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A SINGULAR PERTURBATION PROBLEM IN INTEGRODIFFERENTIAL EQUATIONS *

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Abstract

Consider the singular perturbation problem for

$$\varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) = Au(t;\varepsilon) + \int_0^t K(t-s)Au(s;\varepsilon)\,ds + f(t;\varepsilon)\,,$$

where $t \geq 0$, $u(0;\varepsilon) = u_0(\varepsilon)$, $u'(0;\varepsilon) = u_1(\varepsilon)$, and

$$w'(t) = Aw(t) + \int_0^t K(t-s)Aw(s)\,ds + f(t)\,, \quad t \geq 0\,, \quad w(0) = w_0\,,$$

in a Banach space X when $\varepsilon \to 0$. Here A is the generator of a strongly continuous cosine family and a strongly continuous semigroup, and $K(t)$ is a bounded linear operator for $t \geq 0$. With some convergence conditions on initial data and $f(t;\varepsilon)$ and smoothness conditions on $K(\cdot)$, we prove that when $\varepsilon \to 0$, one has $u(t;\varepsilon) \to w(t)$ and $u'(t;\varepsilon) \to w'(t)$ in X uniformly on $[0,T]$ for any fixed $T > 0$. An application to viscoelasticity is given.

1 Introduction.

Consider the example of vibration of a membrane in a viscous medium, given by

$$\rho v_{tt} + \gamma v_t = \sigma \Delta v, \tag{1}$$

where ρ, γ, and σ are the mass density per unit area of the membrane, the coefficient of viscosity of the medium, and the tension of the membrane, respectively. Fattorini [5] rewrote (1.1) as

$$\varepsilon^2 u_{tt} + u_t = \Delta u, \tag{2}$$

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with $\varepsilon = (\rho\sigma)^{1/2}/\gamma$ and $u(x,(\sigma/\gamma)t) = v(x,t)$. Now, if $\varepsilon \to 0$ (this is case when the medium is highly viscous ($\gamma \gg 1$), or the density ρ is very small), then formally, the "limiting" of (1.2) will be the first order (in t) differential equation

$$u_t = \Delta u. \tag{3}$$

Hence Fattorini [5] formulated the problem for

$$\begin{aligned} \varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) &= Au(t;\varepsilon) + f(t;\varepsilon), \quad t \geq 0, \\ u(0;\varepsilon) &= u_0(\varepsilon), \;\; u'(0;\varepsilon) = u_1(\varepsilon), \end{aligned} \tag{4}$$

and

$$w'(t) = Aw(t) + f(t), \;\; t \geq 0, \;\; w(0) = w_0, \tag{5}$$

when $\varepsilon \to 0$, where A generates a strongly continuous cosine family and also a strongly continuous semigroup in a Banach space X. The behavior of the solution of (1.4) as $\varepsilon \to 0$ was referred to as the singular perturbation.

It was shown in [5] with some convergence conditions on initial data and $f(t;\varepsilon)$ that if $\varepsilon \to 0$, then $u(t;\varepsilon) \to w(t)$ and $u'(t;\varepsilon) \to w'(t)$ (in some sense) for t in compact sets of $(0,\infty)$, or of $[0,\infty)$ if no "initial layer".

We want to generalize these concepts and results to integrodifferential equations

$$\begin{aligned} \varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) &= Au(t;\varepsilon) + \int_0^t K(t-s)Au(s;\varepsilon)ds + f(t;\varepsilon), \quad t \geq 0, \\ u(0;\varepsilon) &= u_0(\varepsilon), \;\; u'(0;\varepsilon) = u_1(\varepsilon), \end{aligned} \tag{6}$$

and

$$w'(t) = Aw(t) + \int_0^t K(t-s)Aw(s)ds + f(t), \;\; t \geq 0, \;\; w(0) = w_0, \tag{7}$$

in a Banach space X, with A the generator of a strongly continuous cosine family and a strongly continuous semigroup, and $K(t)$ a bounded linear operator for $t \geq 0$, and prove, with some convergence conditions on initial data and $f(t;\varepsilon)$ and smoothness conditions on $K(\cdot)$, that when $\varepsilon \to 0$, one has $u(t;\varepsilon) \to w(t)$ and $u'(t;\varepsilon) \to w'(t)$ in X uniformly for $t \in [0,T]$ for any fixed $T > 0$. So that we can apply, for example, to equations in linear viscoelasticity,

$$\begin{aligned} \rho u_{tt}(t;\rho) + u_t(t;\rho) &= \Delta u(t;\rho) + \int_0^t K(t-s)\Delta u(s;\rho)ds + f(t;\rho), \quad t \geq 0, \\ u(0;\rho) &= u_0(\rho), \;\; u_t(0;\rho) = u_1(\rho), \end{aligned} \tag{8}$$

and show that when the density of the material $\rho \to 0$, solutions and their derivatives will converge to solutions and derivatives of the "limiting" heat equation

$$w_t(t) = \Delta w(t) + \int_0^t K(t-s)\Delta w(s)ds + f(t), \;\; t \geq 0, \;\; w(0) = w_0. \tag{9}$$

This result also relates to a concept called “change the type” (from hyperbolic to parabolic).

The main task here is to handle the integral term. What we will do is to formally rewrite (1.6) and (1.7) into equations that look like (1.4) and (1.5), and then estimate $u(t;\varepsilon) - w(t)$ and $u'(t;\varepsilon) - w'(t)$. Other studies in singular perturbations can be found, for example, in Goldstein [6], Hale and Raugel [10], Smith [13], Grimmer and Liu [8], and the references therein.

2 Convergence of Solution and Derivative.

In this paper we make the following hypotheses:

H1. Operator A generates a strongly continuous cosine family $C(\cdot)$ and a strongly continuous semigroup $S(\cdot)$. (See [5].)

H2. For $t \geq 0, K(t), K'(t), K''(t) \in B(X), (B(X) =$ space of all bounded linear operators on X). For $x \in X, Kx, K'x, K''x \in L^1_{loc}(R^+, X)$. Here K', K'' are the strong derivatives.

H3. $f(\cdot;\varepsilon), f \in C^1(R^+, X), f(0) = 0,$ where $\varepsilon > 0, R^+ = [0, \infty)$.

In order to verify the existence of solutions of (1.6) we change it to another more common form. (See [5].) Let

$$u(t;\varepsilon) = e^{-t/2\varepsilon^2} v(t/\varepsilon).$$

Then (1.6) can be replaced by

$$v''(t/\varepsilon) = \Big(A + \frac{1}{4\varepsilon^2}\Big) v(t/\varepsilon) + \int_0^t K(t-s) e^{(t-s)/2\varepsilon^2} A v(s/\varepsilon) ds + e^{t/2\varepsilon^2} f(t;\varepsilon).$$

Now let $h = t/\varepsilon$ and then change h to t to get

$$\begin{aligned} v''(t) &= \Big(A + \frac{1}{4\varepsilon^2}\Big) v(t) + \int_0^t \hat{K}(t-s) A v(s) ds + \hat{f}(t), \qquad (1) \\ v(0;\varepsilon) &= u_0(\varepsilon), \;\; v'(0;\varepsilon) = \frac{1}{2\varepsilon} u_0(\varepsilon) + \varepsilon u_1(\varepsilon), \end{aligned}$$

where $\Big(A + \frac{1}{4\varepsilon^2}\Big)$ generates a strongly continuous cosine family and

$$\hat{K}(t) = \varepsilon K(\varepsilon t) e^{t/2\varepsilon}, \quad \hat{f}(t) = f(\varepsilon t;\varepsilon) e^{t/2\varepsilon}, \;\; t \geq 0.$$

Note that the existence and uniqueness of solutions of (2.1) and (1.7) were obtained in [3, 4, 7, 14, 15], and we are only interested in singular perturbations in this paper, so we may assume that (1.6) and (1.7) have unique solutions $u(t;\varepsilon)$ and $w(t)$ respectively for every $\varepsilon > 0$.

Now we can state and prove the following result concerning the convergence of solutions and derivatives, with the following hypotheses:

H4. $u_0(\varepsilon), u_1(\varepsilon), w_0 \in D(A)$, $u_0(\varepsilon) \to w_0, Au_0(\varepsilon) \to Aw_0$, $u_1(\varepsilon) \to Aw_0$, as $\varepsilon \to 0$.

H5. For any $T > 0, f(t;\varepsilon) \to f(t), f'(t;\varepsilon) \to f'(t)$ in X uniformly for $t \in [0,T]$, as $\varepsilon \to 0$.

Theorem 2.1. Assume that hypotheses (H1) – (H5) are satisfied. Then for any $T > 0, u(t;\varepsilon) \to w(t)$ and $u'(t;\varepsilon) \to w'(t)$ in X uniformly for $t \in [0,T]$, as $\varepsilon \to 0$.

Proof. Define

$$R * H(t) = \int_0^t R(t-s)H(s)ds \text{ and } \delta * H = H.$$

Then we can find the solution F of $F + K + F * K = 0$. (See [1, 2, 11, 12].) So that

$$(\delta + F) * (\delta + K) = \delta. \tag{2}$$

Now write (1.6) as

$$\varepsilon^2 u''(\varepsilon) + u'(\varepsilon) = (\delta + K) * Au(\varepsilon) + f(\varepsilon).$$

Then we have

$$(\delta + F) * \Big[\varepsilon^2 u''(\varepsilon) + u'(\varepsilon)\Big] = Au(\varepsilon) + (\delta + F) * f(\varepsilon)\,.$$

Hence

$$\varepsilon^2 u''(\varepsilon) + u'(\varepsilon) = Au(\varepsilon) + (\delta + F) * f(\varepsilon) - F * \Big[\varepsilon^2 u''(\varepsilon) + u'(\varepsilon)\Big].$$

Integration by parts yields

$$F * u'(t;\varepsilon) = \int_0^t F'(t-s)u(s;\varepsilon)\,ds + F(0)u(t;\varepsilon) - F(t)u_0(\varepsilon),$$

$$\begin{aligned} F * u''(t;\varepsilon) \quad &= \quad \int_0^t F''(t-s)u(s;\varepsilon)\,ds \\ &\quad + F(0)u'(t;\varepsilon) - F(t)u_1(\varepsilon) + F'(0)u(t;\varepsilon) - F'(t)u_0(\varepsilon)\,. \end{aligned}$$

Therefore, (1.6) can be replaced by

$$\begin{aligned} \varepsilon^2 u''(t;\varepsilon) + u'(t;\varepsilon) \quad &= \quad Au(t;\varepsilon) + \hat{f}(t;\varepsilon), \quad t \geq 0, \\ u(0;\varepsilon) \quad &= \quad u_0(\varepsilon), \;\; u'(0;\varepsilon) = u_1(\varepsilon), \end{aligned} \tag{3}$$

with

$$\begin{aligned} \hat{f}(t;\varepsilon) &= (\delta+F)*f(t;\varepsilon) - F*\Big[\varepsilon^2 u''(t;\varepsilon)+u'(t;\varepsilon)\Big] \\ &= (\delta+F)*f(t;\varepsilon) - \int_0^t F'(t-s)u(s;\varepsilon)\,ds \\ &\quad -F(0)u(t;\varepsilon)+F(t)u_0(\varepsilon) - \varepsilon^2\Big[\int_0^t F''(t-s)u(s;\varepsilon)\,ds \\ &\quad +F'(0)u(t;\varepsilon)-F'(t)u_0(\varepsilon)+F(0)u'(t;\varepsilon)-F(t)u_1(\varepsilon)\Big]. \end{aligned} \tag{4}$$

Similarly, (1.7) can be replaced by

$$w'(t) = Aw(t) + \hat{f}(t), \quad t \ge 0, \quad w(0) = w_0, \tag{5}$$

with

$$\begin{aligned} \hat{f}(t) &= (\delta+F)*f(t) - F*w'(t) \\ &= (\delta+F)*f(t) - \int_0^t F'(t-s)w(s)ds - F(0)w(t) + F(t)w_0. \end{aligned} \tag{6}$$

By linearity, we view (2.3) (and (2.5)) as the addition of two solutions such that the first one is with $\hat{f}(t;\varepsilon)$ (and $\hat{f}(t)$) being zero and the second with zero initial data.

For the first solutions for (2.3) and (2.5), it was shown in [5, p.198] that under hypothesis (H4), the solution and the derivative of (2.3) (with $\hat{f}(t;\varepsilon)=0$) converge to solution and the derivative of (2.5) (with $\hat{f}(t)=0$) in X uniformly for $t\in[0,T]$ for any fixed $T>0$, as $\varepsilon\to 0$. So we only need to study (2.3) and (2.5) with zero initial data. We still use $u(t;\varepsilon)$ and $w(t)$ to denote the solutions corresponding to zero initial data since it causes no confusion.

Now, according to formula (7.36) in [5, p.217], the solutions of (2.3) and (2.5) corresponding to zero initial data satisfy

$$\begin{aligned} u'(t;\varepsilon) - w'(t) &= \int_0^t G'(t-s;\varepsilon)\Big[\hat{f}(s;\varepsilon)-\hat{f}(s)\Big]ds + \Big[G(t;\varepsilon)-S(t)\Big]\hat{f}(t) \\ &\quad + \int_0^t \Big[G'(t-s;\varepsilon)-S'(t-s)\Big]\Big[\hat{f}(s)-\hat{f}(t)\Big]\,ds, \end{aligned} \tag{7}$$

where $S(\cdot), C(\cdot)$ are given in (H1), $R(\cdot;\varepsilon), G(\cdot;\varepsilon)$ are linear operators defined in [5] using the Bessel functions, and they have the following properties: For some constants $\alpha,\omega>0$,

P1. $\|C(t)\|, \|S(t)\| \le \alpha e^{\omega^2 t}$, $t\ge 0$, $\varepsilon>0$.

P2. $\|G(t;\varepsilon)\|, \varepsilon^2\|G'(t;\varepsilon)\| \le \alpha e^{\omega^2 t}$, $t\ge 0$, $\varepsilon>0$.

P3. $\varepsilon^2 G'(t;\varepsilon) = e^{-t/2\varepsilon^2} C(t/\varepsilon) + \frac{1}{2}\Big[R(t;\varepsilon) - G(t;\varepsilon)\Big]$.

P4. If $t(\varepsilon) > 0$ for $\varepsilon > 0$ with $t(\varepsilon)/\varepsilon^2 \to \infty$ as $\varepsilon \to 0$, then for every $T > 0$,

$$\lim_{\varepsilon\to 0} \sup_{t(\varepsilon)\le t\le T} \|R(t;\varepsilon)x - S(t)x\| = 0, \quad \lim_{\varepsilon\to 0} \sup_{t(\varepsilon)\le t\le T} \|G(t;\varepsilon)x - S(t)x\| = 0$$

uniformly for x in bounded subsets of X.

Now let $T > 0$ be fixed and consider (2.7) for $t \in [0, T]$. We are going to estimate $u'(t;\varepsilon) - w'(t)$. First we have

$$\begin{aligned}\int_0^t G'(t-s;\varepsilon)\Big[\hat f(s;\varepsilon) - \hat f(s)\Big]\,ds = \\ \int_0^t G'(t-s;\varepsilon)\Big[\hat f(s;\varepsilon) - \hat f(s) + \varepsilon^2 F(0)u'(s;\varepsilon)\Big]\,ds - \\ \int_0^t G'(t-s;\varepsilon)\varepsilon^2 F(0)u'(s;\varepsilon)ds.\end{aligned} \tag{8}$$

$$\begin{aligned}\int_0^t G'(t-s;\varepsilon)\varepsilon^2 F(0)u'(s;\varepsilon)\,ds = \\ \int_0^t G'(t-s;\varepsilon)\varepsilon^2 F(0)\Big[u'(s;\varepsilon) - w'(s)\Big]\,ds + \int_0^t G'(t-s;\varepsilon)\varepsilon^2 F(0)w'(s)\,ds.\end{aligned} \tag{9}$$

Note that from property (P3),

$$\begin{aligned}\int_0^t G'(t-s;\varepsilon)\varepsilon^2 F(0)w'(s)ds = \\ \int_0^t \Big\{e^{-(t-s)/2\varepsilon^2} C((t-s)/\varepsilon) + \frac{1}{2}\Big[R(t-s;\varepsilon) - G(t-s;\varepsilon)\Big]\Big\}F(0)w'(s)\,ds.\end{aligned} \tag{10}$$

Observe that $w'(s)$ is locally bounded, so use property (P4) with $t(\varepsilon) = \varepsilon$ to obtain for any $t, s \in [0, T]$ with $s < t$,

$$\Big[R(t-s;\varepsilon) - G(t-s;\varepsilon)\Big]F(0)w'(s) \to 0, \ \ \varepsilon \to 0. \tag{11}$$

Hence the dominated convergence theorem can be used to prove that

$$\int_0^t \Big[R(t-s;\varepsilon) - G(t-s;\varepsilon)\Big]F(0)w'(s)\,ds \to 0, \ \ \varepsilon \to 0, \tag{12}$$

uniformly for $t \in [0,T]$. Next, assume that $\varepsilon > 0$ is so small that $4\varepsilon\omega^2 \le 1$, then from (P1),

$$\begin{aligned}
&\int_0^t e^{-(t-s)/2\varepsilon^2}\|C((t-s)/\varepsilon)\|ds = \int_0^t e^{-s/2\varepsilon^2}\|C(s/\varepsilon)\|ds \\
&\le \alpha\int_0^t e^{-s/2\varepsilon^2+\omega^2 s/\varepsilon}ds = \Big[2\alpha\varepsilon^2/(1-2\varepsilon\omega^2)\Big]\Big[1-e^{(2\varepsilon\omega^2-1)t/2\varepsilon^2}\Big] \\
&\le 4\alpha\varepsilon^2 \to 0,\ \ \varepsilon\to 0, \qquad (13)
\end{aligned}$$

uniformly for $t \in [0,T]$. Then by (2.9), (2.10), (2.12), (2.13), and property (P2), we obtain

$$\|\int_0^t G'(t-s;\varepsilon)\varepsilon^2 F(0)u'(s;\varepsilon)\,ds\| \le (\text{type } 1) + 0(\varepsilon,[0,T]), \qquad (14)$$

where (type 1) is of the form

$$(\text{constant})\int_0^t \|u'(s;\varepsilon) - w'(s)\|ds, \qquad (15)$$

and $0(\varepsilon,[0,T])$ satisfies

$$0(\varepsilon,[0,T]) \to 0 \text{ as } \ \varepsilon\to 0,\ \text{uniformly for } t\in[0,T]. \qquad (16)$$

Next we have

$$\begin{aligned}
&\int_0^t G'(t-s;\varepsilon)\Big[\hat f(s;\varepsilon) - \hat f(s) + \varepsilon^2 F(0)u'(s;\varepsilon)\Big]\,ds = \\
&\quad G(t;\varepsilon)\Big[\hat f(0;\varepsilon) - \hat f(0) + \varepsilon^2 F(0)u_1(\varepsilon)\Big] \\
&\quad + \int_0^t G(t-s;\varepsilon)\Big[\hat f(s;\varepsilon) - \hat f(s) + \varepsilon^2 F(0)u'(s;\varepsilon)\Big]'ds, \qquad (17)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t G(t-s;\varepsilon)\Big[\hat f(s;\varepsilon) - \hat f(s) + \varepsilon^2 F(0)u'(s;\varepsilon)\Big]'\,ds = \qquad (18) \\
&\quad \int_0^t G(t-s;\varepsilon)\Big\{\Big[f(s;\varepsilon) - f(s)\Big] + \int_0^s F(s-h)\Big[f(h;\varepsilon) - f(h)\Big]\,dh \\
&\quad + F(s)\Big[u_0(\varepsilon) - w_0\Big] - \int_0^s F'(s-h)\Big[u(h;\varepsilon) - w(h)\Big]\,dh \\
&\quad + F(s)\varepsilon^2 u_1(\varepsilon) + \varepsilon^2 F'(s)u_0(\varepsilon) - \Big[\varepsilon^2 F'(0) + F(0)\Big]\Big[u(s;\varepsilon) - w(s)\Big] \\
&\quad - \varepsilon^2 F'(0)w(s) - \varepsilon^2\int_0^s F''(s-h)\Big[u(h;\varepsilon) - w(h)\Big]\,dh
\end{aligned}$$

$$-\varepsilon^2 \int_0^s F''(s-h)w(h)dh\Big\}' ds$$
$$= \int_0^t G(t-s;\varepsilon)\Big\{\Big[f'(s;\varepsilon)-f'(s)\Big] + F(0)\Big[f(s;\varepsilon)-f(s)\Big]$$
$$+ \int_0^s F'(s-h)\Big[f(h;\varepsilon)-f(h)\Big]dh + F'(s)\Big[u_0(\varepsilon)-w_0\Big]$$
$$-F'(0)\Big[u(s;\varepsilon)-w(s)\Big] - \int_0^s F''(s-h)\Big[u(h;\varepsilon)-w(h)\Big]\,dh$$
$$+F'(s)\varepsilon^2 u_1(\varepsilon) + \varepsilon^2 F''(s)u_0(\varepsilon) - \Big[\varepsilon^2 F'(0)+F(0)\Big]\Big[u'(s;\varepsilon)-w'(s)\Big]$$
$$-\varepsilon^2 F'(0)w'(s) - \varepsilon^2 F''(s)\Big[u_0(\varepsilon)-w_0\Big]$$
$$-\varepsilon^2 \int_0^s F''(s-h)\Big[u'(h;\varepsilon)-w'(h)\Big]\,dh$$
$$-\varepsilon^2 F''(s)w_0 - \varepsilon^2 \int_0^s F''(s-h)w'(h)\,dh\Big\}\,ds.$$

Note that with formula (2.14) in [5, p.168], the technique used here can be applied to obtain the convergence of solutions for (2.3) and (2.5). So for simplicity, we omit the details here and assume that under hypotheses (H1) – (H5), one has $u(t;\varepsilon) \to w(t)$ in X uniformly for $t \in [0,T]$. With this remark, property (P2), hypotheses (H1) – (H5), and the fact that $w'(\cdot)$ is locally bounded, we obtain

$$\begin{aligned}\|\int_0^t G(t-s;\varepsilon)\Big[\hat{f}(s;\varepsilon)-\hat{f}(s)+\varepsilon^2 F(0)u'(s;\varepsilon)\Big]' ds\| \\ \le (\text{ type } 1) + 0(\varepsilon,[0,T])\,. \end{aligned} \tag{19}$$

Combine (2.8), (2.14), (2.17), (2.19), and using (H4, H5) and (P2), we get

$$\|\int_0^t G'(t-s;\varepsilon)\Big[\hat{f}(s;\varepsilon)-\hat{f}(s)\Big]ds\| \le (\text{type } 1) + 0(\varepsilon,[0,T]). \tag{20}$$

Next, we have

$$\begin{aligned}&\int_0^t \Big[G'(t-s;\varepsilon)-S'(t-s)\Big]\Big[\hat{f}(s)-\hat{f}(t)\Big]\,ds = \\ &\quad\Big[G(t;\varepsilon)-S(t)\Big]\Big[\hat{f}(0)-\hat{f}(t)\Big] + \int_0^t \Big[G(t-s;\varepsilon)-S(t-s)\Big]\hat{f}'(s)\,ds.\end{aligned} \tag{21}$$

Note that from property (P4), and the fact that $\hat{f}''(t)$ is locally bounded, we have (similar to (2.11) and (2.12))

$$\int_0^t \Big[G(t-s;\varepsilon)-S(t-s)\Big]\hat{f}'(s)ds = 0(\varepsilon,[0,T]),\ t\in[0,T]. \tag{22}$$

Therefore, combining (2.7), (2.20), (2.21), and (2.22), we obtain (by H3, $\hat{f}(0) = f(0) = 0$)

$$\begin{aligned} &\|u'(t;\varepsilon) - w'(t)\| \leq 0(\varepsilon, [0,T]) + (\text{ constant}) \int_0^t \|u'(s;\varepsilon) - w'(s)\| ds \\ &+ \| \Big[G(t;\varepsilon) - S(t)\Big] \hat{f}(t) + \Big[G(t;\varepsilon) - S(t)\Big] \Big[\hat{f}(0) - \hat{f}(t)\Big] \| \\ &= 0(\varepsilon, [0,T]) + (\text{constant}) \int_0^t \|u'(s;\varepsilon) - w'(s)\| ds,\ t \in [0,T]. \end{aligned} \tag{23}$$

Now the Gronwall's inequality ([9]) can be used to obtain

$$\|u'(t;\varepsilon) - w'(t)\| \leq 0(\varepsilon, [0,T]), \quad t \in [0,T]. \tag{24}$$

This proves the theorem. □

Note that the Laplacian Δ in (1.8) with appropriate boundary conditions generates a strongly continuous cosine family and a strongly continuous semigroup, so with some convergence conditions on initial data and $f(t;\varepsilon)$ and smoothness conditions on $K(\cdot)$, Theorem 2.1 can be applied to (1.8) and (1.9). We omit the details here.

Remark. It was pointed out in [5, p.214] that $f(0) = 0$ is almost necessary to obtain the convergence in derivative at $t = 0$. We also need this technical condition in our proof.

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