - (\phi_{E_2} - \phi_{E_2^h}) - (\phi_{E_3} - \phi_{E_3^h})]\, d\mathcal{H}^{n-1}. \tag{IV.6}$$

Adding $\beta_3 \int_{\partial\Omega} (\phi_{E_3} - \phi_{E_3^h})\, d\mathcal{H}^{n-1}$ to both sides, and rearranging terms, we then have

$$\begin{aligned} &\beta_1 \int_{\partial\Omega} (\phi_{E_1} - \phi_{E_1^h})\, d\mathcal{H}^{n-1} + \beta_2 \int_{\partial\Omega} (\phi_{E_2} - \phi_{E_2^h})\, d\mathcal{H}^{n-1} + \beta_3 \int_{\partial\Omega} (\phi_{E_3} - \phi_{E_3^h})\, d\mathcal{H}^{n-1} \\ &\le \left[\beta_2 \int_{\partial\Omega} (\phi_{E_1} - \phi_{E_1^h})\, d\mathcal{H}^{n-1} - \beta_1 \int_{\partial\Omega} (\phi_{E_1} - \phi_{E_1^h})\, d\mathcal{H}^{n-1}\right] \\ &+ \left[\beta_3 \int_{\partial\Omega} (\phi_{E_3} - \phi_{E_3^h})\, d\mathcal{H}^{n-1} - \beta_2 \int_{\partial\Omega} (\phi_{E_3} - \phi_{E_3^h})\, d\mathcal{H}^{n-1}\right]. \end{aligned} \tag{IV.7}$$

Use (iii) on the right hand side of (IV.7) to obtain the desired inequality (IV.4). ■

Theorem IV.0.2 *Suppose the hypotheses of Lemma IV.0.1 hold. In addition, suppose $\partial\Omega$ is Lipschitz with coefficient L, $\gamma_i \ge 0, \gamma_i + \gamma_j > 0$, and $\gamma_i + \gamma_j \ge \sqrt{1 + L^2}|\beta_i - \beta_j|\,(i, j = 1, 2, 3)$.*

Then $\mathcal{F}$ is lower semicontinuous.

Proof. Let $\epsilon > 0, E \subset \Omega$. By (III.11) we have

$$\int_{\partial\Omega} \phi_E \, d\mathcal{H}^{n-1} \leq \sqrt{1+L^2} \int_{\Omega_\epsilon} |D\phi_E| + c \int_{\Omega_\epsilon} \phi_E \, dx, \tag{IV.8}$$

Now by definition of $\mathcal{F}$, and rearranging terms, we obtain

$$\begin{aligned} \mathcal{F}(E) - \mathcal{F}(E^h) &= \sum_{i=1}^{3} \left(\gamma_i \int_\Omega |D\phi_{E_i}| + g\rho_i \int_\Omega x_n \phi_{E_i} \, dx + \beta_i \int_\Omega \phi_{E_i} \, d\mathcal{H}^{n-1} \right) \\ &\quad - \sum_{i=1}^{3} \left(\gamma_i \int_\Omega |D\phi_{E_i^h}| + g\rho_i \int_\Omega x_n \phi_{E_i^h} \, dx + \beta_i \int_\Omega \phi_{E_i^h} \, d\mathcal{H}^{n-1} \right) \\ &= \sum_{i=1}^{3} \left[\gamma_i \left(\int_\Omega |D\phi_{E_i}| - \int_\Omega |D\phi_{E_i^h}| \right) + g\rho_i \int_\Omega x_n (\phi_{E_i} - \phi_{E_i^h}) \, dx \right. \\ &\qquad \left. + \beta_i \int_\Omega (\phi_{E_i} - \phi_{E_i^h}) \, d\mathcal{H}^{n-1} \right]. \end{aligned} \tag{IV.9}$$

We now proceed term by term. For the first term, we have:

$$\begin{aligned} \sum_{i=1}^{3} \gamma_i \left(\int_\Omega |D\phi_{E_i}| - \int_\Omega |D\phi_{E_i^h}| \right) = \sum_{i=1}^{3} \gamma_i \left(\int_{\Omega - \Omega_\epsilon} |D\phi_{E_i}| - \int_{\Omega - \Omega_\epsilon} |D\phi_{E_i^h}| \right. \\ \left. + \int_{\Omega_\epsilon} |D\phi_{E_i}| - \int_{\Omega_\epsilon} |D\phi_{E_i^h}| \right). \end{aligned} \tag{IV.10}$$

The second term ($g\rho_i$) is unchanged. Finally, we treat the third term by Lemma IV.0.1 and equation (III.11). First, we use the fact that

$$\gamma_i \int_\Omega |D\phi_{E_i}| = \gamma_i \int_{\Omega \setminus \Omega_\epsilon} |D\phi_{E_i}| + \gamma_i \int_{\Omega_\epsilon} |D\phi_{E_i}| \tag{IV.11}$$

to obtain

$$
\begin{aligned}
\mathcal{F}(E)-\mathcal{F}(E^h) &= \sum_{i=1}^{3}\Bigg[\gamma_i\left(\int_\Omega |D\phi_{E_i}| - \int_\Omega |D\phi_{E_i^h}|\right) + g\rho_i \int_\Omega x_n(\phi_{E_i}-\phi_{E_i^h})\,dx \\
&\quad +\beta_i \int_\Omega (\phi_{E_i}-\phi_{E_i^h})\,d\mathcal{H}^{n-1}\Bigg] \\
&\le \sum_{i=1}^{3}\Bigg[\gamma_i\left(\int_{\Omega-\Omega_\epsilon} |D\phi_{E_i}| - \int_{\Omega-\Omega_\epsilon} |D\phi_{E_i^h}|\right) + \gamma_i \int_{\Omega_\epsilon}|D\phi_{E_i}| \\
&\qquad +g\rho_i \int_\Omega x_n(\phi_{E_i}-\phi_{E_i^h})\,dx\Bigg] \\
&\quad + \sum_{j=1,3}\Bigg[\sqrt{1+L^2}|\beta_j-\beta_2|\int_{\Omega_\epsilon}|D\phi_{E_j}| \\
&\qquad +(\sqrt{1+L^2}|\beta_j-\beta_2|-\gamma_j)\int_{\Omega_\epsilon}|D\phi_{E_j^h}| \\
&\qquad +c|\beta_j-\beta_2|\int_{\Omega_\epsilon}(\phi_{E_j}-\phi_{E_j^h})\,dx\Bigg] + \gamma_2\int_{\Omega_\epsilon}|D\phi_{E_2^h}|.
\end{aligned}
\tag{IV.12}
$$

To show that the right hand side of the above inequality vanishes as $h \to \infty$, we again treat it term by term:

(i) $\int_{\Omega-\Omega_\epsilon}|D\phi_{E_i}|$ is lower semicontinuous, so
$\gamma_i\left(\int_{\Omega-\Omega_\epsilon}|D\phi_{E_i}| - \int_{\Omega-\Omega_\epsilon}|D\phi_{E_i^h}|\right) \le 0$ as $h\to\infty$.

(ii) $\gamma_i\int_{\Omega_\epsilon}|D\phi_{E_i}|$ and $|\beta_j-\beta_2|\int_{\Omega_\epsilon}|D\phi_{E_j}|$ also vanish as $\epsilon \to 0$, since $\Omega_\epsilon \to \varnothing$.

(iii) $\int_\Omega x_n\phi_{E_i}$ is continuous with respect to our sequence $\{E^h\}$, so both
$g\rho_i\int_\Omega x_n(\phi_{E_i}-\phi_{E_i^h})\,dx$ and $c|\beta_j-\beta_2|\int_{\Omega_\epsilon}(\phi_{E_j}-\phi_{E_j^h})\,dx \to 0$ as $h\to\infty$.

To satisfy the definition of lower semicontinuity, it is sufficient to prove

$$\limsup_{h\to\infty}[\mathcal{F}(E)-\mathcal{F}(E^h)] < 0.$$

Therefore it remains to show the following:

$$\limsup_{h\to\infty}\left\{\sum_{j=1,3}\left[\sqrt{1+L^2}|\beta_j-\beta_2|-\gamma_j\right]\int_{\Omega_\epsilon}|D\phi_{E_j^h}|-\gamma_2\int_{\Omega_\epsilon}|D\phi_{E_2^h}|\right\}\leq 0. \quad \text{(IV.13)}$$

Now this is trivial if $\gamma_j \geq |\beta_j-\beta_2|$, so instead suppose $\gamma_1 < |\beta_1-\beta_2| = \beta_2-\beta_1$. By rearranging terms, and the assumption $\gamma_i+\gamma_j \geq \sqrt{1+L^2}|\beta_i-\beta_j|$, we obtain

$$\begin{aligned}
&\sum_{j=1,3}\left(\sqrt{1+L^2}|\beta_j-\beta_2|-\gamma_j\right)\int_{\Omega_\epsilon}|D\phi_{E_j^h}|-\gamma_2\int_{\Omega_\epsilon}|D\phi_{E_j^h}|\\
&=\left[\sqrt{1+L^2}(\beta_2-\beta_1)-\gamma_1\right]\int_{\Omega_\epsilon}|D\phi_{E_1^h}|+\left[\sqrt{1+L^2}(\beta_3-\beta_2)-\gamma_3\right]\int_{\Omega_\epsilon}|D\phi_{E_3^h}|\\
&\qquad+\gamma_2\int_{\Omega_\epsilon}|D\phi_{E_2^h}|\\
&=\left[\sqrt{1+L^2}(\beta_2-\beta_1)-\gamma_1\right]\int_{\Omega_\epsilon}|D(1-\phi_{E_2^h}-\phi_{E_3^h})|\\
&\qquad+\left[\sqrt{1+L^2}(\beta_3-\beta_2)-\gamma_3\right]\int_{\Omega_\epsilon}|D\phi_{E_3^h}|+\gamma_2\int_{\Omega_\epsilon}|D\phi_{E_2^h}|\\
&=\left[\sqrt{1+L^2}(\beta_2-\beta_1)-\gamma_1-\gamma_2\right]\int_{\Omega_\epsilon}|D\phi_{E_2^h}|\\
&\qquad+\left[\sqrt{1+L^2}(\beta_3-\beta_1)-\gamma_1-\gamma_3\right]\int_{\Omega_\epsilon}|D\phi_{E_3^h}|\\
&\leq\sqrt{1+L^2}(|\beta_1-\beta_2|-|\beta_1-\beta_2|)\int_{\Omega_\epsilon}|D\phi_{E_2^h}|\\
&\qquad+\sqrt{1+L^2}(|\beta_1-\beta_3|-|\beta_1-\beta_3|)\int_{\Omega_\epsilon}|D\phi_{E_3^h}|\\
&=0.
\end{aligned} \quad \text{(IV.14)}$$

Thus $\mathcal{F}$ is lower semicontinuous. ■

V. BEMELMANS, GALDI, AND KYED

A recent paper by Bemelmans, Galdi, and Kyed [1] solves the energy minimization problem in the case of two fluids and one solid. Here, the energy functional is given by

$$\begin{aligned} \mathcal{F}(E) := \alpha \int_{\Omega \backslash \mathcal{B}} |D\phi_E| + \beta \int_{\partial\Omega} \phi_E \, d\mathcal{H}^{n-1} + \tau \int_{\partial\mathcal{B}} \phi_E \, d\mathcal{H}^{n-1} \\ + \rho g \int_{\Omega \backslash \mathcal{B}} x_n \phi_E \, dx + \rho_0 g \int_{\mathcal{B}} x_n \, dx \end{aligned} \tag{V.1}$$

Where $\Omega := G \times [0, p]$ is a bounded cylindrical container in $\mathbb{R}^3$, $\mathcal{B}$ is the solid, α is the coefficient of surface tension, β is the adension energy of the fluid on the container, ρ is the density of the solid, g is the gravitational constant, and τ is the adhesion energy of the fluid on the solid. In general, we denote $\mathcal{B} = \mathcal{B}(c, R)$, where c is a translation in $\mathbb{R}^3$ and R is a rotation. We wish to minimize $\mathcal{F}$ in the class of elements

$$\begin{aligned} \mathcal{C} := \{(c, R, E) \big| c \in \mathbb{R}^3, R \in SO(3) \text{ such that } \mathcal{B}(c, R) \subset \Omega; \\ E \subset \Omega \backslash \mathcal{B}(c, R) \text{measurable with } \mathcal{L}^3(E) = V_0\}, \end{aligned} \tag{V.2}$$

where $SO(3)$ denotes the set of rotations about the origin in $\mathbb{R}^3$. First, we need to prove an analogue of Emmer's equation (III.11) for the case where the solid touches the boundary of the container, since now the boundary may no longer be Lipschitz, as illustrated in figure V.1.

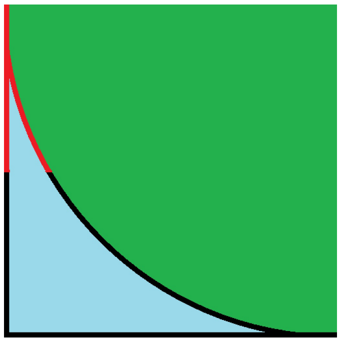

Figure V.1: A cusp is highlighted here. The domain of the fluid is non-Lipschitz at such a cusp.

To mitigate this problem, we first require $\partial\Omega$ to be of class C^2, and $\mathcal{B}(c, R)$ to have a projection $P(c, R)$ into Ω so that

$$\min_{R \in SO(3), x' \in \partial P(c,R)} K(\partial P(c, R), x') > max_{x' \in \partial\Omega} K(\partial\Omega, x'). \tag{V.3}$$

where $K(a, b)$ denotes the curvature of curve a at the point b. This constraint ensures that if $\mathcal{B}$ and $\partial\Omega$ touch, then the point of contact is at only one point p_0. In a neighborhood U_{ϵ_0} of p_0, we can describe $\partial\Omega$ by the graph of a function $p_3 = \omega(p_1, p_2)$, and $\partial\mathcal{B}$ by $p_3 = \beta(p_1, p_2)$, where $(p_1, p_2) = p' \in \mathbb{R}^2$ with $(0, 0) = p_0$. Notice that $\beta(p') \geq \omega(p')$, with equality only on the origin. Also notice the necessity of $\mathcal{B}$ and $\partial\Omega$ touching at a single point: we later integrate ω and β, which would not be possible if the point of contact was instead a line as then ω and β would not be continuous.

Our proof of Lemma III.0.2 breaks down when we use Ω_t, strips of constant width. Instead, we will now use strips of variable width. To that end, we fix an

$\epsilon > 0$ and define

$$\tau(p') = \begin{cases} \frac{\beta(p')-\gamma(p')}{3\epsilon} & if \beta(p') - \omega(p') \leq 3\epsilon \\ 1 & if \beta(p') - \omega(p') \geq 3\epsilon \end{cases}$$

$$\begin{aligned} &\beta^*(\delta) = \{p = (p', p_3) | p_3 = \beta(p') - \delta\tau(y)\} \\ &\omega^*(\delta) = \{p = (p', p_3) | p_3 = \omega(p') - \delta\tau(y)\} \\ &\mathcal{B}^*_\epsilon = \cup_{\delta \in (0,\epsilon)} \beta^*(\delta) \\ &\Omega^*_\epsilon = \cup_{\delta \in (0,\epsilon)} \omega^*(\delta) \end{aligned} \tag{V.4}$$

These last two sets form "strips" of variable length, of $1/3$ the local distance from $\partial\Omega$ to $\mathcal{B}$, and they will replace Ω_ϵ in our version of (III.11):

Lemma V.0.1 *Let $\partial\Omega$ and $\mathcal{B}$ touch in one point as described above. Then for $u \in BV(\Omega \backslash \mathcal{B})$,*

$$\int_{\partial\Omega \cap U_{\epsilon_0}} u \, d\mathcal{H}^{n-1} \leq \sqrt{1 + L^2} \int_{\Omega^*_\epsilon} |Du| + c \int_{\Omega^*_\epsilon} u \, dx. \tag{V.5}$$

Proof. Let $\delta \in (0, \epsilon)$, u_δ be the trace of u on $\omega^*(\delta)$, and $\Gamma_{\epsilon_0} = \partial\Omega \cap U_{\epsilon_0}$. We use the triangle inequality to obtain

$$\int_{\Gamma_{\epsilon_0}} u \, d\mathcal{H}^{n-1} \leq \int_{\Gamma_{\epsilon_0}} |u - u_\delta| \, d\mathcal{H}^{n-1} + \int_{\Gamma_{\epsilon_0}} |u_\delta| \, d\mathcal{H}^{n-1} \tag{V.6}$$

We estimate the first term on the right hand side by $|Du|$, using the definition of the Hausdorff integral and a boundary change:

$$\begin{aligned}\int_{\Gamma_{\epsilon_0}} |u-u_\delta|\, d\mathcal{H}^{n-1} &\leq \int_{A_{\epsilon_0}} |u(y',\omega(y')-u(y',\omega(y')+\delta\tau(y'))|\sqrt{1+|D\omega(y')|^2}\, dy' \\ &\leq \sqrt{1+L^2}\int_{A_{\epsilon_0}}\int_{\omega(y')}^{\omega(y')+\delta\tau(y')} \left|\frac{\partial u}{\partial y_3}\right| (y',t)\, dt\, dy' \\ &\leq \sqrt{1+L^2}\int_{\Omega_\epsilon^*} |Du|.\end{aligned} \tag{V.7}$$

We also use the definition of the Hausdorff integral on the second term of the right hand side of (V.6), as well as adding 0 in the form

$$\begin{aligned}&\int_{A_{\epsilon_0}} |u(y',\omega(y')+\delta\tau(y'))|\sqrt{1+|D(\omega(y')+\delta\tau(y'))|^2}\, dy' \\ &-\int_{A_{\epsilon_0}} |u(y',\omega(y')+\delta\tau(y'))|\sqrt{1+|D(\omega(y')+\delta\tau(y'))|^2}\, dy'\end{aligned} \tag{V.8}$$

to obtain

$$\begin{aligned}\int_{\Gamma_{\epsilon_0}} |u_\delta(y)|\, d\mathcal{H}^{n-1} = &\int_{A_{\epsilon_0}} |u(y',\omega(y')+\delta\tau(y'))|\sqrt{1+|D(\omega(y')+\delta\tau(y'))|^2}\, dy' \\ &+\int_{A_{\epsilon_0}} |u(y',\omega(y')+\delta\tau(y'))| \Big[\sqrt{1+|D\omega(y')|^2} \\ &-\sqrt{1+|D(\omega(y')+\delta\tau(y'))|^2}\Big]\, dy'\end{aligned} \tag{V.9}$$

Now, integrate the right hand side of (V.9) with respect to $\delta \in (0,\epsilon)$. The first term is the definition of the surface area for A_{ϵ_0}. After integrating with $\delta \in (0,\epsilon)$, we obtain the volume of a variable-width strip. This volume is given by $\int_{\Omega_\epsilon^*} |u|\, dy$. Similarly, the second term becomes $C\int_{\Omega_\epsilon^*} |u|\, dy$. We now integrate (V.6) with respect to $\delta \in (0,\epsilon)$ to obtain

$$\epsilon\int_{\Gamma_{\epsilon_0}} u\, d\mathcal{H}^{n-1} \leq \epsilon\sqrt{1+L^2}\int_{\Omega_{\epsilon^*}} |Du| + C_\epsilon\int_{\Omega_{\epsilon^*}} |u|\, dx, \tag{V.10}$$

with $C_\epsilon = 1 + C$. We therefore obtain (V.5). ∎

Theorem V.0.2 *Assume $\partial\mathcal{B}(c,R)$ and $\partial\Omega$ touch in at most one point. Let $\alpha, \rho, \rho_0, \beta, \tau \in \mathbb{R}$ such that $\rho > \rho_0 > 0, \alpha > 0, \alpha - |\beta|\sqrt{1-L^2} < 0, \alpha - |\tau|\sqrt{1+L^2} > 0$. Then there exists an element $(c_0, R_0, E_0) \in \mathcal{C}$ such that*

$$\mathcal{F}(c_0, R_0, E_0) \leq \mathcal{F}(c, R, E) \, \textit{for all} \, (c, R, E) \in \mathcal{C}. \tag{V.11}$$

Remark V.0.3 *Recall that*

$$\mathcal{F}(E) := \alpha \int_{\Omega \backslash \mathcal{B}} |D\phi_E| + \beta \int_{\partial\Omega} \phi_E \, d\mathcal{H}^{n-1} + \tau \int_{\partial\mathcal{B}} \phi_E \, d\mathcal{H}^{n-1}$$
$$+ \rho g \int_{\Omega \backslash \mathcal{B}} x_n \phi_E \, dx + \rho_0 g \int_{\mathcal{B}} x_n \, dx$$

Proof. We first show that $\mathcal{F}$ is bounded below in $\mathcal{C}$. By assumption $\alpha > 0$, so the first term is bounded below. For the second term, consider both the part of Ω covered by U_{ϵ_0}, if it exists, as well as the part outside of the point of contact. For the former, we have by Lemma V.0.1

$$\beta \int_{\partial\Omega \cap U_{\epsilon_0}} \phi_E \, d\mathcal{H}^{n-1} \geq -|\beta|\sqrt{1+L^2} \int_{\Omega_\epsilon^*} |D\phi_E| - |\beta| c \int_{\Omega_\epsilon^*} \phi_E \, dx.$$

For the part of Ω outside the point of contact, we have

$$\beta \int_{\Gamma \backslash U_{\epsilon_0}} \phi_E \, d\mathcal{H}^{n-1} \geq -|\beta|\sqrt{1+L^2} \int_{\Omega_\epsilon \backslash U_{\epsilon_0}} |D\phi_E| - c' \int_{\Omega_\epsilon \backslash U_{\epsilon_0}} \phi_E \, dx.$$

In both cases, the first term is maximized by

$$\alpha \int_{\Omega \backslash \mathcal{B}} |D\phi_E|,$$

and the second is finite since

$$\int_{\Omega_\epsilon^*} \phi_E \, dx \leq \mathcal{L}^3(E) = V_0.$$

For the third term of $\mathcal{F}$, we have

$$\tau \int_{\partial\mathcal{B}} \phi_E \, d\mathcal{H}^{n-1} \geq -|\tau| \, |\partial\mathcal{B}| > -\infty.$$

Finally, the last two terms are clearly positive by nature of g.

Therefore $\mathcal{F}$ is bounded below, and thus there exists a minimizing sequence $\{(C_n, R_n, E_n)\}$ from $\mathcal{C}$. We now need to prove that this sequence is bounded in $\mathcal{C}$. We may assume that $\mathcal{F}(C_n, R_n, E_n) \leq m_0 + 1$, and due to the boundedness of the $\rho g, \rho_0 g$, and τ terms, we have

$$\alpha \int_{\Omega\setminus\mathcal{B}} |D\phi_{E_n}| + \beta\partial\Omega\phi_{E_n} \, d\mathcal{H}^{n-1} \leq m_0 + 1 + c_1.$$

Using (V.5) we get

$$(\alpha - |\beta|)\sqrt{1+L^2} \int_{\Omega\setminus\mathcal{B}} |D\phi_{E_n}| \leq m_0 + 1 + c_1 + cV_0.$$

Due to the assumptions on α, β, V, we get

$$\|\phi_{E_n}\|_{L^1(\Omega)} + \int_{\Omega\setminus\mathcal{B}} |D\phi_{E_n}| \leq C.$$

Therefore ϕ_{E_n} is bounded in $BV(\Omega)$. Now the values of R_n belong to a compact set, so R_n is bounded for all $n \in \mathbb{N}$. Finally, the x_1, x_2 component of c_n are bounded by $diam(\Omega)$, and x_3 is bounded by

$$\rho_0 g \int_{\mathcal{B}} x_3 \, dx \leq m_0 + 1.$$

Therefore our sequence $\{(c_n, R_n, E_n)\}$ is bounded, so there exists a convergent subsequence, which we again denote $\{(c_n, R_n, E_n)\}$ for simplicity. Finally, we now claim that $\mathcal{F}$ is lower semicontinuous with respect to our convergent sequence. First, $\mathcal{B}(c_n, R_n)$ converges uniformly to a $\mathcal{B}(c_0, R_0)$, so the integral $\rho_0 g \int_{\mathcal{B}} x_3\, dx$ is continuous. Also, the integrand $x_3\phi_{E_n}(x)$ in the ρg term is non-negative, therefore the integral is lower semicontinuous via Fatou's Lemma.

Finally, in order to compute the traces of ϕ_{E_n} and ϕ_{E_0} on $\partial\mathcal{B}$, we set $E_n' = T_n E_n$, where T_n denotes the interpolated motion mapping $\mathcal{B}(c_n, R_n)$ to $\mathcal{B}(c_0, R_0)$. This E_n' generally is not an element of $\mathcal{C}$, nor even a subset of Ω. However, restricting$(\phi_{E_n'} - \phi_{E_0})$ to $\partial\mathcal{B}$ allows us to estimate using (V.6). It is then sufficient to show the lower semicontinuity of

$$\mathcal{A}(c, R, E) = \alpha \int_{\Omega\setminus\mathcal{B}} |D\phi_E| + \beta \int_{\partial\Omega} \phi_E \, d\mathcal{H}^{n-1} + \tau \int_{\partial\mathcal{B}} \phi_E \, d\mathcal{H}^{n-1}.$$

Using

$$\tau \int_{\partial\mathcal{B}(c_0,R_0)} \phi_{E_0} \, d\mathcal{H}^{n-1} - \tau \int_{\partial\mathcal{B}(c_n,R_n)} \phi_{E_n} \, d\mathcal{H}^{n-1} = \tau \int_{\partial\mathcal{B}(c_0,R_0)} \phi_{E_0} - \phi_{E_0'} \, d\mathcal{H}^{n-1},$$

as well as (V.6), we have

$$\begin{aligned}
&\mathcal{A}(c_0, R_0, E_0) - \mathcal{A}(c_n, R_n, E_n) \\
&\leq \alpha \int_{\Omega\setminus(\mathcal{B}^*_\epsilon(c_0,R_0)\cup\Omega^*_\epsilon)} |D\phi_{E_0}| + \alpha \int_{\mathcal{B}^*_\epsilon(c_0,R_0)} |D\phi_{E_0}| + \alpha \int_{\Omega^*_\epsilon} |D\phi_{E_0}| \\
&\quad - \alpha \int_{\Omega\setminus(\mathcal{B}^*_\epsilon(c_n,R_n)\cup\Omega^*_\epsilon)} |D\phi_{E_n}| - \alpha \int_{\mathcal{B}^*_\epsilon(c_n,R_n)} |D\phi_{E_n}| - \alpha \int_{\Omega^*_\epsilon} |D\phi_{E_n}| \\
&\quad + |\beta|\sqrt{1+L^2} \int_{\Omega^*_\epsilon} |D(\phi_{E_0} - \phi_{E_n}| + |\beta| c \int_{\Omega^*_\epsilon} |\phi_{E_0} - \phi_{E_n}| \, dx \\
&\quad + |\tau|\sqrt{1+L^2} \int_{\mathcal{B}^*_\epsilon(c_0,R_0)} |D(\phi_{E_0} - \phi_{E_n'}| + |\beta| c \int_{\mathcal{B}^*_\epsilon(c_0,R_0)} |\phi_{E_0} - \phi_{E_n'}| \, dx
\end{aligned} \tag{V.12}$$

Now we use the following on (V.12):

1. $\left(|\beta|\sqrt{1+L^2}-\alpha\right)\int_{\Omega_\epsilon^*}|D\phi_{E_n}| \leq 0$ for all $n \in \mathbb{N}$ by assumption.

2. Since $E_n \to E_0$, then $|\beta|c\int_{\Omega_\epsilon^*}|\phi_{E_0}-\phi_{E_n}|\,dx \to 0$ as $n \to \infty$, for all $\epsilon > 0$.

3. Similarly, using the triangle inequalty results in $\int_{\mathcal{B}_\epsilon^*(c_0,R_0)}|\phi_{E_0}-\phi_{E_n'}|\,dx$ $\leq \int_{\mathcal{B}_\epsilon^*(c_0,R_0)}|\phi_{E_0}-\phi_{E_n}|\,dx + \int_{\mathcal{B}_\epsilon^*(c_0,R_0)}|\phi_{E_n}-\phi_{E_n'}|\,dx$, and both integrals vanish as $n \to \infty$.

4. The Hausdorff measure $\mathcal{H}^2$ is invariant under rigid motions, therefore $|\tau|\sqrt{1+L^2}\int_{\mathcal{B}_\epsilon^*(c_0,R_0)}|D\phi_{E_n'}| - \alpha\int_{\mathcal{B}_\epsilon^*(c_n,R_n)}|D\phi_{E_n}|$ ≤ 0.

5. $\alpha\int_{\mathcal{B}_\epsilon^*(c_0,R_0)}|D\phi_{E_0}|, \alpha\int_{\Omega_\epsilon^*(c_0,R_0)}|D\phi_{E_0}|$ are of order $O(\epsilon)$, for all $\epsilon > 0$.

6. $\int_{\Omega\setminus\mathcal{B}_\epsilon^*(c_0,R_0)}|D\phi_{E_0}| \leq \liminf_{n\to\infty}\int_{\Omega\setminus\mathcal{B}_\epsilon^*(c_n,R_n)}|D\phi_{E_n}|$.

With these applied to $\mathcal{A}$, we obtain the lower semi-continuity of $\mathcal{A}$, and thus of $\mathcal{F}$, as desired. ∎

VI. NEW PROBLEM

We consider a bounded cylindrical container $\Omega := G \times [0, p]$, where G is a bounded, simply connected domain in $\mathbb{R}^2$. Ω is filled with three liquids: we denote the domain of each fluid by $E_i, i = 1, 2, 3$. The density of each fluid is given by ρ_i. In addition, there is also a rigid body $\mathcal{B}$ with density ρ_0 floating inside Ω. The interface between each fluid is assumed to be governed by surface tension. If we denote the surface tension between fluids i, j by α_{ij}, then we will consider the "surface tension coefficient" of each fluid by

$$\alpha_i := \frac{1}{2}(\alpha_{ij} + \alpha_{ik} - \alpha_{jk}), \tag{VI.1}$$

for $i, j, k = 1, 2, 3$ mutually distinct.The energy of the system Ω is assumed to be the following:

- g, the gravitational constant,
- the adhesion energy between fluid i and the boundary $\partial\Omega$ of the container, given by β_i,
- the adhesion energy between fluid i and the surface $\partial\mathcal{B}$ of the solid, given by τ_i.

We study the energy functional in the case where each fluid region is represented by a Caccioppoli set: each fluid's characteristic function is of bounded variation. Let $\mathcal{B}(c, R) := \{y = c + Rx : x \in \mathcal{B}\}$, where $c \in \mathbb{R}^3$ denotes a translation and $R = R(d, \theta) \in SO(3)$ describes a rotation with respect to some axis with unit vector d about some angle θ. The quantities c, R are restricted by requiring that the floating body is contained in Ω. We additionally assume that $\mathcal{B}(c, R)$ is a closed set. Then $E_i \subset \Omega \backslash \mathcal{B}(c,R)$ measurable, so denote $V_i = \mathcal{L}^3(E_i)$ to be the volume of each fluid. We also define $E = \{E_1, E_2, E_3\}$. Lastly, we shall use the notation (x', x_n) for $x \in \mathbb{R}^3$. The energy functional is then

$$\begin{aligned}\mathcal{F}(c,R,E) := \sum_{i=1}^{3} \Bigg(\alpha_i \int_{G\backslash\mathcal{B}(c,R)} |D\phi_{E_i}| + g\rho_i \int_{G\backslash\mathcal{B}(c,R)} x_n \phi_{E_i}\, dx \\ + \beta_i \int_{\partial\Omega} \phi_{E_i}\, d\mathcal{H}^2 + \tau_i \int_{\partial\mathcal{B}} \phi_{E_i}\, d\mathcal{H}^2 \Bigg) + g\rho_0 \int_{\mathcal{B}(c,R)} x_n\, dx,\end{aligned} \tag{VI.2}$$

where

$$\int_{G\backslash\mathcal{B}(c,R)} |D\phi_{E_i}| := sup \left\{ \int_{G\backslash\mathcal{B}(c,R)} \phi_{E_i}\, div(g)\, dx : g \in C_c^1(G\backslash\mathcal{B}(c,R); \mathbb{R}^3), \|g\|_{C^0} \leq 1 \right\}$$

denotes the total variation of ϕ_{E_i}, the two integrals $\partial\Omega$, $\partial\mathcal{B}$ denote the area of the wetted part of the container and the solid respectively. We prove that there is a minimizing configuration $(\mathcal{B}(c,R), E)$ to $\mathcal{F}$ in the class

$$\begin{aligned}\mathcal{C} :=& \{(c,R,E) : c \in \mathbb{R}^3, R \in SO(3) \text{ such that } \mathcal{B}(c,R) \subset G, \\ & E_i \subset G\backslash\mathcal{B}(c,R) \text{ measurable with } \mathcal{L}^3(E) = V_0\}.\end{aligned} \tag{VI.3}$$

We procced via a combination of Massari's proof [3] of the case where there is no solid, and Bemermans, Galdi, and Kyed's approach [1] in the case where there are only two fluids. Both of these in turn are based on Emmer's proof of the two-fluid, no-solid problem [2]. We begin by establishing a lower bound on $\mathcal{F}$, then forming a minimizing sequence of $(E_{i_h})_{h=1}^{\infty}$. We then prove that $\mathcal{F}$ is lower semicontinuous with respect to this sequence. In proving lower semicontinuity of $\mathcal{F}$, we use an adaptation of Emmer [2]. Emmer's original lemma fails in the case when $\mathcal{B}$ and $\partial\Omega$ touch, as the boundary of the fluids may not be Lipschitz. Instead, we require $\partial\Omega$ to be of class C^2 and $\mathcal{B}(c,R)$ to have a projection $P(c,R)$

into G such that

$$\min_{R\in SO(3),x'\in\partial P(c,R)} K(\partial P(c,R),x') > \max_{x'\in\partial\Omega} K(\partial\Omega,x'). \tag{VI.4}$$

This restriction ensures that all possible rotations of $\mathcal{B}$ give a projection curvature strictly larger than that of G, so if our solid and container touch, then they only touch at a single point p_0. Then for a neighbood U_{ϵ_0} of p_0, we can describe the boundaries $\partial\Omega$ and $\partial\mathcal{B}$ by the graphs of the respective functions $y_3 = \omega(y_1, y_2)$ and $y_3 = \beta(y_1, y_2)$, where $(y_1, y_2) \in A_{\epsilon_0} \subset E$ are cartesian coordinates with center $y'_0 = (0, 0)$ in the tangent plane to p_0, A_{ϵ_0} is the domain of both ω and β. We then have $\beta(y') > \omega(y')$ for all $y' \in A_{\epsilon_0}$. Fix $\epsilon > 0$ and define the following:

$$\begin{aligned}
\tau(p') &:= \begin{cases} \frac{\beta(p')-\gamma(p')}{3\epsilon} & if \beta(p') - \omega(p') \leq 3\epsilon \\ 1 & if \beta(p') - \omega(p') \geq 3\epsilon \end{cases} \\
\beta^*(\delta) &:= \{p = (p', p_3) | p_3 = \beta(p') - \delta\tau(y)\} \\
\omega^*(\delta) &:= \{p = (p', p_3) | p_3 = \omega(p') - \delta\tau(y)\} \\
\mathcal{B}^*_\epsilon &:= \cup_{\delta\in(0,\epsilon)}\beta^*(\delta) \\
G^*_\epsilon &:= \cup_{\delta\in(0,\epsilon)}\omega^*(\delta)
\end{aligned} \tag{VI.5}$$

These last two form strips of variable length that we use instead of Emmer's original fixed-length strips:

Lemma VI.0.1 *Let $\partial\Omega$ and $\mathcal{B}$ touch in one point as described in* (VI.4)*. Then for $u \in BV(\Omega\backslash\mathcal{B})$, we have*

$$\int_{\partial\Omega\cap U_{\epsilon_0}} u\, d\mathcal{H}^{n-1} \leq \sqrt{1+L^2}\int_{G^*_\epsilon} |Du| + c\int_{G^*_\epsilon} u\, dx, \tag{VI.6}$$

$$\int_{\partial\mathcal{B}(c,R)\cap U_{\epsilon_0}} u\,d\mathcal{H}^{n-1} \leq \sqrt{1+L^2}\int_{\mathcal{B}^*_\epsilon(c,R)}|Du| + c\int_{\mathcal{B}^*_\epsilon(c,R)} u\,dx. \tag{VI.7}$$

Proof. We begin by showing that the strips $\mathcal{B}^*_\epsilon$ and Ω^*_ϵ are locally Lipschitz at p_0. By assumption both $\omega(y')$ and $\beta(y')$ are regular, therefore Lipschitz. Let L be an upper bound for both $|D\omega|$ and $|D\beta|$. Then near p_0 we have

$$D\tau(y') = \frac{D\beta(y') - D\omega(y')}{3\epsilon} \leq \frac{2L}{3\epsilon} \tag{VI.8}$$

Since $\omega(y')$ and $\beta(y')$ are regular, we can force $L < \epsilon$ by covering $\omega(y')$ and $\beta(y')$ by some appropriate open cover $\mathcal{O}$. We then have $D\tau(y') \leq 2/3 < 1$ for all $y' \in A_{\epsilon_0}$. Now that we have established an upper bound for $D\omega(y')$, $D\beta(y')$, and $D\tau(y')$ for all $y' \in A_{\epsilon_0}$ we can then find an upper bound for $D\beta^*(\delta)$ and $D\omega^*(\delta)$ for $\delta \in (0,\epsilon)$. Therefore the strips $\mathcal{B}^*_\epsilon$ and Ω^*_ϵ are Lipschitz.

Next, let $\delta \in (0,\epsilon)$, u_δ be the trace of u on $\omega^*(\delta)$, and $\Gamma_{\epsilon_0} = \partial\Omega \cap U_{\epsilon_0}$. We use the triangle inequality to obtain

$$\int_{\Gamma_{\epsilon_0}} u\,d\mathcal{H}^{n-1} \leq \int_{\Gamma_{\epsilon_0}} |u-u_\delta|\,d\mathcal{H}^{n-1} + \int_{\Gamma_{\epsilon_0}} |u_\delta|\,d\mathcal{H}^{n-1} \tag{VI.9}$$

We estimate the first term on the right hand side by $|Du|$. Using the definition of the Hausdorff integral, we get

$$\int_{\Gamma_{\epsilon_0}} |u-u_\delta|\,d\mathcal{H}^{n-1} \leq \int_{A_{\epsilon_0}} |u(y',\omega(y')) - u(y',\omega(y')+\delta\tau(y'))|\sqrt{1+|D\omega(y')|^2}\,dy', \tag{VI.10}$$

and by using the Lipschitz property of $\omega(y')$, we have

$$\begin{aligned}&\int_{A_{\epsilon_0}} |u(y',\omega(y')) - u(y',\omega(y')+\delta\tau(y'))|\sqrt{1+|D\omega(y')|^2}\,dy' \\ &\leq \sqrt{1+L^2}\int_{A_{\epsilon_0}} |u(y',\omega(y')) - u(y',\omega(y')+\delta\tau(y'))|\,dy'.\end{aligned} \tag{VI.11}$$

By the Fundamental Theorem of Calculus, we obtain

$$
\begin{aligned}
&\sqrt{1+L^2}\int_{A_{\epsilon_0}} |u(y',\omega(y')) - u(y',\omega(y')+\delta\tau(y'))|\,dy' \\
&\leq \sqrt{1+L^2}\int_{A_{\epsilon_0}} \left|\int_{\omega(y')}^{\omega(y')+\delta\tau(y')} \frac{\partial u}{\partial y_3}(y',t)\,dt\right|\,dy' \\
&\leq \sqrt{1+L^2}\int_{A_{\epsilon_0}} \int_{\omega(y')}^{\omega(y')+\delta\tau(y')} \left|\frac{\partial u}{\partial y_3}(y',t)\right|\,dt\,dy' \\
&\leq \sqrt{1+L^2}\int_{G_\epsilon^*} |Du|.
\end{aligned}
\tag{VI.12}
$$

Use the definition of the Hausdorff integral on the second term of the right hand side of (VI.9), and add 0 in the form

$$
\begin{aligned}
&\int_{A_{\epsilon_0}} |u(y',\omega(y')+\delta\tau(y'))|\sqrt{1+|D(\omega(y')+\delta\tau(y'))|^2}\,dy' \\
&-\int_{A_{\epsilon_0}} |u(y',\omega(y')+\delta\tau(y'))|\sqrt{1+|D(\omega(y')+\delta\tau(y'))|^2}\,dy'
\end{aligned}
\tag{VI.13}
$$

to obtain

$$
\begin{aligned}
\int_{\Gamma_{\epsilon_0}} |u_\delta(y)|\,d\mathcal{H}^{n-1} = &\int_{A_{\epsilon_0}} |u(y',\omega(y')+\delta\tau(y'))|\sqrt{1+|D(\omega(y')+\delta\tau(y'))|^2}\,dy' \\
&+\int_{A_{\epsilon_0}} |u(y',\omega(y')+\delta\tau(y'))|\Big[\sqrt{1+|D\omega(y')|^2} \\
&-\sqrt{1+|D(\omega(y')+\delta\tau(y'))|^2}\Big]\,dy'
\end{aligned}
\tag{VI.14}
$$

We now integrate the right hand side of (VI.14) with respect to $\delta \in (0,\epsilon)$. Before integrating, the first term is the definition of the surface area for A_{ϵ_0}. After integrating with respect to δ from 0 to ϵ, we obtain the volume of a variable-width strip, given by $\int_{G_\epsilon^*}|u|\,dy$. The second term is also surface area, but shortened proportionally to $\delta\tau(y')$. By the same logic as before, the second term becomes $C\int_{G_\epsilon^*}|u|\,dy$. We now integrate (VI.9) with respect to $\delta \in (0,\epsilon)$ to obtain

$$\epsilon \int_{\Gamma_{\epsilon_0}} u\, d\mathcal{H}^{n-1} \leq \epsilon\sqrt{1+L^2}\int_{G_{\epsilon^*}} |Du| + C_\epsilon \int_{G_{\epsilon^*}} |u|\,dx, \tag{VI.15}$$

with $C_\epsilon = 1 + C$. We therefore obtain (VI.6). Equation (VI.7) can be proven similarly. ■

Lemma VI.0.2 *If $\alpha_i, \rho_i, \rho_0, \beta_i, \tau_i \in \mathbb{R}$ such that*

$$\begin{aligned} &\rho_i \geq 0,\ \rho_0 > 0, \\ &\alpha_i \geq 0,\ \alpha_i + \alpha_j > 0, \\ &\alpha_i + \alpha_j \geq \sqrt{1+L^2}|\beta_i - \beta_j|, \end{aligned} \tag{VI.16}$$

and $\partial\mathcal{B}(c,R)$ and $\partial\Omega$ touch in at most one point as in (VI.4)*, then $\mathcal{F}$ is bounded below.*

Proof. Index each fluid so that the β_i are in increasing order. Then, from Lemma VI.0.1 we have, near the point of contact,

$$\beta_i \int_{\partial\Omega\cap U_{\epsilon_0}} \phi_{E_i}\, d\mathcal{H}^{n-1} \geq -|\beta_i|\sqrt{1+L^2}\int_{G_\epsilon^*} |D\phi_{E_i}| - |\beta_i| c \int_{G_\epsilon^*} \phi_{E_i}\,dx.$$

Away from the point of contact, $\tau(y') = 1$, and our strips G_ϵ^* become $G_\epsilon\backslash U_{\epsilon_0}$, which have constant width. Then Lemma VI.0.1 becomes Emmer's original lemma [2]:

$$\beta_i \int_{\Gamma\backslash U_{\epsilon_0}} \phi_{E_i}\, d\mathcal{H}^{n-1} \geq -|\beta_i|\sqrt{1+L^2}\int_{G_\epsilon\backslash U_{\epsilon_0}} |D\phi_{E_i}| - c' \int_{G_\epsilon\backslash U_{\epsilon_0}} \phi_{E_i}\,dx.$$

In the right hand side of the previous two inequalities, by the assumption

$\alpha_i + \alpha_j \geq \sqrt{1+L^2}|\beta_i - \beta_j|$, the first term is maximized by

$$\alpha_i \int_{G \setminus \mathcal{B}} |D\phi_{E_i}|,$$

. The second is finite since

$$-\int_{G^*_\epsilon} \phi_{E_i}\, dx \geq -\mathcal{L}^3(E_i) \geq -\mathcal{L}^3(E) > -\infty.$$

For the third term of $\mathcal{F}$ we have

$$\tau_i \int_{\partial\mathcal{B}} \phi_{E_i}\, d\mathcal{H}^{n-1} \geq -|\tau_i|\,|\partial\mathcal{B}| > -\infty,$$

and the last two gravitational terms are clearly bounded below by some $r \leq 0$. Therefore $\mathcal{F}$ is bounded below. ■

Lemma VI.0.3 *Define*

$$\mathcal{A}(c,R,E) := \sum_{i=1}^{3} \left[\alpha_i \int_{G\setminus\mathcal{B}} |D\phi_{E_i}| + \beta_i \int_{\partial\Omega} \phi_{E_i}\, d\mathcal{H}^{n-1} + \tau_i \int_{\partial\mathcal{B}} \phi_{E_i}\, d\mathcal{H}^{n-1} \right]. \tag{VI.17}$$

Suppose that the hypotheses of Lemma VI.0.2 are satisfied. In addition suppose that we have a sequence $\{(c_j,R_j,E_j)\} \subset \mathcal{C}$ *and a* $(c_0,R_0,E_0) \in \mathcal{C}$ *with* $(c_j,R_j,E_j) \to (c_0,R_0,E_0)$ *a.e. Then* $\mathcal{A}(c_0,R_0,E_0) \leq \liminf_j \mathcal{A}(c_j,R_j,E_j)$.

Proof. In order to show our desired result, we subtract $\liminf_j \mathcal{A}(c_j,R_j,E_j)$ from both sides to obtain

$$\mathcal{A}(c_0,R_0,E_0) - \left[\liminf_j \mathcal{A}(c_j,R_j,E_j)\right] \leq 0.$$

Using the fact that $\liminf_k(a_k) = -\limsup_k(-a_k)$, we get

$$\limsup_j \left[\mathcal{A}(c_0, R_0, E_0) - \mathcal{A}(c_j, R_j, E_j)\right] \leq 0,$$

which we will use to show our conclusion.

Note that the β_i terms are labeled in increasing order, so that $\beta_1 < \beta_2 < \beta_3$. We then obtain the following:

$$\begin{aligned}
&\mathcal{A}(c_0, R_0, E_0) - \mathcal{A}(c_j, R_j, E_j)\\
&= \sum_{i=1}^{3}\Bigg[\alpha_i \int_{G\setminus\mathcal{B}} (|D\phi_{E_i^0}| - |D\phi_{E_i^j}|) + \beta_i \int_{\partial\Omega} (\phi_{E_i^0} - \phi_{E_i^j})\, d\mathcal{H}^{n-1}\\
&\quad +\tau_i \int_{\partial\mathcal{B}} (\phi_{E_i^0} - \phi E_i^j)\, d\mathcal{H}^{n-1}\Bigg]\\
&\leq \sum_{i=1}^{3}\left[\alpha_i \left(\int_{G\setminus\Omega_\epsilon^*} |D\phi_{E_i^0}| - \int_{G\setminus\Omega_\epsilon^*} |D\phi_{E_i^j}|\right) + \alpha_i \int_{G_\epsilon} |D\phi_{E_i^0}|\right]\\
&\quad + \sum_{k=1,3}\Bigg[\sqrt{1+L^2}|\beta_k - \beta_2| \int_{G_\epsilon^*} |D\phi_{E_k^0}|\\
&\qquad + (\sqrt{1+L^2}|\beta_k - \beta_2| - \alpha_k) \int_{G_\epsilon^*} |D\phi_{E_k^h}|\\
&\qquad + c|\beta_k - \beta_2| \int_{G_\epsilon^*} |\phi_{E_k^0} - \phi_{E_k^j}|\, dx\Bigg] + \alpha_2 \int_{G_\epsilon^*} |D\phi_{E_2^j}|\\
&\quad + \sum_{k=1,3}\Bigg[\sqrt{1+L^2}|\tau_k - \tau_2| \int_{\mathcal{B}_\epsilon^*} |D\phi_{E_k^0}|\\
&\qquad + (\sqrt{1+L^2}|\tau_k - \tau_2| - \alpha_k) \int_{\mathcal{B}_\epsilon^*} |D\phi_{E_k^j}|\\
&\qquad + c|\tau_k - \tau_2| \int_{\mathcal{B}_\epsilon^*} |\phi_{E_k^0} - \phi_{E_k^j}|\, dx\Bigg] + \alpha_2 \int_{G_\epsilon^*} |D\phi_{E_2^j}|
\end{aligned} \tag{VI.18}$$

Note that if we did not assume the τ_i were ordered, we could replace the last sum in the right hand side of (VI.18) with a different sum that excludes the middle τ_i.Therefore, without loss of generality assume that $\tau_1 \leq \tau_2 \leq \tau_3$, as we have for the β terms. We now show that the right hand side of (VI.18) vanishes as $j \to \infty$. As $\epsilon \to 0$, then:

- As $j \to \infty$, we have $E_i^j \to E_i^0$, so $\alpha_i \left(\int_{G-G_\epsilon} |D\phi_{E_i^0}| - \int_{G-G_\epsilon} |D\phi_{E_i^j}| \right)$ vanishes.

- Similarly, both $g\rho_i \int_G x_n(\phi_{E_i} - \phi_{E_i^h})\, dx$ and $c|\beta_j - \beta_2| \int_{G_\epsilon} |\phi_{E_j} - \phi_{E_j^h}|\, dx \to 0$ as $j \to \infty$

- $\gamma_i \int_{G_\epsilon} |D\phi_{E_i}|$ and $|\beta_j - \beta_2| \int_{G_\epsilon} |D\phi_{E_j}|$ clearly vanish as $\epsilon \to 0$.

Therefore, it remains to show the following:

$$\limsup_{h\to\infty} \Bigg\{ \sum_{k=1,3} \left[\left(\sqrt{1+L^2}|\beta_k - \beta_2| - \alpha_k \right) \int_{G_\epsilon^*} |D\phi_{E_k^j}| \right] + \alpha_2 \int_{G_\epsilon} |D\phi_{E_2^j}| \\ + \sum_{k=1,3} \left[\left(\sqrt{1+L^2}|\tau_k - \tau_2| - \alpha_k \right) \int_{\mathcal{B}_\epsilon^*} |D\phi_{E_k^j}|) \right] + \alpha_2 \int_{G_\epsilon} |D\phi_{E_2^j}| \Bigg\} \leq 0. \tag{VI.19}$$

This is trivial if $\alpha_k \geq |\beta_k - \beta_2|$, so instead suppose $\gamma_1 < |\beta_1 - \beta_2| = \beta_2 - \beta_1$. By rearranging terms, and the assumption $\alpha_i + \alpha_j \geq \sqrt{1+L^2}|\beta_i - \beta_j|$, we obtain

$$\begin{aligned}
&\sum_{k=1,3} \left(\sqrt{1+L^2}|\beta_j - \beta_2| - \alpha_k\right) \int_{G_\epsilon} |D\phi_{E_k^j}| - \alpha_2 \int_{G_\epsilon} |D\phi_{E_k^j}| \\
&= \left[\sqrt{1+L^2}(\beta_2 - \beta_1) - \alpha_1\right] \int_{G_\epsilon} |D\phi_{E_1^j}| + \left[\sqrt{1+L^2}(\beta_3 - \beta_2) - \alpha_3\right] \int_{G_\epsilon} |D\phi_{E_3^j}| \\
&\qquad + \alpha_2 \int_{G_\epsilon} |D\phi_{E_2^j}| \\
&= \left[\sqrt{1+L^2}(\beta_2 - \beta_1) - \alpha_1\right] \int_{G_\epsilon} |D(1 - \phi_{E_2^j} - \phi_{E_3^j})| \\
&\qquad + \left[\sqrt{1+L^2}(\beta_3 - \beta_2) - \alpha_3\right] \int_{G_\epsilon} |D\phi_{E_3^j}| + \alpha_2 \int_{G_\epsilon} |D\phi_{E_2^j}| \\
&= \left[\sqrt{1+L^2}(\beta_2 - \beta_1) - \alpha_1 - \alpha_2\right] \int_{G_\epsilon} |D\phi_{E_2^j}| \\
&\qquad + \left[\sqrt{1+L^2}(\beta_3 - \beta_1) - \alpha_1 - \alpha_3\right] \int_{G_\epsilon} |D\phi_{E_3^j}| \\
&\leq \sqrt{1+L^2}(|\beta_1 - \beta_2| - |\beta_1 - \beta_2|) \int_{G_\epsilon} |D\phi_{E_2^j}| \\
&\qquad + \sqrt{1+L^2}(|\beta_1 - \beta_3| - |\beta_1 - \beta_3|) \int_{G_\epsilon} |D\phi_{E_3^j}| \\
&= 0.
\end{aligned} \tag{VI.20}$$

This takes care of the first sum of (VI.19). The second sum is handled in the same way. Therefore $\mathcal{A}$ is lower semicontinuous. ∎

Lemma VI.0.4 *Suppose that the hypotheses of Lemma VI.0.3 are satisfied. Then* $\mathcal{F}(c_0, R_0, E_0) \leq \liminf_j \mathcal{F}(c_j, R_j, E_j)$.

Proof. By assumption $\mathcal{B}(c, R)$ is a rigid object, and by hypothesis we have a convergent sequence $(c_j, R_j, E_j) \to (c_0, R_0, E_0)$. Thus $\mathcal{B}(c_j, R_j)$ converges uniformly to $\mathcal{B}(c_0, R_0)$, so our $\rho_0 g$ term is continuous by the Uniform Limit Theorem. For the $\rho_i g$ terms, the integrand is non-negative, therefore the integral is lower semicontinuous via Fatou's Lemma. We then only need to show the lower semicontinuity of $\mathcal{A}(c, R, E)$, which follows from Lemma VI.0.3. ∎

Theorem VI.0.5 *If* $\alpha_i, \rho_i, \rho_0, \beta_i, \tau_i \in \mathbb{R}$ *such that* $\rho_i \geq 0, \rho_0 > 0, \alpha_i \geq 0, \alpha_i + \alpha_j > 0, \alpha_i + \alpha_j \geq \sqrt{1+L^2}|\beta_i - \beta_j|$, *and* $\partial\mathcal{B}(c, R)$ *and* $\partial\Omega$ *touch in at most one point as in* (VI.4)*, then there exists a* $(c_0, R_0, E_0) \in \mathcal{C}$ *such that for all* $(c, R, E) \in \mathcal{C}$,

$$\mathcal{F}(c_0, R_0, E_0) \leq \mathcal{F}(c, R, E).$$

Proof. By Lemma VI.0.2, $\mathcal{F}$ is bounded from below, and we can form a minimizing sequence $\{C_j, R_j, E_j\} \subset \mathcal{C}$. We may assume that $\mathcal{F}(C_j, R_j, E_j) \leq m_0 + 1$, and due to the boundedness of the $\rho_i g, \rho_0 g$, and τ_i terms, we have

$$\alpha_i \int_{G\setminus\mathcal{B}} |D\phi_{E_{j_i}}| + \beta\partial\Omega\phi_{E_{j_i}}\, d\mathcal{H}^{n-1} \leq m_0 + 1 + c_1.$$

Using (VI.6) we get

$$(\alpha_i - |\beta_i|)\sqrt{1+L^2} \int_{G\setminus\mathcal{B}} |D\phi_{E_{j_i}}| \leq m_0 + 1 + c_1 + cV_0.$$

Due to the assumptions on α_i, β_i, V, we get

$$\|\phi_{E_{j_i}}\|_{L^1(G)} + \int_{G\setminus\mathcal{B}} |D\phi_{E_{j_i}}| \leq C, \text{ i} = 1,\ 2,\ 3. \tag{VI.21}$$

Therefore ϕ_{E_i}, $i = 1, 2, 3$ are bounded in $BV(\Omega)$. Now the values of R_j belong to a compact set, so $|R_j|$ is bounded in $\mathbb{R}$ for all $n \in \mathbb{N}$. Finally, the x_1, x_2 component of c_j are bounded by $diam(G)$, and x_3 is bounded since

$$0 \leq \rho_0 g \int_{\mathcal{B}(c_j, R_j)} x_3\, dx \leq m_0 + 1,$$

and everything but x_3 is fixed.

Therefore our sequence is bounded in the sense that $|c_j| + |R_j| + \|\phi_{E_j}\|_{BV(\Omega)} \leq C$ for all $n \in \mathbb{N}$. Since our sequence was a minimizing sequence, we can form a minimizing convergent subsequence, which we again call $\{(c_j, R_j, E_j)\}$. By Lemma VI.0.4, $\mathcal{F}$ is lower semi-continuous with respect to this sequence, hence the sequence converges in $\mathcal{C}$ to some (c_0, R_0, E_0), as desired. ■

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