BOUNDING THE INDEX OF A NORMAL SUBGROUP WITH A LARGE CLASS SIZE

by

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I. INTRODUCTION

This thesis considers an extension of a problem in finite group theory. Finite group theory is the study of finite sets of objects related to one another by an associative binary operation i.e. multiplication, addition, or function composition. Often, these structures represent various forms of symmetries, and one may study these symmetries using a group. In this thesis, we study a generalization of a group theoretic analogue to a character theoretic parameter. The original group theoretic analogue studied by A. Harrison was inspired by a series of publications starting with N. Snyder's paper and an extension of Snyder's parameter was the subject of C. Durfee's dissertation [2]. For an in depth explanation of the group theoretic analogue see A. Harrison [5] and for the original parameter see N. Snyder [10].

In [2], Durfee studied a parameter that was essentially Snyder's e relative to a fixed normal subgroup. The parameter was defined as:

Definition. *Let N be a normal subgroup of a finite group G. Let ϕ and θ be irreducible characters of G and N, respectively, such that θ is fixed by the conjugation action of G and ϕ restricts to a multiple of θ on N. Let $d = \frac{\phi(1)}{\theta(1)}$, and define e by $|G/N| = d(d+e)$.*

Durfee's parameter e is always a non-negative integer. In her dissertation, she studies the case where $e = 1$ and $e = 2$ and then proceeds to study this parameter when the group is either supersolvable or nilpotent. This will give us direction for our work. Her main result, in her paper [3], was the following:

Theorem 1. *Let N be a normal subgroup of a finite group G, where G/N is solvable and let ϕ be an irreducible character of N that is G-invariant. Let θ be an irreducible character of N that is a multiple of θ and let $d = \frac{\phi(1)}{\theta(1)}$. Write $|G : N| = d(d+e)$ for some non-negative integer e. If $e > 1$ and $d > e^5 - e$, then we can find* *groups X and Y such that:*

1. $N \leq X \lhd Y \leq G$
2. $|Y/X| = (\frac{d}{e})(\frac{d}{e}+1)$
3. *Y/X is either a group of order 2 or is a 2-transitive Frobenius group.*

There is a close connection between characters and conjugacy classes that often motivates a problem for one based on a result of the other. Theorems produced in one line of research often generate similar statements in the other. Due to the aforementioned research, a group theoretic analog to Durfee's parameter is studied in this thesis which was motivated by A. Harrison's paper [5] and his thesis [4].

Our paper continues where A. Harrison's paper ended, and we defer to his paper for the construction of the non-relative parameter [5]. Our approach is to study the relative parameter corresponding to a fixed minimal normal subgroup N of a finite group G by placing an upper bound on $|G : N|$ in terms of the relative parameter. We then classify those groups in which $|G : N|$ achieves equality with the upper bound. We finish our paper by determining what conditions the relative parameter would possibly need to ensure a Frobenius group.

Our paper begins with a review of relevant material and we devote the last chapter to our main results. An understanding of group theory is assumed, although where possible we try to present all supporting material. For certain results we will omit proof and send the reader to the appropriate source for proof, because including proof would lengthen this document significantly. All conjectures were tested in GAP (Groups, Algorithms and Programming), and portions of the relevant code are attached in the appendix. Lastly, our notation will closely follow the notation used in Isaac's *Finite Group Theory* [7] and other notation will be introduced as needed.

II. NILPOTENT, P-POWER, & SUPERSOLVABLE GROUPS

Nilpotent groups and p-groups

All groups are assumed to be finite from here on. We begin our discussion with p-groups and nilpotent groups, the latter of which plays a significant role in the results of the thesis. While some familiarity with the properties of these groups is assumed, we will review the relevant properties.

Now, p-groups are not typical of groups in general, but they play a prominent role and are ubiquitous in the study of group theory. This is of course a consequence of Sylow theory. The structure of p-groups is different when compared with groups in general, but the number of p-groups is quite large when compared to the number of groups which are not of prime power order. For example, there are 51 isomorphism types of groups of order 2^5 and 267 isomorphism types of groups of order 2^6. This continues to grow at a large rate. To contrast this, we have that there are only 175 isomorphism types of groups with order between 2^5 and 2^6. So the amount of p-groups is quite staggering when compared to how many groups exists when the group order is a product of different primes.

A fundamental fact about p-groups is that they have nontrivial centers which intersect normal subgroups non-trivially.

Theorem 2. *Let P be a p-group and let N be a non-identity normal subgroup of P. Then $|N \cap Z(P)| > 1$. In particular, if P is nontrivial, then $|Z(P)| > 1$.*

Proof. Let P act on N via conjugation. Now, observe that $N \cap Z(P)$ is the set of elements of N that lie in orbits of size 1. By the fundamental counting principle, every orbit has p-power size. Therefore, each nontrivial orbit has size divisible by p. Since the set $N - (N \cap Z(P))$ is a union of such orbits, we have that $|N \cap Z(P)| \equiv |N| \equiv 0 \ (mod\ p)$.

Now $N \cap Z(P)$ contains the identity element, and so $|N \cap Z(P)| > 0$. It follows that $|N \cap Z(P)| \geq p > 1$, and hence $N \cap Z(P)$ is nontrivial as desired. The final assertion follows by taking $N = P$ $\square$

We will now discuss what it means for a group to be nilpotent. A group G is said to be *nilpotent* if it has a series defined as follows:

$$1 = N_0 \lhd N_1 \lhd N_2 \lhd \cdots \lhd N_{k-1} \lhd N_k = G$$

such that N_{j-1} is normal in N_j and N_j/N_{j-1} is contained in $Z(G/N_{j-1})$, for $j = 1, 2, ..., k$. We call this type of series a *central series.*

These groups admit spectacular properties and provide useful tools for us to make use of. It is worth observing that subgroups and quotient groups of nilpotent groups are themselves nilpotent.

Now, given a group G, one can attempt to construct a central series as follows. Begin by defining $Z_0 = 1$ and $Z_1 = Z(G)$. The *second center* Z_2 is defined to be the unique subgroup such that $Z_2/Z_1 = Z(G/Z_1)$. Continue this process inductively by defining Z_n such that $Z_n/Z_{n-1} = Z(G/Z_{n-1})$. This chain of normal subgroups:

$$Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \cdots$$

constructed this way is called the *upper central series* of G. Now, this may or may not be the central series for G. But if $Z_r = G$ for some r, then this is in fact a central series and G is nilpotent. Conversely, if G is nilpotent then the upper central series of G is a central series of G [7].

Now, it is simple to show that p-groups are nilpotent, which is indicative of why we choose to discuss them together. So by studying general nilpotent groups we may obtain additional information about p-groups. We make use of a property regarding nilpotent groups that is similar to Theorem (2).

Theorem 3. *If G is a nilpotent group and $N \triangleleft G$ then $N \cap Z(G)$ is non-trivial. Furthermore, if N is a minimal normal subgroup of G then N is of prime order.*

We provide proof in the appendix. We now arrive at a useful characterization of nilpotent groups.

Theorem 4. *Let G be a group. Then the following are equivalent:*

1. *G is nilpotent.*
2. *$N_G(H) > H$ for every proper subgroup $H < G$.*
3. *Every maximal subgroup of G is normal.*
4. *Every Sylow subgroup of G is normal.*
5. *G is the direct product of its nontrivial Sylow subgroups.*

In statement (3), a *maximal* subgroup M of a group G is a proper subgroup, such that no proper subgroup K strictly contains M. Now, to prove this, we need a theorem and a lemma [7].

Theorem 5. (**Frattini**) *Let N be normal in G, and suppose that $P \in Syl_p(N)$. Then $G = N_G(P)N$.*

Proof. Let $g \in G$. Since $P^g \subseteq N$ it follows that $P^g \in Syl_p(N)$. Now, by the Sylow conjugacy theorem applied in N, we deduce that $(P^g)^n = P$, for some element $n \in N$. Since $P^{gn} = P$, we have $gn \in N_G(P)$, and so $g \in N_G(P)n^{-1} \subseteq N_G(P)N$. The result follows. $\square$

Lemma 1. *Let $\mathcal{O}$ be a collection of normal subgroups of a group G, and assume the orders of members of $\mathcal{O}$ are pairwise coprime. Then the product $H = \prod \mathcal{O}$ of members of $\mathcal{O}$ is direct.*

Proof. Certainly $|H| \leq \prod |X|$. Also, by LaGrange's theorem, $|X|$ divides $|H|$, for every member X of $\mathcal{O}$, and since the orders of the members of $\mathcal{O}$ are pairwise co-prime, it follows that $\prod |X|$ divides $|H|$. So, we have $|H| = \prod |X|$.

To see that $H = \prod \mathcal{O}$ is direct it suffices to show that:

$$X \cap \prod \{Y \in \mathcal{O} \, | Y \neq X\} = 1$$

for every member $X \in \mathcal{O}$. This follows since by the previous paragraph, the order of $\prod Y$ for $Y \neq X$ is equal to $\prod |Y|$, and this is co-prime to $|X|$ □

We are now ready to prove Theorem (4).

Proof. **Theorem 4.** We will begin by showing (1) implies (2). Let H be a subgroup of G. Since G is nilpotent, it has a central series, and thus there is some k, where $0 \leq k < r$, such that $N_k \subseteq H$ but $N_{k+1} \nsubseteq H$. Since the subgroups N_i form a central series, we have:

$$N_{k+1}/N_k \subseteq Z(G/N_k) \subseteq N_{G/N_k}(H/N_k) = N_G(H)/N_k$$

where equality holds because $N_k \subseteq H$. Now since $N_{k+1} \subseteq N_G(H)$, the claim follows.

That (2) implies (3) is immediate, since if M is a maximal subgroup of G then its normalizer must be G and so $M \triangleleft G$.

Now assume (3), we will show (4) by contradiction. Suppose there is a Sylow subgroup of G, which is not normal in G. Let $P \in Syl_p(G)$, be such a Sylow subgroup. Clearly, $N_G(P)$ must be contained in some maximal subgroup M of G, which by our hypothesis is normal in G. Now by Fratinni, we have that $G = N_G(P)M \subseteq M$, and this is a contradiction. Thus $P \triangleleft G$.

By Lemma (1), we have that (5) follows from (4). Now, suppose (5). It is clear that (4) holds, and it follows that (4) also holds for every homomorphic image of G. Now, since (4) implies (5), we see that every homomorphic image of G is a direct product of p-groups, for various primes p. The center of a direct product is the direct product of the centers of the factors, and since nontrivial p-groups have nontrivial centers, it follows that every non-identity homomorphic image of G has a nontrivial center. So, it follows that if $Z_i \subset G$ then $Z_{i+1}/Z_i = Z(G/Z_i)$ is nontrivial and thus $Z_i \subset Z_{i+1}$. Thus, G must appear as a member of the upper central series, and since the proper terms of the upper central series are strictly increasing. This proves the claim. □

Supersolvable

This section addresses a weaker group property then when a group is nilpotent, it is termed supersolvable. A group is said to be *supersolvable* if there exists normal subgroups N_i with

$$1 = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_{r-1} \subseteq N_r = G$$

and where each N_i/N_{i-1} is cyclic, for $1 \leq i \leq r$. We have not, and will not discuss solvable groups in this thesis, but we are inclined to say clearly supersolvable groups are solvable, and it is routine to check that subgroups and factor groups of supersolvable groups are supersolvable. The relevant information for solvable groups can be found in *Finite Group Theory* [7].

We can say significantly more about supersolvable groups. In particular, every minimal normal subgroup of a supersolvable group G has prime order. Note that a nontrivial subgroup of a group is termed a *minimal* normal subgroup if it is normal and the only normal subgroup properly contained in it is the identity subgroup. Let N be a minimal normal subgroup of supersolvable group G. Since all supersolvable groups are solvable then N is an elementary abelian p-group, for some prime p. So, there exists some i, where $1 \leq i \leq r$, such that $N \cap N_i = N$. By the minimality of N, we have $N \cap N_{i-1} = 1$. By the homomorphism theorems, we have $N \cong N_i/N_{i-1}$. Since G is supersolvable, we may conclude that N is an elementary abelian p-group which is cyclic. So, by definition of a elementary abelian group, a generator of N must have order p, where p is some prime. So, we may conclude that N is of prime order.

Certainly this is not the only property that these groups possess. In fact every maximal subgroup of a supersolvable group has prime index. To see this take a maximal subgroup M of a supersolvable group G. Now, there exists a minimal normal subgroup N of prime order p. If P is a subgroup of M, then M/P is a maximal subgroup of G/P and by induction one would see that M is of prime index in G. Otherwise, G splits over P and the claim holds. This brings us to the following theorem which we make use of in one result in the last section.

Theorem 6. *If n is a divisor of the order of a supersolvable group G, then G has a subgroup of order n.*

Proof. We will prove the claim by induction, and we will induct on the order of G. Let G be a supersolvable group of order 1, then the claim holds and so our induction begins. Let G be a supersolvable group of order m and N a minimal normal subgroup of G. We have $|N| = p$, for some prime p. Now, let n be any divisor of the order G. We have two cases: either $p \mid n$ or $p \nmid n$.

1. Suppose $p \mid n$. Then $n/p \mid m/p$. Since G/N is supersolvable of order less then m, then by our inductive hypothesis there exists a subgroup H/N of order n/p. So by the homomorphism theorems, H is a subgroup of G of order n.

2. Suppose that $p \nmid n$. Since G/N is supersolvable of order less then m, by our inductive hypothesis there exists a subgroup H/N of order n. So by the homomorphism theorems, H is a subgroup of G of order pn. Now, since $N \triangleleft H$ and $|N|$ is coprime to $|H : N|$, we have that N is complemented in H by M. Which gives us that $MN = H$, and the subgroup we seek is M.

This proves the claim. $\square$

III. DIHEDRAL, QUATERNION, & SEMI-DIHEDRAL GROUPS

The following groups are involved in the conclusion of one of our theorems, thus we are inclined to discuss some basic properties. Familiarity with these groups is assumed.

Dihedral Groups

A group D is said to be *dihedral* if it contains a nontrivial cyclic subgroup C of index two such that every element of $D - C$ is an involution.

Now, suppose $C = < c >$ is a nontrivial cyclic subgroup of D with index two and that for all $t \in D - C$ we have that t is an involution. (Since $|D : C| = 2$, it follows that C is normal in D and D is of even order; further, we may conclude that dihedral groups have order at least four). Then it follows that every element of $D - C$ is an involution if and only if $c^t = c^{-1}$.

Let $x \in C$ and $y \in D - C$. Then $y = at$ for some element $a \in C$, and we have $x^y = x^{at} = x^t = x^{-1}$. Also, ct is an involution, and the group $< ct, t >$ properly contains C, and thus is the whole group D. Finally, ct and t are distinct since $c \neq 1$. So any dihedral group must have two generators. Note that it is also true that a dihedral group is determined, up to isomorphism, by its order.

Now, a dihedral group of order $2n$ has the following presentation:

$$D_{2n} = \left\{a, b : a^n = b^2 = 1, b^{-1}ab = a^{-1}\right\}$$

These groups are well-understood and may be viewed as a semi-direct product of $C_n \rtimes_\phi C_2$, where C_n is the cyclic group of order n and $\phi : C_2 \to Aut(C_n)$ is the map that sends the identity element in C_2 to the identity in $Aut(C_n)$, and the remaining element in C_2 to the automorphism $c \to c^{-1}$. Furthermore, $(1, t)$ has the desired conjugation action on $(c, 1)$.

For more information refer to *Finite Group Theory* [7]. We present the next proposition to contrast it with the subsequent sections.

Proposition 1. *Let G be dihedral of order $2n$, where $n > 2$. Then we have the following:*

1. *The subgroup $< a >$ is of index two.*
2. *If n is even then $Z(G)$ is cyclic of order 2 and if n is odd then the $Z(G)$ is trivial.*
3. *The group $G/Z(G)$ is a dihedral group.*

Proof. We will begin by proving (1). Let G be dihedral and $a \in G$ such that the order of a is n. Since the order of a is n, then we have $|G : < a > | = 2$. This shows (1).

We now show (2). Let $b \in G$ such that the order of b is 2 and let a be as above. Now, every element of $G - < a >$ can be written as ba^i for some i, such that $0 \le i \le 2^{n-1}$. Every member of G that can be written this way is not in $Z(G)$; otherwise we would have that the order of a is 2 and this is a contradiction since the order of a is n. Now, if $a^i \in Z(G)$, for $0 \le i \le n-1$, then $b^{-1}a^i b = a^i$ and we have $a^{2i} = 1$. From this, we may conclude that $n \mid 2i$. If n is even, then we have that $\frac{n}{2} \mid i$, and it follows that either $i = 0$ or $i = \frac{n}{2}$. If n is odd, then $n \mid i$ and we have that $i = 0$. This shows (2).

We now show (3). If n is odd then $Z(G)$ is trivial and the claim holds. So suppose that n is even. We have that $Z(G)$ is a subgroup of C. Now, using the "bar convention", we know that $\overline{C}$ is cyclic and that $[\overline{D} : \overline{C}] = [D : C] = 2$. Furthermore, for all $b \in D - C$, $\overline{b} \in \overline{D} - \overline{C}$, and $\overline{b}^2 = \overline{bb} = \overline{1}$. Hence, $\overline{D}$ is dihedral. $\square$

Semi-Dihedral Groups

The next group we discuss are the semi-dihedral groups, typically denoted as SD. A group SD is said to be *semi-dihedral* if it contains a nontrivial cyclic subgroup C of order n with n divisible by 8 such that C has a unique involution z. It follows that C has a unique automorphism σ such that $c^\sigma = c^{-1}z$, for every generator c of C, and that the order of σ is two. Now, let SD and C be as above. We may write $SD = C \rtimes < \sigma >$, and we have that half the elements of $SD - C$ have order 2 and the other half have order 4. Furthermore, the elements of order 2 in $SD - C$ form a single conjugacy class of SD, and similarly for the elements of order 4 [7].

The semi-dihedral groups, denoted SD_{2^n} for some natural number $n > 3$, have the following presentation:

$$SD_{2^n} = \left\{a, b : a^{2^{n-1}} = b^2 = 1, (ba)^2 = a^{2^{n-2}}\right\}$$

We present a similar proposition as before.

Proposition 2. *Let G be semi-sihedral. Then we have the following:*

1. *The subgroup $< a >$ is of index two in G.*

2. *The $Z(G)$ is cyclic of order two.*

Proof. We will begin by showing (1). Let G be semi-dihedral and $a \in G$, such that the order of a is 2^{n-1}. Since the order of a is 2^{n-1}, we have $|G : < a > | = 2$. This shows (1).

We now show (2). Let $b \in G$ such that the order of b is 2 and let a be as above. Since $a^{2^{n-2}} = b$, then we have that $a^{2^{n-2}}$ commutes with all of G. So, it follows that $a^{2^{n-2}} \in Z(G)$. Now, if $a^i \in Z(G)$, for $0 \leq i \leq 2^{n-1}$, we would have that it must commute with b. This gives us that $b^{-1}a^ib = a^i$ and we may conclude, by the defining relations on G, that $a^{(2^{n-2}-1)i} = 1$. So, we have $2^{n-1} \mid (2^{n-2} - 1)i$. Which implies that a^i is a power of $a^{2^{n-1}}$, so the only power of a in the center is $a^{2^{n-2}}$.

Every member of $G - < a >$, can be written as ba^i for some i such that $0 \leq i \leq 2^{n-1}$. Now, each member of G with this form is not in $Z(G)$; otherwise we have $b^2ba^ib = a^i$, which gives us $a^i = a^{2^{n-2}}$. This proves (2) and completes the proof. □

Generalized-Quaternion Groups

The last type of group we are concerned with are the generalized quaternion groups. The generalized quaternions are similar to that of the semi-dihedral groups. To see this, take a semi-dihedral group SD and let B be a subgroup of the cyclic subgroup C of SD, with the property that $|C : B| = 2$. Now, the elements of order 4 in $SD - C$ form a coset of B and if we let Q be the union of this coset and B, then Q is a subgroup of n. This group Q is the *generalized quaternion* group, typically denoted as Q_{2^n} where $n \geq 3$; we will drop the adjective "generalized" from here on and refer to these groups as quaternion. Note that since all elements of $Q - B$ have order 4, the involution in B is the unique involution in Q [7].

The quaternion group has the following presentation:

$$Q_{2^n} = \left\{a, b : a^{2^{n-2}} = b^2, a^{2^{n-1}} = 1, b^{-1}ab = a^{-1}\right\}$$

Observe that the group structure for the quaternion group is similar to that of the dihedral and semi-dihedral group and thus is of no surprise that properties of these groups are related. We present a similar proposition to those as before.

Proposition 3. *Let G be generalized quaternion. Then we have the following:*

1. *The subgroup $< a >$ is of index* 2.

2. *The center of G is cyclic of order 2.*

3. *The group $G/Z(G)$ is isomorphic to the dihedral group of order 2^{n-1}.*

Proof. We will begin by showing (1). Let G be generalized quaternion and $a \in G$, such that a has order 2^{n-1}. Since the order of a is 2^{n-1}, then we have $|G : < a > | = 2$. This proves (1).

We now show (2). Let $b \in G$ such that the order of b is 4 and let a be as above. Since $a^{2^{n-2}} = b^2$, then we have that $a^{2^{n-2}}$ commutes with all of G. So, it follows that $a^{2^{n-2}} \in Z(G)$. Now, no power of a is in $Z(G)$ except for the one previously mentioned; since if $a^i \in Z(G)$, for $0 \le i \le 2^{n-1}$, then we would have that $(b^{-1}a^ib) = a^i$ which gives us that $a^{2i} = 1$. It follows that $2^{n-2} \mid i$, which implies that a^i is a power of $a^{2^{n-2}}$.

Every member of $G- < a >$ can be written as ba^i, where $0 \le i \le 2^{n-1}$, and none of these members are in $Z(G)$. Since, if an element of this presentation were in $Z(G)$, we would have $a^{2i} = 1$. Which is an identical argument as above. This proves (2).

We now show (3). Using the "bar convention", the factor group $G/Z(G)$ has generators $\overline{a}$ and $\overline{b}$ such that $\overline{a}^{2^{n-2}} = \overline{1}$, $\overline{b}^2 = \overline{1}$, and $\overline{bab}^{-1} = \overline{a}^{-1}$. Therefore, the factor $G/Z(G)$ is the homomorphic image of $D_{2^{n-1}}$. This completes the proof. □

All three of these groups have similar properties. For example, all three groups have analogous generators, their centers are cyclic of order 2, and all three have derived subgroups of index 4 (we did not prove this fact; but it can easily be shown). Furthermore, there are exactly four isomorphism classes of non-abelian groups of order 2^n which contain a cyclic subgroup of index two, three of which we have just discussed. The three groups we have briefly discussed are also the only groups of order 2^n, up to isomorphism, that have a nilpotency class of $n - 1$. But, these groups have differences though. For example, half the elements in a dihedral group have order 2 and half the elements in a generalized quaternion group have order 4. In fact, there is only one element of order 2 in a quaternion group. It is the nontrivial element in the center. Also, it is of worth observing that every subgroup of a quaternion group is cyclic or generalized quaternion. This is the case since there is a unique subgroup of order 2 in a quaternion group, so every nontrivial subgroup has a unique subgroup of order 2.

IV. FROBENIUS GROUPS

Frobenius groups play a role in our thesis, so we are inclined to include the relevant information

We begin with group actions. Let G be a group and Ω a nonempty set. Suppose we have a rule that determines an element of Ω, denoted $\omega \cdot g$, whenever we are given a point $\omega \in \Omega$ and an element $g \in G$. We say this rule defines an *action* of G on Ω if this rule also satisfies:

1. $\omega \cdot 1 = \omega$ for all $\omega \in \Omega$ and
2. $(\omega \cdot g) \cdot h = \omega \cdot (gh)$ for all $\omega \in \Omega$ and for all $g, h \in G$.

For an example, let G be a group and $\Omega = G$ and consider conjugation. Define, $\omega \cdot g = \omega^g = g^{-1}\omega g$, where $\omega \in \Omega$ and $g \in G$. We have that $\omega \cdot 1 = \omega^1 = \omega$ and $(\omega \cdot g) \cdot h = (\omega^g) \cdot h = (\omega^g)^h = \omega^{gh} = \omega \cdot (gh)$. Therefore, conjugation does indeed define an action. Another standard example of an action would be right multiplication, when we take $\Omega = G$.

Now, let N and M be groups and suppose that M acts on N and in addition $(xy) \cdot m = (x \cdot m)(y \cdot m)$ for all $x, y \in N$ and $m \in M$. The action of M on N is said to be *Frobenius* if $n^m \neq n$ whenever $n \in N$ and $m \in M$ are non-identity elements.

For an example of such an action, consider the dihedral groups of order not divisible by 4. Take N to be the cyclic group of order n, where n is odd, and M to be the cyclic group of order 2. Now, suppose M acts on N via inversion, i.e. $n^m = n^{-1}$. Since, N has odd order, then $n = n^{-1}$ if and only if $n = 1$. Thus, the action of M on N is Frobenius.

Now, a group M is said to be a *Frobenius complement* if it has a Frobenius action on some non-identity group N. Similarly, a group N is said to be a *Frobenius kernel* if it admits a Frobenius action by some non-identity group M. These types of groups admit spectacular properties that we omit since we have no need for them. For our purposes, we need the Frobenius kernel to be abelian, but in general this is not the case. In generality though, Frobenius kernels are always nilpotent. Recall the following theorem [7].

Theorem 7. *Suppose that M is a Frobenius complement. Then each Sylow subgroup of M is cyclic or generalized quaternion.*

Characterization of Frobenius groups

Given a normal subgroup N of a group G, a subgroup M of G is a complement for N in G if $NM = G$ and $N \cap M = 1$. If this situation occurs, we say G is the semi-direct product of N and M, written as $G = N \rtimes M$. In this situation, we also say that G is a semi-direct product of M acting on N. We now consider the semi-direct product of G, where M acts on N; if M is given the Frobenius action then what useful information can be determined about G? We take a moment to mention that when we consider semi-direct products of G, we view N and M as subgroups of G, where N is normal in G and M is complemented by N in G. The following theorem answers our previous question [7].

Theorem 8. *Let N be a normal subgroup of a group G, and suppose that M is a complement for N in G. The following are equivalent:*

1. *The conjugation action of M on N is Frobenius.*
2. *$M \cap M^g = 1$ for all $g \in G - M$*
3. *$\boldsymbol{C}_G(m) \subseteq M$ for all non-identity elements $m \in M$*
4. *$\boldsymbol{C}_G(n) \subseteq N$ for all non-identity elements $n \in N$*

If both N and M are nontrivial in the above situation then we say G is Frobenius, and that N and M are the Frobenius kernel and Frobenius complement, respectively. To prove Theorem (8), we need the following lemma [7].

Lemma 2. *Let N be a subgroup of a group G, and suppose that $N \cap N^g = 1$ for all $g \in G - N$. Let X be the subset of G consisting of those elements that are not conjugate in G to any nonidentity element of N. Then $|X| = |G|/|N|$.*

Proof. If $N = 1$, then N has no non-identity elements, and so $X = G$. So suppose that $|N| > 1$, we see that $N = N_G(N)$ since if $N^x = N$, then $|N \cap N^x| = |N| > 1$, and thus $x \in N$. It follows that N has exactly $|G : N|$ distinct conjugates in G. Furthermore, since each of these conjugates satisfies the condition we assumed about N, no two of them can have a non-trivial intersection. The conjugates of N, therefore account for a total of $|G : N|(|N| - 1)$ non-identity elements of G, and these are exactly the elements of G that are conjugate to non-identity elements of N. We conclude that $|X| = |G| - |G : N|(|N| - 1) = |G : N|$, as required. □

Corollary 1. *Let N be a normal subgroup of a group G, and suppose that M is a complement for N in G such that $N \cap N^g = 1$ for all $g \in G - N$. Then N is exactly the set X from above.*

Proof. Since $N \cap M = 1$, it is also true that $N \cap M^g = 1$ for all $g \in G$, and thus no element of N can be conjugate to a non identity element of M. It follows that $N \subseteq X$. By our previous lemma, $|X| = |G|/|M| = |N|$, and the result follows. □

We are now ready to prove Theorem (8) [7].

Proof. **(Theorem 8)** We will begin by proving that (1) implies (2). To prove (2), suppose that $|M \cap M^g| > 1$ for some element $g \in G$. Since $G = MN$, we have that $g = mn$ with $m \in M$ and $n \in N$. Thus, we have $M^g = M^{mn} = M^n$. Then $|M^n \cap M| > 1$, and so we can choose a non-identity element $b^n \in M^n \cap M$, where $b^n \in M$ and $b \in M$. So, we have $b^{-1}b^n \in M$, and since $N \lhd G$, we also have $b^{-1}b^n \in N$. Then $b^{-1}b^n \in M \cap N = 1$, and hence b centralizes n. But the action of M on N is Frobenius by (1) and $b \neq 1$, and so it follows that $n = 1$. Thus $g \in M$ and this shows (1).

Now, suppose (2), and let $1 \neq m \in M$. If $x \in C_G(M)$, then $m \in M \cap M^x$, and since this intersection is nontrivial it follows by our hypothesis that $m \in M$. Thus $C_G(M) \subseteq M$, and this proves (3).

Next, we show that (3) implies (1). If $1 \neq m \in M$, then by (3), we have $C_N(m) = N \cap C_G(m) \subseteq N \cap M = 1$, and thus the action of M on N is Frobenius.

Now suppose (1) and (2), we aim to prove (4). By (2) and our previous corollary we have that every element of G outside of N lies in some conjugate of M, so some non-identity element of $C_G(n)$ lies in a conjugate of M. For an appropriate conjugate a of n, some non-identity element of $C_G(a)$ lies in M. Since, $a \in N$ and the action of M on N is Frobenius, it follows that $a = 1$, and thus $n = 1$. This proves (4).

Finally, we will show (4) implies (1). Let $1 \neq n \in N$, then

$$\begin{aligned} C_M(n) &= M \cap C_G(n) \\ &\subseteq M \cap N \\ &= 1 \end{aligned}$$

and thus the action of M on N is Frobenius. $\square$

V. AN ANALOGUE TO DURFEE'S PARAMETER

We begin the main results of the thesis. This chapter is devoted to a generalization of a group theoretic analog to Durfee's parameter. Recall, that the group theoretic analog to Snyder's parameter has the irreducible character degrees in his definition replaced by the square roots of conjugacy class sizes [4]. Through this replacement, we arrive at the following definition:

Definition. *Let G be a group. The parameter e is defined as follows:*

$$e = \min\left\{\left(|C_G(x)| - 1\right)\sqrt{|G : C_G(x)|} : x \in G\right\}$$

We are concerned with a generalization of this parameter which incorporates a normal subgroup. After an initial definition and an example we will analyze the properties of this generalized parameter, as well as the range of values this generalized parameter can take. The generalized parameter is defined as the following:

Definition. *Let G be a group and N a normal subgroup of G. The parameter e_N is defined as follows:*

$$e_N = \min\left\{\left(\frac{|C_G(x)|}{|C_N(x)|} - 1\right)\sqrt{\frac{|G|}{|NC_G(x)|}} : x \in G\right\}$$

For simplicity, we use the notation that Harrison used in his thesis [5]; which is letting $k_N(x)$ denote the radicand $c_N(x)$ denote $|C_G(x) : C_N(x)|$. We also refer to the "e" parameter as the *non-relative parameter* and the "e_N" parameter as the *relative parameter*; since the relative parameter is dependent on the normal subgroup.

We begin by showing that the relative parameter is indeed a generalization of the non-relative parameter. Let G be a group and N a normal subgroup of G and $x \in G$ such that $e_N = (c_N(x) - 1)\sqrt{k_N(x)}$. Observe the following:

$$\begin{aligned}
e_N &= \left(c_N(x) - 1\right)\sqrt{k_N(x)} \\
&= c_N(x)\sqrt{k_N(x)} - \sqrt{k_N(x)} \\
&= c_N(x)\frac{k_N(x)}{\sqrt{k_N(x)}} - \sqrt{k_N(x)} \\
&= \frac{|G:N|}{\sqrt{k_N(x)}} - \sqrt{k_N(x)} \\
\sqrt{k_N(x)}\left(e_N + \sqrt{k_N(x)}\right) &= |G:N|
\end{aligned}$$

When we let N be the trivial subgroup we have that:

$$\begin{aligned}
\sqrt{k}(e_N + \sqrt{k}) &= |G| \\
\sqrt{k}(e + \sqrt{k}) &= |G|
\end{aligned}$$

where k is the largest class size. This is, of course, the non-relative parameter by definition. It is clear that the relative parameter is indeed a generalization of the non-relative parameter by the above. But, there is a significant difference when studying the relative parameter to that of when studying the non-relative parameter; it is that e_N may be zero for certain non-trivial groups and their corresponding normal subgroups. We provide two examples of this occurring; one when e_N is zero and another when e_N is non-zero, both of which will occur in the same group. Let G be the cyclic group of order 4, and let N be the whole group. We have trivially that $c_N(x) = 1$, since G is abelian, and we may conclude that $e_N = 0$. Now, let N be any other normal subgroup in G. Since G is an abelian, we have $|C_G(x)| = |G|$ and $|C_N(x)| = |N|$, for all $x \in G$. Thus, it must be that $c_N(x) = |G : N|$ which gives us that $c_N(x) > 1$, and this implies that e_N is non-zero, since $k_N(x)$ is clearly always non-zero.

In fact, for abelian groups calculating the relative parameter is as trivial as calculating non-relative parameter [4]. To see this, let G be an abelian group and $x \in G$ such that $e_N = (c_N(x) - 1)\sqrt{k_N(x)}$. Since the group is abelian, we have $|C_G(x)| = |G|$ and $|C_N(x)| = |N|$. We also have $NC_G(x) = NG = G$. This gives us $k_N(x) = 1$ and $c_N(x) = |G : N|$. So we have the following:

$$\begin{aligned} e_N &= (c_N(x) - 1)\sqrt{k_N(x)} \\ &= |G : N| - 1 \end{aligned}$$

A more interesting example in which e_N is zero is when $G = S_3$ and $N = A_3$. Take, x to be the 3-cycle $(1\ 2\ 3)$. Then $C_G(x) = A_3$ and since $N = A_3$, then we have that $C_N(x) = A_3$. This gives us $c_N(x) = 1$, and we may conclude that $e_N = 0$ for this normal subgroup of G. We illustrate this second example of when the relative parameter is zero to demonstrate that the relative parameter can be zero in both abelian and non-abelian groups. In fact, the relative parameter is zero quite often.

A natural question to ask is "When is relative parameter zero?" To answer this question we introduce the following definition. Let G be a group with $N \triangleleft G$ and $x \in G$. The conjugacy class x^G is said to be *non-split* when $x^G = x^N$.

For the remainder of this section we will use the following representation of $k_N(x)$ as it is more useful. Observe the following:

$$\begin{aligned} k_N(x) &= \frac{|G|}{|NC_G(x)|} \\ &= \frac{|G|}{\frac{|N||C_G(x)|}{|N \cap C_G(x)|}} \end{aligned}$$

$$= \frac{|G|}{\frac{|N||C_G(x)|}{|C_N(x)|}}$$
$$= \frac{|G : C_G(x)|}{|N : C_N(x)|}$$

Writing $k_N(x)$ this way is more useful since we may use the size of the conjugacy class in which x belongs to.

Thanks to Harrison [4], we have two useful lemmas regarding the behavior of $k_N(x)$ and $c_N(x)$ which simplifies working with the relative parameter.

Lemma 3. *Let G be a group with a normal subgroup N. For $x \in G$, we have that $c_N(x)$ and $k_N(x)$ are positive integers. The conjugacy class of x is non-split if and only if $k_N(x) = 1$.*

Proof. The first portion of the lemma follows directly from the fact that both $c_N(x)$ and $k_N(x)$ are indices of groups. For the next portion, by above, we have:

$$k_N(x) = \frac{|G : C_G(x)|}{|N : C_N(x)|}$$

and thus $k_N(x) = 1$ if and only if $|x^N| = |x^G|$. But, since x^N is contained x^G, it must be the case that $x^N = x^G$ and thus x^G is non-split. □

Note that $e_N \geq 0$, since $c_N(x)$ and $k_N(x)$ are both greater than 1. We digress though; we can now answer when the relative parameter is zero.

Lemma 4. *Let G be a group with a normal subgroup N. The parameter $e_N = 0$ if and only if $C_G(x) \leq N$, for some $x \in G$. In particular, $e_N > 0$ if N contains no centralizers $C_G(x)$ for any $x \in G$.*

Proof. Suppose $e_N = 0$. Then there exists an $x \in G$ such that $(c_N(x)-1)\sqrt{k_N(x)} = 0$. Since $k_N(x) \geq 1$, we must have $c_N(x) = 1$. Then $C_G(x) = C_N(x)$ and $C_G(x) \leq N$.

Reversing this chain of logic gives the converse. $\square$

Now, the previous two lemmas give useful conditions to study the relative parameter.

We hope to mimic the results from Harrison [4]. As was the case with the non-relative parameter; we can bound the index of a normal subgroup with the relative parameter in the exact same fashion.

Theorem 9. *Let G be a group with a normal subgroup N and $x \in G$ such that $e_N = (c_N(x) - 1)\sqrt{k_N(x)}$ and $e_N > 0$. Then $|G : N| \leq 2(e_N)^2$.*

Proof. Let G be a group with $N \triangleleft G$ and $x \in G$ such that $e_N = (c_N(x) - 1)\sqrt{k_N(x)}$. We have the following equality:

$$
\begin{aligned}
e_N &= (c_N(x) - 1)\sqrt{k_N(x)} \\
(e_n)^2 &= (c_N(x) - 1)^2 \frac{|G : C_G(x)|}{|N : C_N(x)|} \\
(e_n)^2 &= (c_N(x) - 1)^2 \frac{|G : N|}{|C_G(x) : C_N(x)|} \\
(e_n)^2 &= \frac{(c_N(x) - 1)^2}{c_N(x)} |G : N| \\
(e_n)^2 &= \frac{(c_N(x) - 1)^2}{c_N(x)} |G : N| \\
(e_n)^2 \frac{c_N(x)}{(c_N(x) - 1)^2} &= |G : N|
\end{aligned}
$$

Since $\frac{c_N(x)}{(c_N(x)-1)^2}$ decreases as $c_N(x)$ increases, we need only look at the minimal value of $c_N(x)$. By hypothesis e_N is non-zero. So it follows that the minimal integer value for $c_N(x)$ is 2, since $c_N(x)$ is always an integer and larger than 1.

This gives us the following inequality:

$$|G : N| \leq 2(e_N)^2$$

and completes our proof. $\square$

As with the non-relative parameter; using the above theorem we may classify certain groups that correspond to a particular e_N. Since the normal subgroup may vary within a group, the only way in which we may classify groups attaining certain e_N efficiently is by fixing the normal subgroup and then classifying the groups having this e_N. To see this we state the following theorem:

Theorem 10. *Let G be a group with a minimal normal subgroup N such that $|N| = 2$ and $1 \leq e_N \leq 2$. Then one of the following holds:*

1. *$e_N = 1$ if and only if G is cyclic of order 4 or is Klein-4.*
2. *$e_N = \sqrt{2}$ if and only if G is the dihedral group of order 8 or the quaternion group.*
3. *$e_N = 2$ if and only if G is cyclic of order 6.*

Proof. We will begin by proving the statements (1) – (3) individually and will end the proof by showing that there are no other possible groups with $1 \leq e_N \leq 2$ under our conditions. Let G be a group and N be a minimal normal subgroup of G such that $|N| = 2$. For the remainder of the proof let $x \in G$ such that:

$$e_N = (c_N(x) - 1)\sqrt{k_N(x)} \tag{1}$$

Suppose that $e_N = 1$. Since $e_N = 1$ and $|N| = 2$, then we have, by (1), that $C_N(x) = 2$ and $k_N(x) = 1$. So, by rearranging the terms in $k_N(x)$, we may conclude that $|G| = 4$. Up to isomorphism, there are two groups of order 4, both of which have normal subgroups of order 2. Conversely, if G is one of the two groups of order 4, then $e_N = 1$, since abelian groups have the property that $e_N = |G : N| - 1$.

Now, suppose that $e_N = \sqrt{2}$. Since $e_N = \sqrt{2}$ and and $|N| = 2$, then we have, by (1), that $c_N(x) = 2$ and $k_N(x) = 2$. As before, by rearranging the terms in $k_N(x)$, we may conclude that $|G| = 8$. Up to isomorphism, we have exactly five groups of order 8, three of which are abelian and hence have an integer valued e_N. Thus G must be dihedral or quaternion of order 8. Conversely, for D_8, we have that the minimal normal subgroup of order 2 in D_8 is $Z(G)$ and that any involution realizes e_N. Let $x \in G$, be any involution. Now, $|C_G(x)| = 4$ and $|C_N(x)| = 2$ and since $|G : N| = 4$ then $k_N(x) = 2$ and we have that $e_N = \sqrt{2}$. The case of the quaternions is identical except the element realizing e_N is of order 4.

Now, suppose that $e_N = 2$. We have two cases either $c_N(x) = 3$ and $k_N(x) = 1$ or $c_N(x) = 2$ and $k_N(x) = 4$. Suppose the former, that is $c_N(x) = 3$ and $k_N(x) = 1$. By rearranging the terms in $k_N(x)$, we may conclude that $|G| = 6$. Up to isomorphism, we have two groups of order 6, they are S_3 and C_6. In S_3, there does not exist any normal subgroup of order 2, so G must be C_6. Now suppose that $c_N(x) = 2$ and $k_N(x) = 4$. By rearranging the terms in $k_N(x)$, we may conclude that $|G| = 8$ and we have already consider groups of this order; none of which have an $e_N = 2$. Conversely, suppose that G is C_6. Since this group is abelian, we have that $|G : N| - 1 = e_N$, so we may conclude that $e_N = 2$.

We will now show there are no other groups which attain this parameter under the given assumptions. By our hypothesis and Theorem (9), we only have to consider groups of even order less than or equal to 8. Furthermore, we need only consider groups of order 2 since we have already consider all other groups of even order less than 8. Up to isomorphism, we have only one group of order 2, and it is

C_2. Since this group is abelian, we have that $e_N = 0$; this completes our proof. $\square$

Of course the natural question to ask is which groups have the index of their minimal normal subgroups attaining the bound in Theorem (9). By assuming some additional properties, we may simplify our task greatly.

Theorem 11. (Main) Let G be nilpotent. The following are equivalent.

(a) There exists a minimal normal subgroup N, satisfying the following: If $x \in G$ realizing the relative parameter then e_N is non-zero, x^G is split, and $|G : N| = 2(e_N)^2$.

(b) The group G is one of the following groups:

- Dihedral of order 2^n, where $n \geq 2$,
- Quaternion of order 2^n, where $n \geq 2$,
- Semi-Dihedral of order 2^n, where $n \geq 3$.

Before we prove our main theorem, we need two additional theorems [11].

Theorem 12. *Let G be a group with a non-cyclic subgroup T of order 4 which is its own centralizer in G. If K is the largest normal subgroup of odd order in G, then G/K is isomorphic with $PSL(3,3)$, M_{11}, $GL(2,3)$, $H(q)$,$PGL(2,q)$, $PSL(2,q)$ (q odd), A_7, or a 2-group of dihedral or semi-dihedral type.*

Theorem 13. *Let G be a group with a cyclic subgroup T of order 4 which is its own centralizer in G. If K is the largest normal subgroup of odd order in G, then G/K is isomorphic with $SL(2,3)$, $SL(2,5)$, A_7, $PSL(2,1)$, $PSL(2,9)$, $PGL(2,3)$, $PGL(2,5)$, $H(9)$, J, a 2-group of semi-dihedral or generalized quaternion type, a dihedral group of order 8, or a cyclic group of order 4.*

Proof. **(Main)** We will begin by showing (a) implies (b). We first determine a value for $c_N(x)$. By definition of e_N, we have the following:

$$\begin{aligned}
|G:N| &= 2(e_N)^2 \\
|G:N| &= 2\Big(c_N(x) - 1\Big)^2\Big(\sqrt{k_N(x)}\Big)^2 \\
|G:N| &= 2\Big(c_N(x) - 1\Big)^2\Big(\frac{|G:C_G(x)|}{|N:C_N(x)|}\Big) \\
|G:N| &= 2\Big(c_N(x) - 1\Big)^2 \frac{|G:N|}{|C_G(x):C_N(x)|} \\
1 &= 2\Big((c_N(x))^2 - 2c_N(x) + 1\Big)\frac{1}{c_N(x)} \\
0 &= 2(c_N(x))^2 - 5c_N(x) + 2
\end{aligned}$$

Solving for $c_N(x)$ gives us that $c_N(x)$ is 2 or $\frac{1}{2}$ and, by Lemma (1), we may conclude that $c_N(x) = 2$.

Now, since G is a nilpotent group, we have by Theorem (3) that N is of prime order p and hence is cyclic. We aim to show $p = 2$, we will show this by contradiction. Suppose that p is an odd prime. Let $M = NC_G(x)$; now N is contained in the center of the group by the minimality of N and the center, of course, is contained in $C_G(x)$, so it follows that $M = C_G(x)$. Observe that $|M| = 2p$.

Now, let $S \in Syl_2(M)$. Since every subgroup of a nilpotent group is nilpotent, then by Theorem (4) we have that M is a direct product of S and N. Furthermore, since p is an odd prime, then M is in fact cyclic. Also, $S = < s >$ and the order of s is two. Since $M = S \times N$ then we can write x as sn, for some $n \in N$. So we have that $C_G(s) = C_G(sn) = M$ and by replacing x with s we may assume that the order of x is 2. We may do this replacement since $n \in Z(G)$.

Now, let $Q \in Syl_2(G)$ such that $x \in Q$; since x is an involution and $|M| = 2p$ and p is odd, then clearly $C_Q(x) = < x >$. By the Fundamental Theorem of Abelian groups we have that $Q = < x >$. Now, since Q is an abelian Sylow subgroup we have that it must be contained in the center. This is the case since G may be written as a direct product of its Sylow subgroups and so we have that every member of Q commutes component-wise; hence Q is contained in $Z(G)$. This is a contradiction since, by hypothesis, x^G is split.

Before proceeding, we make the observation that $|Z(G)| = 2$. By the argument above we have that $p = 2$, so $|C_G(x)| = 4$. Consider the following chain of subgroups:

$$N \leq Z(G) \leq C_G(x)$$

Either $Z(G)$ is N or $C_G(x)$. If the $Z(G)$ where equal to $C_G(x)$, then the conjugacy class of x would be non-split which contradicts our hypothesis. So $Z(G) = N$ and $|Z(G)| = 2$.

The fact that G is a 2-group follows immediately from the order of the center and G being nilpotent. To see this observe that G is a direct product of its Sylow subgroups, and thus $Z(G)$ is the direct product of the centers of the Sylow subgroups. By Theorem (2) and $|Z(G)| = 2$, it must be that G is a 2-group.

Now, we have that $|C_G(x)| = 4$; up to isomorphism, we have that $C_G(x)$ is either cyclic of order 4 or Klein-4. Also, we clearly have that $C_G(x)$ is its own centralizer in G. If $C_G(x)$ is cyclic of order 4, then by Theorem (13) and G being a 2-group, we have G must either be semi-dihedral, quaternion, or dihedral of order 8. Else, $C_G(x)$ is Klein-4 and so by Theorem (12) and G being a 2-group, we have that G is dihedral. Observe that G can not be cyclic of order 4 since x^G is split.

We now show the converse and will do so by cases. In each of the three cases the minimal normal subgroup we seek is $Z(G)$ and we will denote it as N in all three cases. Note that for each of the above groups G we have $C_N(x) = N$, for every

$x \in G$. We now have our first case.

1. Suppose that G is dihedral of order 2^n where $n \geq 3$, and $a \in G$ is the element such that $|G :< a > | = 2$. Now, it is clear that no centralizer is contained in $Z(G)$ since $|Z(G)| = 2$; hence e_N is non-zero. We have that $Z(G) = \left\{1, a^{2^{n-2}}\right\}$. Let x be any involution not contained in the center of G. Now, the conjugacy class of x is split since the normal subgroup we selected was the center and $x \notin Z(G)$. Furthermore, we have that $C_G(x) = \left\{1, x, a^{2^{n-2}}, xa^{2^{n-2}}\right\}$. To see this suppose that some other a^i centralizes x, where $1 < i < 2^n$ and $i \neq 2^{n-2}$. Now, observe the following:

$$
\begin{aligned}
xa^i &= a^i x \\
x^{-1}a^i x &= a^i \\
a^{-i} &= a^i
\end{aligned}
$$

and this implies that $a^i = a^{2^{n-2}}$, which is a contradiction. Now, suppose some other involution centralizes x, we may write the involution as xa^i, where $1 < i < 2^n$ and $i \neq 2^{n-2}$. Observe the following:

$$
\begin{aligned}
x(xa^i) &= (a^i x)x \\
x^{-1}a^i x &= a^i \\
a^{-i} &= a^i
\end{aligned}
$$

Which is the same contradiction as before; hence $|C_G(x)| = 4$. We have the following:

$$2^{n-1} = 2^{n-1}$$

$$
\begin{aligned}
2^{n-1} &= 2(2^{n-2}) \\
2^{n-1} &= 2\Big(2-1\Big)^2\Big(\frac{2^n}{4}\Big) \\
2^{n-1} &= 2\Big(\frac{|C_G(x)|}{|C_N(x)|}-1\Big)^2\Big(\sqrt{\frac{|G|}{|C_G(x)|}}\Big)^2 \\
2^{n-1} &= 2\Big(\frac{|C_G(x)|}{|C_N(x)|}-1\Big)^2\Big(\sqrt{\frac{|G|}{|NC_G(x)|}}\Big)^2 \\
2^{n-1} &= 2\Big(c_N(x)-1\Big)^2\Big(\sqrt{k_N(x)}\Big)^2 \\
|G:N| &= 2(e_N)^2
\end{aligned}
$$

Which proves the case when G is dihedral.

2. Now, suppose G is quaternion of order order 2^n where $n \geq 3$, and $a \in G$ is the element such that $|G :< a > | = 2$. Now, it is clear that no centralizer is contained in $Z(G)$; hence e_N is non-zero. We also have that $Z(G) = \{1, a^{2^{n-2}}\}$. Let $x \in G$ such that the order of x is four. Now, we have that x^G is split since the normal subgroup we selected was the center and $x \notin Z(G)$. Furthermore, we have that $C_G(x) = \{1, x, a^{2^{n-2}}, xa^{2^{n-2}}\}$. To see this suppose that some other element a^i centralizes x, where $1 < i < 2^n$ and $i \neq 2^{n-2}$. Now, observe the following:

$$
\begin{aligned}
xa^i &= a^i x \\
x^{-1}a^i x &= a^i \\
a^{-i} &= a^i
\end{aligned}
$$

and this means that $a^i = a^{2^{n-2}}$, which is a contradiction. Now, suppose some other element of order 4 centralizes x, we may write this element as xa^i, where

$1 < i < 2^n$ and $i \neq 2^{n-2}$. Observe the following:

$$
\begin{aligned}
x(xa^i) &= (a^i x)x \\
x^{-1}a^i x &= a^i \\
a^{-i} &= a^i
\end{aligned}
$$

Which is the same contradiction as before; hence $|C_G(x)| = 4$. We have the following:

$$
\begin{aligned}
2^{n-1} &= 2^{n-1} \\
2^{n-1} &= 2(2^{n-2}) \\
2^{n-1} &= 2\Big(2-1\Big)^2\Big(\frac{2^n}{4}\Big) \\
2^{n-1} &= 2\Big(\frac{|C_G(x)|}{|C_N(x)|}-1\Big)^2\Big(\sqrt{\frac{|G|}{|C_G(x)|}}\Big)^2 \\
2^{n-1} &= 2\Big(\frac{|C_G(x)|}{|C_N(x)|}-1\Big)^2\Big(\sqrt{\frac{|G|}{|NC_G(x)|}}\Big)^2 \\
2^{n-1} &= 2\Big(c_N(x)-1\Big)^2\Big(\sqrt{k_N(x)}\Big)^2 \\
|G:N| &= 2(e_N)^2
\end{aligned}
$$

Which proves the case when G is quaternion.

3. This case that G is semi-dihedral is identical to that of the dihedral group, with the obvious changes.

This shows that (b) implies (a) and completes the proof. □

We have an immediate corollary.

Corollary 2. *Let G be a nilpotent group with a minimal normal subgroup N. There* *is no $x \in G$ realizing e_N where $c_N(x) = 2$ and $k_N(x) = 2m$, such that m is a square-free product of odd primes.*

Proof. We will prove the claim by contradiction. Let G be a nilpotent group with a minimal normal subgroup N and $x \in G$ realizing e_N as above. By hypothesis, we have that $c_N(x) = 2$. So, by rearranging terms in $k_N(x)$, we have $|G : N| = 4m$. Now, by substituting e_N into $|G : N|$, we arrive at the following equality:

$$|G : N| = 2(e_N)^2$$

By Theorem (11), we have that G is a 2-group, and so m is even, which is a contradiction. □

To contrast the difference of our main theorem with that of Harrison's, observe that when the group order achieved equality with the upper bound in the non-relative parameter we concluded that the group must be a dihedral group of order not divisible by 4 i.e. a Frobenius group. But the conclusion of our theorem gives groups that are not Frobenius. So it is of interest to pursue which conditions on the relative parameter are required to conclude that the group is Frobenius.

Before doing so we introduce some new notation. We may view the relative parameter inside of a subgroup M of a group G. We define this new parameter identically to that of the relative parameter.

Definition. *Let G be a group and M a subgroup of G with $N \triangleleft M$. The parameter $e_{N,M}$ is defined as:*

$$e_{N,M} = \min\left\{\left(\frac{|C_M(x)|}{|C_N(x)|} - 1\right)\sqrt{\frac{|M|}{|NC_M(x)|}} : x \in M\right\}$$

We denote $\dfrac{|C_M(x)|}{|C_N(x)|}$ as $c_N^M(x)$ and $\dfrac{|M|}{|NC_M(x)|}$ as $k_N^M(x)$, whenever convenient. Observe that when $M = G$, we recover the relative parameter. We provide a partial answer to which conditions may be necessary to ensure the groups is Frobenius when working with the relative parameter.

Theorem 14. *Let G be a supersolvable group with a trivial center and suppose that p is the smallest prime dividing $|G|$. Furthermore, let G have the property that its Sylow p-subgroups are maximal. Then, $e_{N,P} = |P : N| - 1$ for all minimal normal subgroups N contained in the Sylow p-subgroups P of G if and only if G is Frobenius with abelian kernel of prime order q and cyclic complement P.*

Proof. Let $x \in P$ realizing $e_{N,P}$. By hypothesis, we have that $e_{N,P} = |P : N| - 1$. Now, we claim P is abelianf; we will prove this by contradiction. Suppose that P is non-abelian and let $y \in P - Z(P)$. Since P is a p-group and $N \triangleleft P$, then by Theorem (2) we have $N \cap Z(P)$ is non-trivial. By the minimality of N, we may conclude that N is a subgroup of $Z(P)$. Clearly, $|C_N(y)| = |N|$. Observe the following:

$$\begin{aligned}
|C_P(y)| &< |P| \\
\frac{|C_P(y)|}{|N|} &< \frac{|P|}{|N|} \\
\frac{\sqrt{|C_P(y)|}}{|N|} - \frac{1}{\sqrt{|C_P(y)|}} &< \frac{\sqrt{|P|}}{|N|} - \frac{1}{\sqrt{|P|}} \\
\left(\frac{\sqrt{|C_P(y)|}}{|N|} - \frac{1}{\sqrt{|C_P(y)|}}\right)\sqrt{|P|} &< \left(\frac{\sqrt{|P|}}{|N|} - \frac{1}{\sqrt{|P|}}\right)\sqrt{|P|} \\
\frac{|C_P(y)|}{|N|}\sqrt{\frac{|P|}{|C_P(y)|}} - \sqrt{\frac{|P|}{|C_P(y)|}} &< \frac{|P|}{|N|} - 1 \\
\left(\frac{|C_P(y)|}{|N|} - 1\right)\sqrt{\frac{|P|}{|C_P(y)|}} &< |P : N| - 1
\end{aligned}$$

$$\left(\frac{|C_P(y)|}{|C_N(y)|} - 1\right)\sqrt{\frac{|P|}{|NC_P(y)|}} < |P:N| - 1$$
$$\left(\frac{|C_P(y)|}{|C_N(y)|} - 1\right)\sqrt{\frac{|P|}{|NC_P(y)|}} < e_{N,P}$$

which is, of course, a contradiction to the fact that $e_{N,P}$ is minimal. Thus, we may conclude that P is abelian.

Now, by Theorem (6), there exists a subgroup M whose order is co-prime to $|G : P|$. Note that since G is supersolvable, then $|M| = q$, for some prime q. Since P is abelian, we may conclude that $M \triangleleft G$; otherwise we would have that a member of M is contained in some Sylow p-subgroup of G. Since $M \triangleleft G$, then G is the semi-direct product of P acting on N. Now, if G were not Frobenius then there would exist a $p \in P - 1$, such that $C_G(p) \nsubseteq P$. But since M is of prime order, every member of M commutes with p and since P is abelian, then p must also commute with all of P, contradicting our trivial center. Hence, G is Frobenius and by Theorem (7), we may conclude that P is cyclic.

The converse of the statement follows directly since P is an abelian subgroup. This completes the proof. $\square$

An example of a group satisfying the above conditions is S_3. This group has a trivial center and every Sylow 2-subgroup P is cyclic and maximal, all of which are generated by the 2-cycles, and the only minimal normal subgroup contained inside of the Sylow 2-subgroups is themselves. Since P is abelian, we have $|P : N| - 1 = e_N$. So we have that G is Frobenius with complement C_2 and kernel C_3. Another example would be the general affine group of degree one over the field of five elements.

Now, there is strong evidence computationally via GAP to support that supersolvability is not needed and that we only need the group to be solvable. But the condition of solvability is necessary to ensure that G splits over some subgroup. In particular, G would spilt over a Hall-π subgroup (which we have not discussed in this document). Also, assuming certain conditions on the relative parameter regarding the normal subgroups contained in a maximal subgroup is necessary to ensure that the Frobenius complement and kernel are self-centralizing and in determining when the center of the group is trivial.

Conjecture. Let G be a finite solvable group and suppose that p is the smallest prime dividing $|G|$. Then, $e_{N,P} = |P : N| - 1$ for all minimal normal subgroups N contained in the Sylow p-subgroups P of G if and only if G is Frobenius with cyclic complement that is a Sylow p-subgroup and abelian kernel.

APPENDIX

In this appendix we provide proof for Theorem (3), as well as categorize those groups who have a corresponding e value between 2 and 3. We also provide some corollaries to Harrison's results in his original thesis [4].

Theorem. If G is a nilpotent group and $N \triangleleft G$, then $N \cap Z(G)$ is non trivial. Furthermore, if N is a minimal normal subgroup of G then N is of prime order.

Proof. Let G be a nilpotent group with $N \triangleleft G$. Consider the upper central series of G. By definition of the upper central series, there exists some i, where $1 \leq i \leq r$, such that $N \cap Z_i$ is trivial and that $N \cap Z_{i+1}$ is non-trivial. We have the following chain of containment:

$$\begin{aligned}[G, N \cap Z_{i+1}] &\subseteq [G, N] \cap [G, Z_{i+1}] \\ &\subseteq N \cap Z_i \\ &= 1\end{aligned}$$

which means that $[G, N \cap Z_{i+1}]$ is in fact trivial. This implies that that $N \cap Z_{i+1}$ is contained in $Z(G) \cap N$, which proves the first claim.

Let N be a minimal normal subgroup of G. Since $N \triangleleft G$, then $N \cap Z(G)$ is non-trivial and by the minimality of N, we may conclude that N is contained in $Z(G)$. Since any subgroup of $Z(G)$ is normal in G, then the only possibility for a minimal subgroup is a cyclic group of prime order. $\square$

Theorem 15. *Let G be the direct product of H and K, where H and K are groups. Then we have:*

$$e_G = \frac{|H||K| - hk}{\sqrt{hk}}$$

where h and k are the largest conjugacy classes in H and K, respectively.

Proof. Let $G = H \times K$, where H and K are groups, and let h and k denote the largest conjugacy classes in H and K, respectively. We have that the largest conjugacy class in G is of size hk and $|G| = |H||K|$. By definition we have the following equality:

$$|H||K| = hk + \sqrt{(e_G)(hk)}$$

Solving for e_G gives the result. □

Theorem 16. *Let G be a group such that $2 < e < 3$. Then one of the following holds:*

1. $e = \sqrt{5}$ *if and only if G is the dihedral group of order* 10.
2. $e = \sqrt{7}$ *if and only if G is the dihedral group of order* 14.

Proof. We will begin by proving the statements (1) and (2) individually and will end the proof by showing that there are no other possible groups with $2 < e < 3$. For the remainder of the proof, let $x \in G$, such that:

$$e = (|C(x)| - 1)\sqrt{|G : C(x)|}$$

We will begin by showing (1). Suppose, that $e = \sqrt{5}$. We have, by hypothesis, that $|C(x)| = 2$ and $|G : C_G(x)| = 5$, thus $|G| = 10$. Up to isomorphism, we have one non-abelian group and one abelian group of order 10, these two groups are the dihedral group and the cyclic group respectively. Since the $|C(x)| < 10$, then G must be the dihedral group. Conversely, the size of the largest conjugacy class in D_{10} is 5 and the corresponding member of D_{10} that minimizes the parameter is any involution.

We now show (2). Suppose that $e = \sqrt{7}$. We have, by assumption, that $x|C(x)| = 2$ and $|G : C_G(x)| = 7$, and so we may conclude that $|G| = 14$. As before, we have up to isomorphism only one non-abelian group and one abelian group of order 14, this is the dihedral group and the cyclic group, respectively. Since the $|C(x)| < 14$, then G must be the dihedral group. Conversely, the size of the largest conjugacy class in D_{14} is 7 and the corresponding member of D_{14} that minimizes the parameter is any involution.

We will now show that these are the only values of e, in which $2 < e < 3$. Let k denote the largest conjugacy class of G. By Harrison, we have that $|G| \leq 2e^2$. The largest possible order G could have is 18. Before continuing, observe that there are no non-abelian groups of order 2 – 5, 7, 9, 11, 13, 15, and 17. Every abelian group with order varying through the aforementioned orders all results in a e that does not satisfy our conditions except for C_4 and Klein-4. Now, D_8 and Q_8 both give a $e = 3\sqrt{2}$, which is larger than three. Also, there is one abelian and one non-abelian group of order 10 and 14, both of which we have already considered.

We have only three groups of order 12 which are non-abelian. These groups are D_6, A_4, and $Dic(12)$. Where the size of the largest conjugacy class in each of these groups is 3, 4, and 3, respectively. All three groups have an e that does not satisfy our conditions.

We now consider the non-abelian groups of order 16. Up to isomorphism, we have 9 non-abelian groups. The size of the largest conjugacy class in all of these groups is 2 or 4. Calculating the e with either value gives a parameter larger than three. □

Theorem 17. *There is no group G with $e = \sqrt{2m}$, where m is a square free product of odd primes.*

Proof. We will prove the claim by contradiction. Suppose there exists a group G with $e = \sqrt{2m}$, where m is a square free product of odd primes. Now, by definition, we have the following equality:

$$|G| = k + \sqrt{2mk}, \tag{2}$$

where k is the size of the largest conjugacy class in G.

By Harrison, we have $|G| \leq 2e^2$; by assumption it follows that $|G| \leq 4m$. We now have the following:

$$k + \sqrt{2mk} \leq 4m$$

Solving the inequality for k, gives us that $k \in [0, 2m]$.

For (2) to evaluate to an integer, k must be a product of an odd power of 2 and a odd power of all the prime factors of m. We are inclined to say clearly there is no such integer less than $2m$ satisfying this condition except for 0. Thus, we have two solutions, either $k = 0$ or $k = 2m$. Now since k is the size of a conjugacy class, it must be that $k = 2m$. This gives:

$$\begin{aligned} |G| &= 4m \\ &= 2(\sqrt{2m})^2 \\ &= 2e^2 \end{aligned}$$

Which is a contradiction to Harrison. $\square$

Theorem 18. *Let G be a group such that the smallest prime dividing $|G|$ is p and p^2 does not divide $|G|$. Then $|G| = \frac{p}{(p-1)^2}e^2$ if and only if G is Frobenius with complement C_p. In particular, if G is Frobenius with complement C_p and abelian kernel M, then $e_N = (p-1)\sqrt{\frac{k}{|N|}}$.*

Proof. The bi-conditional statement can be found in Harrison [4]. We will prove the second claim; suppose G is Frobenius with complement C_p and abelian kernel M. Let $N \lhd G$. Furthermore, we may assume that $N \subseteq M$. Let $x \in G$ such that $e_N = (c_N(x) - 1)\sqrt{k_N(x)}$. Note that $|G| = pk$, and that the smallest $|C_G(x)|$ could be is p. We have two cases; either $x \in M$ or $x \in C_p$. Suppose that $x \in M$; then we have $c_N(x) = |M : N|$ and $k_N(x) = |G : M|$. This gives us:

$$\begin{aligned} e_N &= (c_N(x) - 1)\sqrt{k_N(x)} \\ &= (|M : N| - 1)\sqrt{|G : M|} \end{aligned}$$

Now, suppose that $x \in C_p$; then we have $c_N(x) = p$ and $k_N(x) = |M : N|$. This gives us:

$$e_N = (p-1)\sqrt{|M : N|}$$

Note that $e_N = (p-1)\sqrt{|M : N|} = (p-1)\sqrt{\frac{k}{|N|}}$. We now show that $e_N = (p-1)\sqrt{\frac{k}{|N|}}$ is the true minimum. Let $\beta = |M : N|$. Since p is the smallest prime dividing $|G|$, we have that $p < \beta$. Now observe the following:

$$\frac{\beta - p}{p\beta} < \beta - p$$

$$\frac{1}{p} - \frac{1}{\beta} < \beta - p$$
$$p + \frac{1}{p} < \beta + \frac{1}{\beta}$$
$$\frac{(p-1)^2}{p} < \frac{(\beta-1)^2}{\beta}$$
$$(p-1)\sqrt{\beta} < (\beta-1)\sqrt{p}$$
$$(p-1)\sqrt{|M:N|} < (|M:N|-1)\sqrt{p}$$
$$(p-1)\sqrt{\frac{k}{|N|}} < (|M:N|-1)\sqrt{|G:M|}$$

Which proves the claim. $\square$

We include various functions which can be implemented in GAP. All of these functions are part of a larger piece of code that was implemented to study various properties regarding the relative parameter. The function calculates the relative parameter for all normal subgroups of a given group.

```
CalcRP := function(n,m)
local G,y,N,e,x,C,CN,b,i,I,j,l;
i:=1;
G:= SmallGroup(n,m);
y:=NormalSubgroups(G);
e:=Elements(G);
l:=[Size(G)];
for i in [1..Size(y)] do
N:=y[i];
for j in [1..Size(G)] do
x:=e[j];
C:=Centralizer(G,x);
CN:=Centralizer(N,x);
I:=Intersection(N,C);
b:=((Size(C)/Size(CN)) - 1)*((Size(C)/Size(CN)) - 1)*
     ( (Size(G)/( (Size(N)*Size(C))/Size(I)))) ;
l[j]:=b;
od;
Print("Your relative parameter is the square root of ", b,"
         with normal subgroup is N[",i,"]", "\n");
od;
```

```
return 0;
end;
```

This function determines when a group is Frobenius.

```
IsFrobenius := function(m,n)
local G,cc,hh,i,x,y,e;
e:=0;
i:=2;
G:=SmallGroup(m,n);
cc:=Size(List(ConjugacyClasses(G)));
hh:=NormalSubgroups(G);
while i  < Size(hh) do
x:= Size(List(ConjugacyClasses(FactorGroup(G, hh[i]))));
y:= (Size(List(ConjugacyClasses(hh[i]))) -1)/Index(G,hh[i]);
if cc = x+y then
Print("Your group is frobenius, with kernel ", hh[i], "\n");
Print("with size ", Order(hh[i]), "\n");
e:=1;
fi;
i:=i+1;
od;
if e =0 then
Print("Your group is not frobenius!", "\n");
fi;
return 0;
end;
```

This function determines the relative parameter for a minimal normal subgroup for all groups whose order is less then a given non-relative parameter.

```
CalcRPonIntervals := function(p)
local G,nn,N,e,x,C,CN,I,b,i,j,k,l,w,u;
j:=4;
w:=p*p;
while j<= (2*w+1) do
l:=1;
k:=NumberSmallGroups(j);
while l<=k do
G := SmallGroup(j,l);
nn := NormalSubgroups(G);
N := nn[1];
e := Elements(G);
u:=1000;
for i in [1..Size(e)] do
x:=e[i];
C:=Centralizer(G,x);
CN:=Centralizer(N,x);
I:=Intersection(N,C);
b:=( (Size(C)/Size(CN)) - 1)*((Size(C)/Size(CN)) - 1)*
     ((Size(G)/( (Size(N)*Size(C))/Size(I))) );
u:=Minimum(u,b);
od;
Print("Your Group is SmallGroup(", j, ",",l,")","\n");
```

```
Print("Your bound is the square root of ", u,"\n");
Print("The index of the Normal Subgroup is ", Index(G,N),"\n");
l:=l+1;
od;
j:=j+1;
od;
return 0;
end;
```

This function ignores p-groups when calculating relative parameters on an interval.

```
Check := function(k)
local i;
for i in [1..NumberSmallGroups(k)] do
if IsPrimePowerInt(k) = false then
if IsAbelian(SmallGroup(k,i)) = false  then
GiveMe(SmallGroup(k,i));
fi;
fi;
if IsPrimePowerInt(k) = true and GcdInt(k,2)>2 then
if IsAbelian(SmallGroup(k,i)) = false  then
GiveMe(SmallGroup(k,i));
fi;
fi;
if IsPrime(k) = true  then
GiveMe(SmallGroup(k,i));
```

```
fi;
Print("This is SmallGroup(",k,",",i,")","\n","\n");
od;
return 0;
end;
```

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