On a Class of Elliptic Systems in $\mathbb{R}^N$ *

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Abstract

We consider a class of variational systems in $\mathbb{R}^N$ of the form

$$
\begin{align*}
-\Delta u + a(x)u &= F_u(x, u, v) \\
-\Delta v + b(x)v &= F_v(x, u, v),
\end{align*}
$$

where $a, b : \mathbb{R}^N \to \mathbb{R}$ are continuous functions which are coercive; i.e., $a(x)$ and $b(x)$ approach plus infinity as $x$ approaches plus infinity. Under appropriate growth and regularity conditions on the nonlinearities $F_u(\cdot)$ and $F_v(\cdot)$, the (weak) solutions are precisely the critical points of a related functional defined on a Hilbert space of functions $u, v$ in $H^1(\mathbb{R}^N)$.

By considering a class of potentials $F(x, u, v)$ which are nonquadratic at infinity, we show that a weak version of the Palais-Smale condition holds true and that a nontrivial solution can be obtained by the Generalized Mountain Pass Theorem.

Our approach allows situations in which $a(\cdot)$ and $b(\cdot)$ may assume negative values, and the potential $F(x, s)$ may grow either faster or slower than $|s|^2$.

1 Introduction

In this paper we consider a class of semilinear elliptic systems in $\mathbb{R}^N$ of the form

$$(P) \quad \begin{cases} 
-\Delta u + a(x)u & = f(x, u, v) \text{ in } \mathbb{R}^N \\
-\Delta v + b(x)v & = g(x, u, v) \text{ in } \mathbb{R}^N,
\end{cases}$$

where $a, b : \mathbb{R}^N \to \mathbb{R}$ are continuous functions satisfying $a(x) \geq a_0$, $b(x) \geq b_0$ $\forall x \in \mathbb{R}^N$ and such that $\lim_{|x| \to \infty} a(x) = \lim_{|x| \to \infty} b(x) = +\infty$. The nonlinearities $f, g : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$ are also continuous with $f(x, 0, 0) = g(x, 0, 0) \equiv 0$, so that $(u, v) \equiv (0, 0)$ solves $(P)$ and we therefore must look for nontrivial solutions. We shall consider the variational situation in which $(f, g) = \nabla F$ for some

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A $C^1$ function $F: \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$, where $\nabla F$ stands for the gradient of $F$ in the variables $U = (u, v) \in \mathbb{R}^2$.

In the scalar case $-\Delta u + a(x)u = f(x, u)$, among other results, P. Rabinowitz [14] showed existence of a nontrivial solution $u \in W^{1,2}(\mathbb{R}^N, \mathbb{R})$ under the assumption that $f(x, u)$ was superlinear with subcritical growth. This was done by a mountain-pass type argument [1] applied to the pertinent functional

$$I(u) = \int_{\mathbb{R}^N} \left(\frac{1}{2}(|\nabla u|^2 + a(x)u^2) - F(x, u)\right) dx,$$

without the use of the Palais-Smale condition, which was not clear to hold true. On the other hand, Ding and Li showed in [8] existence of a nontrivial solution without the use of the Palais-Smale condition, which was not clear to hold true.

Motivated by these results and using some recent ideas from [7, 6], our purpose in this paper is twofold. First we consider a class of potentials $F(x, u, v)$ which we call nonquadratic at infinity (cf. [7, 6]) and show that a weaker version of the Palais-Smale condition holds true so that a nontrivial solution of $(P)$ can be obtained by a variant of the Generalized Mountain-Pass Theorem [12]. Such an existence result partially extends and, in fact, complements the above mentioned results of Rabinowitz and Ding-Li. Secondly we show that, under the hypotheses of superlinearity used in [14, 8], the Palais-Smale condition is indeed satisfied so that the standard Mountain-Pass Theorem can be used to prove those results. More precisely, we will prove Theorems 1.1 and 1.2 below, where the following hypotheses will be used:

(A0) $a, b \in C(\mathbb{R}^N)$, $a(x) \geq a_0, b(x) \geq b_0$ for some positive constants $a_0, b_0$, and all $x \in \mathbb{R}^N$.

(A1) $a(x) \to +\infty, b(x) \to +\infty$ as $|x| \to \infty$.

(F0) $|\nabla f(x, U)| + |\nabla g(x, U)| \leq c(1 + |U|^{p-1})$ for all $(x, U) \in \mathbb{R}^N \times \mathbb{R}^2$, where $f, g \in C^1(\mathbb{R}^N \times \mathbb{R}^2)$, $c > 0$ and $1 \leq p < (N + 2)/(N - 2)$ if $N \geq 3$ (or $1 \leq p < \infty$ if $N = 1, 2$).

(F1)$_\mu$ $U \cdot \nabla F(x, U) \geq \mu F(x, U) > 0$ for all $(x, U) = \mathbb{R}^N \times \mathbb{R}^2 \setminus \{(0, 0)\}$.

(F2)$_\nu$ $U \cdot \nabla F(x, U) - 2F(x, U) \geq a |U|^{\nu} > 0$ for all $(x, U) = \mathbb{R}^N \times \mathbb{R}^2 \setminus \{(0, 0)\}$.

In what follows, we let $0 < \lambda_1 < \lambda_2 < \ldots$ denote the distinct eigenvalues of the problem $-\Delta U + A(x)U = \lambda U$, $x \in \mathbb{R}^N$, where $U = (u, v)$, $\Delta = \text{diag}(\Delta, \Delta)$ and $A(x) = \text{diag}(a(x), b(x))$.

**Theorem 1.1:** Suppose $(A_0), (A_1)$ and $(F_0), (F_2)_\nu$ are satisfied with $\nu > \frac{N}{2}(p - 1)$ if $N \geq 2$ (or $\nu > p - 1$ if $N = 1$). If, in addition, we have

(F3) $\limsup_{|U| \to 0} \frac{2F(x, U)}{|U|^2} \leq \alpha < \lambda_k < \beta \leq \liminf_{|U| \to \infty} \frac{2F(x, U)}{|U|^2}$ unif. for $x \in \mathbb{R}^N$, where $A_0 = (a_i, b_i)$.
Theorem 1.2 If \((A_0), (A_1), (F_0), (F_1)\) are satisfied with \(\mu > 2\), then the functional \(I\) associated with problem \((P)\) satisfies the Palais-Smale condition and \((P)\) has a nonzero weak solution \(U \in C^1(\mathbb{R}^N, \mathbb{R}^2) \cap W^{1,2}(\mathbb{R}^N, \mathbb{R}^2)\).

Remark 1.3 In the case that \(a, b \in C^1(\mathbb{R}^N)\) and \(f, g \in C^2(\mathbb{R}^N, \mathbb{R}^2)\) then, by standard bootstrap arguments, the weak \(C^1\) solution \(U\) above is indeed a classical solution of \((P)\).

Remark 1.4 Conditions \((F_3), (F_4)\) represent a crossing of the eigenvalue \(\lambda_k\) by the nonlinearity \((f, g)\). On the other hand, when \(f\) and \(g\) are x-independent, a simple calculation shows that \((F_1)_\mu\) with \(\mu > 2\) implies \(\lim_{|U| \to 0} F(U)/|U|^2 = 0\) and \(\lim_{|U| \to \infty} F(U)/|U|^2 = +\infty\), so that all eigenvalues are crossed in this case; in particular, \((F_3), (F_4)\) are automatically satisfied with \(k = 1\) (and letting \(\lambda_0 = 0\)). Also, it is not hard to show (see Remark 2.5) that \((F_1)_\mu\) implies \((F_2)_\mu\) provided that we have \(\lim \inf_{|U| \to 0} F(U)/|U|^4 \geq a > 0\). In this case, when \(p \leq 1 + 4/N\) and \(N \geq 3\) in \((F_0)\), Theorem 1.1 above extends Theorem 1.7 in [14].

Remark 1.5 It will be clear from the proof of Theorem 1.1 that a similar result holds with \((F_2)_\mu\) replaced by its “dual”

\[
(F_2)_\mu^- \qquad U \cdot \nabla F(x, U) - 2F(x, U) \leq -a |U|^\mu < 0
\]

for all \(x \in \mathbb{R}^N, U \in \mathbb{R}^2 \setminus \{(0, 0)\}\).

2 Proofs of Theorems 1.1 and 1.2

Let \(H^1 = H^1(\mathbb{R}^N, \mathbb{R}^2)\) denote the Sobolev space of pairs \(U = (u, v)\) of \(L^2\)-functions \(u, v : \mathbb{R}^N \to \mathbb{R}\) with weak derivatives \(\partial u/\partial x_j, \partial v/\partial x_j\) \((j = 1, \ldots, N)\) also in \(L^2(\mathbb{R}^N)\), endowed with its usual norm

\[
\|U\|_{H^1}^2 = \int (|\nabla u|^2 + |u|^2 + |\nabla v|^2 + |v|^2) \, dx.
\]

Throughout this paper, unless specified otherwise, all integrals are understood to be taken over all of \(\mathbb{R}^N\). Given continuous functions \(a, b : \mathbb{R}^N \to \mathbb{R}\) satisfying \(a(x) \geq a_0 > 0, b(x) \geq b_0 > 0\) \(\forall x \in \mathbb{R}^N\), we consider the subspace \(E \subset H^1\) defined by

\[
E = \{U = (u, v) \in H^1 : \int (|\nabla u|^2 + |\nabla v|^2 + a(x)|u|^2 + b(x)|v|^2) \, dx < \infty\}
\]
and endowed with the norm
$$\|U\|^2 = \int ((\nabla u)^2 + (\nabla v)^2 + a(x)|u|^2 + b(x)|v|^2) \, dx.$$ 

Since $a(x) \geq a_0 > 0$, $b(x) \geq b_0 > 0$, we clearly have the continuous embedding $E \hookrightarrow H^1$. We also recall that Sobolev’s Theorem gives the continuous embeddings $H^1 \hookrightarrow L^q(\mathbb{R}^N, \mathbb{R}^2)$ for all $2 \leq q \leq 2^* := 2N/(N-2)$, if $N \geq 3$ (respectively, $2 \leq q < \infty$ if $N = 1, 2$).

Now, let us consider the functional $I : E \rightarrow \mathbb{R}$ given by

$$I(u,v) = \frac{1}{2}((\nabla u)^2 + (\nabla v)^2 + a(x)|u|^2 + b(x)|v|^2) \, dx - \int F(x,u,v) \, dx$$

$$= \frac{1}{2}\|U\|^2 - N(U). \tag{1}$$

Assuming the growth condition $(F_0)$, it can be shown (cf. Theorem A.VI in [4]) that the functional $N$ is indeed well-defined and of class $C^1$ on $H^1$ and (hence) on the space $E$, with

$$\langle \nabla N(U), \Phi \rangle = \int (f(x,u,v)\varphi + g(x,u,v)\psi) \, dx \tag{2}$$

for all $U = (u,v)$, $\Phi = (\varphi, \psi) \in E$, where we are denoting by $\langle \cdot, \cdot \rangle$ the inner product on $E$. In fact, one can say more when both functions $a(x), b(x)$ are coercive, that is, when condition $(A_1)$ is also satisfied.

**Proposition 2.1** (i) If $(A_0)$ and $(A_1)$ hold true, then the embedding $E \hookrightarrow L^2(\mathbb{R}^N, \mathbb{R}^2)$ is compact.

(ii) Under conditions $(A_0), (A_1)$ and $(F_0)$ the mapping $\nabla N : E \rightarrow E$ is compact.

**Proof of (i)** We will show that $U_m \rightharpoonup 0$ strongly in $L^2(\mathbb{R}^N, \mathbb{R}^2)$ whenever $U_m \rightharpoonup 0$ weakly in $E$. Indeed, let $C > 0$ be such that $\|U_m\| \leq C$. Given $\varepsilon > 0$, pick $R > 0$ such that $a(x) \geq 2C^2/\varepsilon$, $b(x) \geq 2C^2/\varepsilon$ for all $|x| \geq R$ and denote by $B_R$ the ball of radius $R$ in $\mathbb{R}^N$. Then, since the restriction operator $U \rightarrow U|_{B_R}$ is continuous from $H^1(\mathbb{R}^N, \mathbb{R}^2)$ into $H^1(B_R, \mathbb{R}^2)$, we also have that $U_m \rightharpoonup 0$ weakly in $H^1(B_R, \mathbb{R}^2)$. In particular, the compact embedding $H^1(B_R, \mathbb{R}^2) \hookrightarrow L^2(B_R, \mathbb{R}^2)$ implies that for some natural number $m_0$,

$$\int_{B_R} (|u_m|^2 + |v_m|^2) \, dx \leq \frac{\varepsilon}{2} \quad \forall m \geq m_0. \tag{3}$$

On the other hand, by our choice of $R > 0$, we clearly have

$$\frac{2}{\varepsilon} \int_{\mathbb{R}^N \setminus B_R} (|u_m|^2 + |v_m|^2) \, dx \leq \frac{1}{C^2} \int_{\mathbb{R}^N \setminus B_R} (a(x)|u_m|^2 + b(x)|v_m|^2) \, dx$$

$$\leq \frac{1}{C^2} \|U_m\|^2 \leq 1. \tag{4}$$
Combining (3) and (4) we obtain that \( |U_m|_{L^2}^2 \leq \epsilon \) for all \( m \geq m_0 \).

**Proof of (ii)** We assume \( N \geq 3 \), the case \( N = 1, 2 \) being similar. Assumption (F0) implies

\[
|f(x, U) - f(x, \bar{U})| \leq \left( a_1 + b_1 \left( |U|^{p-1} + |\bar{U}|^{p-1} \right) \right) |U - \bar{U}|, \tag{5}
\]

for all \( x \in \mathbb{R}^N \), \( U, \bar{U} \in \mathbb{R}^2 \), with a similar estimate holding true for \( g(x, U) \).

Now, letting \( 2^* = 2N/(N - 2) \), \( p_1 = 2^*/(p - 1) \), \( p_2 = p_3 = 2p_1/(p_1 - 1) \) and recalling that \( p < (N + 2)/(N - 2) = 2^* - 1 \) in (F0), we have that \( p_1, p_2, p_3 > 1 \) with \( p_2, p_3 < 2^* \) and \( p_1^{*+1} + p_2^{-1} + p_3^{-1} = 1 \). Therefore, (5) and Hölder’s inequality give

\[
\int |(f(x, U) - f(x, \bar{U}))\varphi| \, dx \\
\leq A_1|U - \bar{U}|_{L^2} |\varphi|_{L^2} + B_1 \left( |U|_{L^{2^*}}^{p_1} + |\bar{U}|_{L^{2^*}}^{p_1} \right) |U - \bar{U}|_{L^{p_2}} |\varphi|_{L^{p_2}}, \tag{6}
\]

for all \( \varphi \in H^1(\mathbb{R}^N) \), with a similar estimate also holding for \( g(x, U) \), namely,

\[
\int |(g(x, U) - g(x, \bar{U}))\psi| \, dx \\
\leq A_2|U - \bar{U}|_{L^2} |\psi|_{L^2} + B_2 \left( |U|_{L^{2^*}}^{p_2} + |\bar{U}|_{L^{2^*}}^{p_2} \right) |U - \bar{U}|_{L^{p_2}} |\psi|_{L^{p_2}}, \tag{7}
\]

for all \( \psi \in H^1(\mathbb{R}^N) \). From these, letting \((\varphi, \psi) = \nabla N(U) - \nabla N(\bar{U})\), we obtain

\[
\|\nabla N(U) - \nabla N(\bar{U})\| \leq A|U - \bar{U}|_{L^2} + B \left( |U|_{L^{2^*}}^{p_1} + |\bar{U}|_{L^{2^*}}^{p_1} \right) |U - \bar{U}|_{L^{p_2}}. \tag{8}
\]

On the other hand, using the continuous embedding \( E \hookrightarrow L^q(\mathbb{R}^N, \mathbb{R}^2), \ 2 \leq q \leq 2^* \), together with the interpolation inequality (where \( 1/q = \sigma/2 + (1 - \sigma)/2^* \))

\[
|U|_{L^q} \leq |U|_{L^{2^*}}^{1-\sigma} |U|_{L^{2^*}}^{\sigma} \quad \forall U \in L^2 \cap L^{2^*}
\]

and the fact (proved in (i)) that the embedding \( E \hookrightarrow L^2 \) is compact, we infer that the embeddings \( E \hookrightarrow L^q \) are also compact for \( 2 \leq q < 2^* \). Therefore, using (8) and recalling that \( p_2 < 2^* \), we conclude that \( \nabla N(U_m) \rightharpoonup \nabla N(U) \) strongly in \( E \) whenever \( U_m \rightharpoonup U \) weakly in \( E \). The proof of Proposition 2.1 is complete.

**Remark 2.2** Let \( H = l^2(\mathbb{N}) \) be the Hilbert space of square-summable sequences \( a = (a_j)_{j \in \mathbb{N}} \) with its usual norm \( |a|_H^2 = \sum a_j^2 \). As is well-known, given a sequence \( \{ \epsilon_j \} \subset \mathbb{R}_+ \) with \( \lim_{j \to \infty} \epsilon_j = 0 \), the operator \( T : H \to H \) defined by \( (Ta)_j = \epsilon_j a_j \) is a compact operator. This fact can also be stated by saying that, given a positive sequence \( \{ M_j \} \) with \( \lim_{j \to \infty} M_j = +\infty \), the embedding \( E \hookrightarrow H \) is compact, where \( E = \{ a = (a_j) \in H : \|a\|^2 = \sum M_j a_j^2 < \infty \} \). Proposition 2.1 (i) above is an expression of this fact to our present situation.

We learned from P. Rabinowitz that similar versions of Proposition 2.1 (i) were also proved in [11, 8].
Next we recall a compactness condition of the Palais-Smale type which was introduced by Cerami in [5]. It was subsequently used by Bartolo-Benci-Fortunato [2] to prove a deformation theorem (Thm 1.3 in [2]) and, as a consequence, general minimax results as in Benci-Rabinowitz [3].

**Definition 2.3** A functional \( I \in C^1(E, \mathbb{R}) \) is said to satisfy condition \((C)\) if any sequence \( \{U_m\} \subset E \) such that \( I(U_m) \) is bounded and \( 1 + \|U_m\| \|\nabla I(U_m)\| \to 0 \) possesses a convergent subsequence.

Note that \((C)\) is implied by the usual Palais-Smale condition \((PS)\): Any sequence \( \{U_m\} \subset E \) such that \( I(U_m) \) is bounded and \( \|\nabla I(U_m)\| \to 0 \) possesses a convergent subsequence.

In our case, where \( I(U) = q(U) - N(U) \) is a perturbation of the quadratic form \( q(U) = \frac{1}{2} \|U\|^2 \), it turns out that if \( N \) is superquadratic at infinity in the sense of \((F1)\), then \( I \) satisfies the usual Palais-Smale condition \((PS)\). In fact, we will show it suffices that \( I \) be nonquadratic at infinity in the sense of \((F2)\) for condition \((C)\) to be satisfied.

**Proposition 2.4** Assume that \((A_0), (A_1)\) and \((F_0)\) hold true. Then:

(i) Condition \((F1)\) implies \((PS)\) whenever \( \mu > 2 \);

(ii) Condition \((F2)\) implies \((C)\) whenever \( \nu > \frac{N}{2}(p-1) \) if \( N \geq 2 \) (or \( \nu > p-1 \) if \( N = 1, 2 \)).

**Proof of (i)** Let \( \{U_m\} \subset E \) be such that \( |I(U_m)| \leq K \) and \( \|\nabla I(U_m)\| = \epsilon_m \to 0 \). Then,

\[
\frac{\mu}{2} - 1 \|U_m\|^2 = \mu I(U_m) - \langle \nabla I(U_m), U_m \rangle + \int \mu F(x, U_m) - U_m \cdot \nabla F(x, U_m) \, dx \\
\leq \mu K + \epsilon_m \|U_m\|
\]

in view of \((F1)\), so that \( \|U_m\| \) is bounded. Since \( \nabla I(U) = U - \nabla N(U) \) and \( \nabla N : E \to E \) is a compact mapping by Proposition 2.1 (ii), we conclude as usual that \( \{U_m\} \) possesses a convergent subsequence.

**Proof of (ii)** We will assume \( N \geq 3 \) since the proof is similar for \( N = 1, 2 \). Recall that \((F_0)\) gives

\[
|F(x, U)| \leq C_1|U|^2 + C_2|U|^{p+1} \quad \forall x \in \mathbb{R}^N, \quad \forall U \in \mathbb{R}^2,
\]

where \( p+1 < 2^* \) and, without loss of generality, we may assume that \( p+1 > \nu \). Thus, we have the interpolation inequality

\[
|U|_{L^{p+1}} \leq |U|^{1-\frac{1}{\nu}}|U|^\frac{\nu}{L^{2^*}} \quad \forall U \in L^\nu \cap L^{2^*},
\]
where \(1/(p+1) = (1-t)/\nu + t/(2^*)\). Using the Sobolev embedding \(E \hookrightarrow L^{2^*}\), we obtain
\[
|U|_{L^{p+1}} \leq C|U|^{1-t}_{L^{p+2}}\|U\|^{t} \quad \forall U \in L^{p} \cap E.
\] (10)
Now, let \(\{U_m\} \subset E\) be such that \(I(U_m)\) is bounded and \((1+\|U_m\|\|\nabla I(U_m)\|) \to 0\). Using \((F_2)_\nu\) we obtain
\[
a|U_m|_{L^\nu} \leq 2I(U_m) - \langle \nabla I(U_m), U_m \rangle \leq K_1,
\]
hence
\[
|U_m|_{L^\nu} \leq K_2 \quad \forall m \in \mathbb{N}.
\] (11)
In particular, writing \(Q_m(x) = U_m(x) \cdot \nabla F(x, U_m(x)) - 2F(x, U_m(x))\), we have that
\[
\limsup \int Q_m(x) \, dx \leq K_1.
\] (12)
On the other hand, using (9) and (10), we obtain the estimate
\[
\frac{1}{2}\|U_m\|^2 - I(U_m) = \int F(x, U_m(x)) \, dx \leq C_1|U_m|^2_{L^2} + C_2 C^{p+1} |U_m|^{(1-t)(p+1)} \|U_m\|^{t(p+1)},
\]
so that (11) implies
\[
\|U_m\|^2 \leq K_3 + K_4|U_m|^2_{L^2} + K_5\|U_m\|^{t(p+1)},
\] (13)
where a simple calculation shows that \(t(p+1) < 2\) since \(\nu > \frac{N}{2}(p-1)\). Finally, we prove the claim below, which implies that \(\{U_m\}\) possesses a convergent subsequence as before.

**Claim:** \(\{U_m\}\) has a bounded subsequence in \(E\).

Suppose, by contradiction, that \(\|U_m\| \to \infty\). Letting \(W_m = U_m/\|U_m\|\) and using the compact embedding \(E \hookrightarrow L^2\), we conclude that there exists \(W \in E\) such that \(W_m \rightharpoonup W\) weakly in \(E\), \(W_m \to W\) strongly in \(L^2\) and \(W_m(x) \to W(x)\) a.e. \(x \in \mathbb{R}^N\). Now, dividing by \(\|U_m\|^2\) in (13) and passing to the limit (recalling that \(t(p+1) < 2\)), we obtain
\[
1 \leq K_4|\hat{W}|^2_{L^2},
\]
so that \(\hat{W} \neq 0\) and the set \(S = \{x \in \mathbb{R}^N : |\hat{W}(x)| \neq 0\}\) has a positive measure. Thus, since \(Q_m(x) \geq a|U_m(x)|^{\nu} \geq 0\) and \(|U_m(x)| \to \infty\) for \(x \in S\), an application of Fatou’s Lemma gives
\[
\lim \int Q_m(x) \, dx \geq \lim \int_S Q_m(x) \, dx = \infty,
\]
which contradicts (12). The proof of Proposition 2.4 is complete. \(\square\)
Remark 2.5 Consider the x-independent case. For simplicity, let $H(U) = F(U)/|U|^\mu$, and $K(U) = |U \cdot \nabla F(U) - 2F(U)|/|U|^\mu$. Then, it is easy to see that $(F_1)_\mu$ implies

$$r \mapsto H(rU)$$

is nondecreasing in $r \in (0, +\infty)$ (for any $|U| = 1$),

$$K(U) \geq (\mu - 2) \inf_{|V|=r} H(V) \quad \forall |U| \geq r > 0.$$  

In particular, since $H(U) > 0$ for $(0, 0) \neq U \in \mathbb{R}^2$, the limits $a_+(U) = \lim_{r \to 0+} H(rU)$ will exist and $a_+(U) \geq 0$. Therefore, in the case that $a_+ = \inf_{|U|=1} a_+(U) > 0$, the above estimate shows that condition $(F_2,\mu)$ holds with $a = (\mu - 2)a_+ > 0$.

Now, before proving Theorems 1.1 and 1.2, we will make a small digression regarding a useful lower estimate for the functional $N(U) = \int_{\Omega} F(x, U)dx$ when the potential is a (continuous) function $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\liminf_{|U| \to \infty} \frac{F(x, U)}{|U|^2} \geq b > -\infty \quad \text{uniformly for } x \in \Omega,$$  

with $\Omega \subset \mathbb{R}^N$ an arbitrary domain. Of course, we are also assuming that $F$ satisfies

$$|F(x, U)| \leq C_1 |U|^2 + C_2 |U|^q,$$  

for some $2 \leq q < \infty$, and that we have a continuous embedding $E \hookrightarrow L^2(\Omega) \cap L^q(\Omega)$, so that $N$ is well-defined on the space $E$.

Let $\hat{b} < b$ be given. Then, by (14), there exists $R > 0$ such that

$$F(x, U) \geq \hat{b}|U|^2 \quad \forall x \in \Omega \text{ and } |U| \geq R,$$  

hence

$$F(x, U) \geq \hat{b}|U|^2 - \hat{M} \quad \forall x \in \Omega \text{ and } U \in \mathbb{R}^2,$$  

in view of (15). The above clearly gives the following lower estimate for the functional $N$,

$$N(U) \geq \hat{b}|U|_{L^2}^2 - \hat{M}\text{meas}(\Omega) \quad \forall U \in E,$$  

which is meaningful only when $\text{meas}(\Omega) < \infty$, in which case it implies

$$\liminf_{\|U\| \to \infty} \frac{N(U) - \hat{b}|U|_{L^2}^2}{\|U\|^2} \geq 0.$$  

We will show next that, even in the case of a general domain $\Omega \subset \mathbb{R}^N$, the above lower bound still holds provided $E$ is compactly embedded in $L^2(\Omega)$.

Proposition 2.6 Assume (14), (15) and that the embedding $E \hookrightarrow L^2(\Omega)$ is compact. Then (17) holds true.
whereas, if denote by\( M \) pact, there exists \( ^{W} \) where we are denoting \( Q \) which is in contradiction with (18). The proof of Proposition 2.6 is complete.

On the other hand, we observe that \( \Omega \) since, on \( \widehat{W}(x) = 0 \) we clearly have \( |W_m(x)| \to 0 \), whereas, if \( |\widehat{W}(x)| > 0 \), we have \( |U_m(x)| = \|U_m\||W_m(x)| \to +\infty \) so that \( \chi_m(x) = 0 \) for all \( m \) large. Therefore, by Lebesgue’s theorem, we conclude that

\[
\int_{\Omega} H_m(x) \, dx \to 0 ,
\]

which is in contradiction with (18). The proof of Proposition 2.6 is complete.

**Proof** In view of (16) and denoting \( \Omega_R(U) = \{ x \in \Omega : |U(x)| < R \} \), we can write

\[
N(U) \geq \hat{b} \int_{\Omega \setminus \Omega_R(U)} |U|^2 \, dx + \int_{\Omega_R(U)} F(x, U) \, dx
\]

\[
= \hat{b} |U|_{L^2}^2 + \int_{\Omega_R(U)} [F(x, U) - \hat{b}|U|^2] \, dx .
\]

Therefore, it suffices to show that \( \liminf_{\|U\| \to \infty} N_R(U)/\|U\|^2 \geq 0 \), where

\[
N_R(U) = \int_{\Omega_R(U)} [F(x, U) - \hat{b}|U|^2] \, dx .
\]

We claim that \( \lim_{\|U\| \to \infty} N_R(U)/\|U\|^2 = 0 \). Indeed, by contradiction, suppose that there exists \( \delta_0 > 0 \) and a sequence \( \{U_m\} \subseteq E \) such that \( \|U_m\| \to \infty \) and

\[
|\int_{0 < |U_m| < R} [Q(x, U_m) - \hat{b}|U_m|^2] \, dx| \geq \delta_0 \|U_m\|^2 \quad \forall m \in \mathbb{N} ,
\]

where we are denoting \( Q(x, U) = F(x, U)/|U|^2 \), \( U \neq (0, 0) \). By taking a subsequence, if necessary, we may assume that the above holds without the absolute value (the case where \( N_R(U_m) < 0 \) is entirely similar). Now, let us define \( W_m = U_m/\|U_m\| \). Then, since \( \|W_m\| = 1 \) and the embedding \( E \hookrightarrow L^2 \) is compact, there exists \( \hat{W} \in E \) such that, for a suitable subsequence (which we still denote by \( \{W_m\} \)), we have

\[
W_m \rightharpoonup \hat{W} \quad \text{weakly in } E ,
\]

\[
W_m \to \hat{W} \quad \text{strongly in } L^2(\Omega) ,
\]

\[
W_m(x) \to \hat{W}(x) \quad \text{a. e. } x \in \Omega ,
\]

\[
|W_m(x)| \leq h(x) \in L^2(\Omega) .
\]

Therefore, letting \( H_m(x) = |Q_m(x, U_m(x)) - \hat{b}|\chi_m(x)|W_m(x)|^2| \) where \( \chi_m \) is the characteristic function of the set \( \Omega_R(U_m) = \{ x \in \Omega : 0 < |U_m(x)| < R \} \), we have

\[
\int_{\Omega} H_m(x) \, dx \geq \delta_0 > 0 \quad \forall m \in \mathbb{N} . \quad (18)
\]

On the other hand, we observe that \( |H_m(x)| \leq (|\hat{b}| + M_R)h(x)^2 \in L^1(\Omega) \), where \( M_R = \max_{|x| \leq R} |Q(x, U)| < \infty \) in view of (15). Moreover, \( H_m(x) \to 0 \) a. e. \( x \in \Omega \) since, on \( \widehat{W}(x) = 0 \) we clearly have \( |W_m(x)| \to 0 \), whereas, if \( |\widehat{W}(x)| > 0 \), we have \( |U_m(x)| = \|U_m\||W_m(x)| \to +\infty \) so that \( \chi_m(x) = 0 \) for all \( m \) large. Therefore, by Lebesgue’s theorem, we conclude that

\[
\int_{\Omega} H_m(x) \, dx \to 0 ,
\]
Proof of Theorem 1.2 In view of Proposition 2.4 (i), it suffices to check that the conditions of the Mountain-Pass Theorem [1] are satisfied. Indeed, it is easy to see that the global assumption $(F_1)_\mu$ implies

\[(i) \quad F(x, U) \geq \min_{|V|=1} F(x, V)|U|^\mu > 0 \quad \forall x \in \mathbb{R}^N \text{ and } |U| \geq 1, \]

\[(ii) \quad 0 < F(x, U) \leq \max_{|V|=1} F(x, V)|U|^\mu \quad \forall x \in \mathbb{R}^N \text{ and } 0 < |U| \leq 1, \]

where $\max_{|V|=1} |F(x, V)| \leq C$ in view of $(F_0)$. In particular, $(19)(ii)$ shows that
\[
\lim_{|U| \to 0} \frac{F(x, U)}{|U|^2} = 0 \text{ uniformly for } x \in \mathbb{R}^N, \tag{20}
\]

and $(19)(i)$ shows that, given any bounded set $S \subset \mathbb{R}^N$, there exists $\tilde{C} = \tilde{C}(S)$, $\tilde{C} > 0$ with
\[
F(x, U) \geq \tilde{C}|U|^\mu \quad \forall x \in S \text{ and } |U| \geq 1, \tag{21}
\]

Now, using the embedding $E \hookrightarrow L^2$, it is clear from (20) that
\[
\inf \frac{I(U)}{|U|^{r}} > 0 \quad \text{for all } r > 0 \text{ sufficiently small.}
\]

On the other hand, (21) shows that there exist many $e \in E$ such that $I(e) < 0$ (For instance, take $e = \rho \Phi$ with $0 \neq \Phi \in C^1(\mathbb{R}^N, \mathbb{R}^2)$ having compact support and $\rho > 0$ being sufficiently large). Therefore, the geometry of the mountain-pass theorem holds true and we can conclude the existence of a critical point $\tilde{U} \in E$ of the functional $I$ with $I(\tilde{U}) > 0$. In other words, problem $(P)$ has a nonzero weak solution $\tilde{U} \in H^1$ such that $b(x)^{1/2}\tilde{U} \in L^2$. Moreover, by the regularity theory, we also have $\tilde{U} \in C^1$. The proof of Theorem 1.2 is complete. □

Remark 2.7 It should be observed that, in our present case, we did not use the (system) analogue of assumption $f(x,0) = f_u(x,0) = 0$ made in [14], since the global condition $(F_1)_\mu$ already implies (2.20).

Proof of Theorem 1.1 Notice that, given $\gamma \in \mathbb{R}$, we can write (2.1) as
\[
I(U) = \frac{1}{2}(U - \gamma TU, U) - N_\gamma(U), \tag{22}
\]

where $N_\gamma(U) := N(U) - \frac{1}{2}\gamma|U|^2_{L^2}$ and $T : E \to E$ is defined by $\langle TU, \Phi \rangle = (U, \Phi)_{L^2} \quad \forall U, \Phi \in E$, so that $T$ is a compact operator in view of Proposition 2.1 (i). In fact, it is easy to see that $T$ is a positive operator and its eigenvalues $\{\tau_j\}_{j \in \mathbb{N}}$ are the reciprocals of the eigenvalues of the eigenvalue problem $-\Delta U + A(x)U = \lambda_j U$, $x \in \mathbb{R}^N$, that is, $\tau_j = 1/\lambda_j$. We denote by $E^+_\gamma, E^0_\gamma$ and $E^-_\gamma$
the subspaces of $E$ where $I - \gamma T$ is positive definite, zero and negative definite, respectively, and let $m_\gamma > 0$ be such that

$$\frac{1}{2} \langle U - \gamma TU, U \rangle \geq m_\gamma \|U\|^2 \quad \forall U \in E^+_\gamma,$$

$$\frac{1}{2} \langle U - \gamma TU, U \rangle \leq -m_\gamma \|U\|^2 \quad \forall U \in E^-_\gamma.$$

Also, we define the subspaces $E^+_\gamma = E^+_{\lambda_{k-1}}$ and $E^-_\gamma = E^-_{\lambda_{k-1}} \oplus E^0_{\lambda_{k-1}}$, so that $E = E^+ \oplus E^-$. 

Now, recalling the crossing condition ($F_3$), pick $\alpha < \beta$ so that $0 < \alpha < \lambda_k < \beta$ and $A < \beta$. Then, there exists $\delta > 0$ such that

$$F(x, U) \leq \frac{1}{2} \alpha |U|^2 \quad \forall |U| \leq \delta,$$

so that $F(x, U) \leq \frac{1}{2} \alpha |U|^2 + M|x|^{p+1} \quad \forall x \in \mathbb{R}^N$ and $U \in \mathbb{R}^2$ and, hence,

$$I(U) \geq \frac{1}{2} (\|U\|^2 - \alpha |U|_{L^2}^2) - \tilde{M} \|U\|^{p+1} \quad \forall U \in E. \quad (23)$$

From (23), letting $\tilde{m} = m_\alpha$, it follows that

$$I(U) \geq \tilde{m} \|U\|^2 - \tilde{M} \|U\|^{p+1} = (\tilde{m} - \tilde{M} \|U\|^{p-1}) \|U\|^2 \quad (24)$$

for all $U \in E^+$. Since we may assume $p > 1$ in $(F_0)$, we can find $\omega, \rho > 0$ such that

$$I(U) \geq \omega \quad \forall U \in E^+, \quad |U| = \rho. \quad (25)$$

On the other hand, we obtain from $(F_4)$ that

$$I(U) \leq \frac{1}{2} (\|U\|^2 - \lambda_{k-1} |U|_{L^2}^2) \leq 0 \quad \forall U \in E^-,$$

and, since $(F_0)$ and $(F_3)$ imply that (15) and (14) hold with $b = \frac{1}{2} \beta > \frac{1}{2} \tilde{\beta}$, we obtain from Proposition 2.6 that, given $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$N(U) \geq \frac{1}{2} \tilde{\beta} |U|_{L^2}^2 - \epsilon \|U\|^2 \quad \forall \|U\| \geq R_\epsilon,$$

hence

$$I(U) \leq \frac{1}{2} (\|U\|^2 - \tilde{\beta} |U|_{L^2}^2) + \epsilon \|U\|^2 \quad \forall \|U\| \geq R_\epsilon.$$

Therefore, as $\frac{1}{2} (\|U\|^2 - \tilde{\beta} |U|_{L^2}^2) \leq -m_{\tilde{\beta}} \|U\|^2 \forall U \in E^- \oplus E^0_{\lambda_k}$, we can pick $0 < \epsilon < m_{\tilde{\beta}}$ to get

$$I(U) \leq (-m_{\tilde{\beta}} + \epsilon) \|U\|^2 < 0 \quad \forall \|U\| \geq R_\epsilon, \quad U \in E^- \oplus E^0_{\lambda_k}. \quad (27)$$
Estimates (25)-(27) show that the functional $I$ exhibits the geometry required by the Generalized Mountain-Pass Theorem (Thm 5.3 in [12]). Moreover, as shown in [2], a deformation theorem can be proved with condition (C) replacing the Palais-Smale condition ($PS$) and it turns out that the Generalized Mountain-Pass Theorem holds true under condition (C) (see [10] for details). Thus, in view of Proposition 2.4 (ii), we may conclude from (25)-(27) that $I$ possesses a critical point $\hat{U} \in E$ with $I(\hat{U}) \geq \omega > 0$. In particular, $\hat{U} \neq 0$ since $I(0) = 0$ by (24) and (26). The proof of Theorem 1.1 is now complete.

3 Final Comments

In this section we make some comments regarding extensions of problem (P), the global assumptions ($F_1$), ($F_2$), and we present a simple example which illustrates the difference between these assumptions.

1) Using the method of [7], we could extend our results to include noncooperative systems of the form

$$
\begin{align*}
-\Delta u + a(x)u + \delta v &= f(x, u, v) \quad \text{in } \mathbb{R}^N \\
-\Delta v - \delta u + b(x)v &= -g(x, u, v) \quad \text{in } \mathbb{R}^N,
\end{align*}
$$

where $\delta > 0$ is given and $(f, g) = \nabla F$. In this case the corresponding functional $I : E \to \mathbb{R}$ is strongly indefinite and care should be taken in proving the required linking condition of the Generalized Mountain-Pass Theorem.

2) In the scalar case, it is well known that problem (P) arises naturally in connection with standing wave solutions of nonlinear Schrödinger Equations (see [4, 15])

$$
i \frac{\partial \phi}{\partial t} = -\Delta \phi + V(x)\phi + g(|\phi|^2)\phi, \quad x \in \mathbb{R}^N, \quad t > 0,
$$

that is, when one seeks time-periodic solutions of the form $\phi(x, t) = e^{-\imath \omega t}u(x)$ for some $\omega \in \mathbb{R}$. Indeed, in this case the function $u(x)$ must satisfy $-\Delta u + a(x)u = f(u)$ with $a(x) = V(x) - \omega$ and $f(u) = -g(|u|^2)u$. The corresponding functional is then given by

$$
I(u) = \int_{\mathbb{R}^N} \frac{1}{2}||\nabla u||^2 + a(x)u^2 + G(|u|^2) \, dx,
$$

where $G(s) = \int_0^s g(\sigma) \, d\sigma$.

3) As already noted in Remark 2.5, condition ($F_1$)$_{\mu}$ with $\mu > 2$ implies ($F_2$)$_{\mu}$ provided that

$$
\lim_{|U| \to 0} \frac{F(x, U)}{|U|^\mu} \geq a_+ > 0,
$$
where we recall that the above limit is always nonnegative. One basic difference between these two global hypotheses is that, unlike \((F_1)_\mu\), condition \((F_2)_\nu\) is insensitive to quadratic terms. In particular, the coercive weight functions \(a(x), b(x)\) in problem \((P)\) do not have to be uniformly bounded away from zero.

**4**) Aside from showing the possibility of trading the superquadraticity condition \((F_1)_\mu\) for the nonquadraticity condition \((F_2)_\nu\), our approach shows that, in the “coercive” case, problem \((P)\) behaves as if it were posed in a bounded domain \(\Omega \subset \mathbb{R}^N\). We should mention that the more general case, in which \(a(x)\) and \(b(x)\) satisfy \((A_\lambda)\) but are not necessarily coercive, may indeed lack the “compactness” needed in our approach. In the scalar situation, by using comparison arguments, such a case was also treated by Rabinowitz in [14] under additional assumptions on \(f(x, u)\).

**5**) Finally, we present an example that illustrates the difference between \((F_1)_\mu\) and \((F_2)_\nu\). Let
\[
F_1(u) = u^2(\log |u| - 1), \quad \text{for } |u| \geq 1.
\]
It is not hard to show that \(F_1\) can be extended to all of \(\mathbb{R}\) as a function \(F : \mathbb{R} \to \mathbb{R}\) of class \(C^2\) such that \(F^{(j)}(0) = 0\) for all \(j \in \mathbb{N}\) and, for suitable \(m > 0\) and \(a > 0\), the function \(\tilde{F}(u) = F(u) - m\) satisfies
\[
u \tilde{F}'(u) - 2\tilde{F}(u) \geq a|u| \quad \forall u \in \mathbb{R}.
\]
(For instance, define \(F(u) = -e^{1-(1/|u|)}\) for \(0 < |u| < 1\).) Therefore, in this example \(\tilde{F}\) satisfies \((F_2)_\nu\) with \(\nu = 1\) but it is not superquadratic and \((F_1)_\mu\) cannot hold with \(\mu > 2\).

**References**


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