Electronic Journal of Differential Equations, Vol. **1996**(1996), No. 09, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp (login: ftp) 147.26.103.110 or 129.120.3.113

ON ELLIPTIC EQUATIONS IN R^N WITH CRITICAL EXPONENTS *

C.O. Alves, J.V. Goncalves, & O.H. Miyagaki

Abstract

In this note we use variational arguments –namely Ekeland's Principle and the Mountain Pass Theorem– to study the equation

$$-\Delta u + a(x)u = \lambda u^q + u^{2^*-1} \text{ in } \mathbb{R}^N .$$

The main concern is overcoming compactness difficulties due both to the unboundedness of the domain $\mathbb{R}^N$, and the presence of the critical exponent $2^* = 2N/(N-2)$.

1 Introduction

In this note we use variational methods to explore existence of weak solutions for the problem

$$(*) \qquad \begin{cases} -\Delta u + a(x)u = \lambda u^q + u^{2^*-1} \text{ in } \mathbb{R}^N \\ \int a(x)u^2 < \infty, \quad \int |\nabla u|^2 < \infty \\ u \geq 0, \quad u \not\equiv 0 \end{cases}$$

where a is a nonnegative L^∞_{loc} function, $\lambda \geq 0$, $0 < q \leq 1$ and 2^* is the critical exponent, $2^* = 2N/(N-2)$, for $N \geq 3$.

This problem has been explored by many authors including Brézis & Nirenberg [6], Ambrosetti-Brézis & Cerami [1], Guedda & Veron [9] (see also their references) for the case of elliptic equations in bounded domains. As far as unbounded domains are concerned we recall the work by Benci & Cerami [12], Noussair-Swanson & Jianfu [3], Jianfu & Xiping [14], Egnell [7], Azorero & Alonso [4], Miyagaki [10].

*1991 *Mathematics Subject Classifications:* 35J20, 35K20.
Key words and phrases: Elliptic equations, unbounded domains, critical exponents, variational methods.

Submitted: August 7, 1996. Published October 22, 1996.
Partially supported by CNPq/Brasil.

In this work we shall assume the following condition on a.

$$(a_1) \qquad a(x) > 0,\ x \in B^c_{R_o} \text{ and } \int_{B^c_{R_o}} \frac{1}{a} < \infty\,.$$

Our main results are the following:

Theorem 1 *Let $0 < q < 1$ and assume (a_1). Then there exists $\lambda^* > 0$ such that $(*)$ has a solution for $0 < \lambda < \lambda^*$, $N \geq 3$.*

Theorem 2 *Let $N \geq 4$ and $q = 1$. Assume (a_1) and $a(x) = 0, \quad x \in B_{2r_0}$ for some $r_0 \in (0, R_0/2)$. Then there is $\lambda^* > 0$ such that $(*)$ has a solution for $0 < \lambda < \lambda^*$.*

These theorems complement the results in [10] in the sense that here the function a is allowed to vanish on a ball B_{R_o}. Actually in Theorem 2, we require a to vanish on B_{2r_0}. In addition we consider the case $0 < q \leq 1$ while in [10], $a > 0$ is continuous and $q \in (1, 2^*)$.

2 Preliminaries

Let

$$E = \left\{ u \in \mathcal{D}^{1,2} \mid \int au^2 < \infty \right\}$$

with inner product and norm given by

$$\langle u, v \rangle = \int (\nabla u \nabla v + auv)\,, \quad \|u\|^2 = \int \left(|\nabla u|^2 + au^2\right).$$

Recall that $\mathcal{D}^{1,2}$ is the closure of $\mathcal{C}^\infty_o$ with respect to the gradient norm $\|u\|_1^2 = \int |\nabla u|^2$. Moreover

$$\mathcal{D}^{1,2} = \left\{ u \in L^{2^*} \mid \partial_i u \in L^2 \right\}$$

and the norm

$$\|u\|' \equiv |u|_{L^{2^*}} + |\nabla u|_{L^2}$$

is equivalent to the $\mathcal{D}^{1,2}$ norm. In addition $\mathcal{D}^{1,2} \to L^{2^*}$.

The following lemma is a variant of a result by Willem & Omana [13] and by Costa [2].

Lemma 1 *Assume (a_1). Then $E \to L^s$ for $1 \leq s \leq 2^*$ and $E \hookrightarrow L^s$ for $1 \leq s < 2^*$.*

We shall look for the critical points of the functional

$$I(u) = \frac{1}{2}\int \left(|\nabla u|^2 + a|u|^2\right) - \frac{1}{q+1}\int \lambda u_+^{q+1} - \frac{1}{2^*}\int u_+^{2^*}$$

in the Hilbert space E.

Using standard techniques we can show that $I \in C^1(E, \mathbb{R})$, and that its derivative is given by

$$\langle I'(u), v\rangle = \int (\nabla u \nabla v + auv) - \lambda \int u_+^q v - \int u_+^{2^*-1} v\,.$$

Therefore, the critical points of I are the weak solutions of $(*)$.

The following auxiliary result concerns the geometry of I.

Lemma 2 *If a satisfies* (a_1) *and if* $0 < q \leq 1$ *then there exists* $\lambda^* > 0$ *such that if* $0 < \lambda < \lambda^*$ *then*

(i) $$I(u) \geq r, \quad \|u\| = \rho, \quad \textit{for some} \quad r, \rho > 0$$

If in addition $\phi \geq 0$, $\phi \not\equiv 0$, *and* $\phi \in E$ *then*

(ii) $$I(t\phi) \to -\infty \quad \textit{as} \quad t \to \infty$$

(iii) $$I(t\phi) < 0, \quad \textit{for small } t > 0, \ \textit{and } 0 < q < 1.$$

3 Proofs

For the sake of completeness, we present a proof of Lemma 3, which is based on the proof in [2].

Proof of Lemma 3. At first let $R > R_o$. Then we have

$$\int_{B_R^c} |u| = \int_{B_R^c} \frac{a^{1/2}|u|}{a^{1/2}} \leq \left(\int_{B_R^c} \frac{1}{a}\right)^{1/2} \left(\int_{B_R^c} a|u|^2\right)^{1/2} \leq C\|u\|$$

which shows that

$$|u|_{L^1} \leq C\|u\|, \quad u \in E.$$

Now using the interpolation inequality

$$|u|_s \leq |u|_1^{\alpha}|u|_r^{1-\alpha}, \quad \alpha + \frac{1-\alpha}{r} = \frac{1}{s}, \ 1 \leq s \leq r \leq 2^*, \quad 0 \leq \alpha \leq 1$$

and the embedding $E \to L^{2^*}$, we infer that $E \to L^s$, $1 \leq s \leq 2^*$.

On the other hand, for sufficiently large $R > 0$ we have

$$\int_{B_R^c} \frac{1}{a} < \epsilon\,.$$

So if $u_n \rightharpoonup 0$ in E, then for large n

$$\int_{B_R^c} |u_n| \le C \int_{B_R^c} \frac{1}{a} \le \epsilon\,.$$

Using compact Sobolev embeddings we also have

$$u_n \to 0 \text{ in } L^1(B_R)$$

so that $u_n \to 0$ in L^1. Using again the interpolation inequality stated above, one concludes the proof of Lemma 3.

Proof of Lemma 4.

Verification of (i)**.** From the continuous embedding in Lemma 3, we have

$$\begin{aligned} I(u) &\ge \tfrac{1}{2}\|u\|^2 - \tfrac{\lambda c_{q+1}}{q+1}\|u\|^{q+1} - \tfrac{S^{-2^*/2}}{2^*}\|u\|^{2^*} \\ &\ge \|u\|^{q+1}\left(\tfrac{1}{2}\|u\|^{2-(q+1)} - \tfrac{\lambda c_{q+1}}{q+1} - \tfrac{S^{-2^*/2}}{2^*}\|u\|^{2^*-(q+1)}\right) \end{aligned}$$

where S is the best constant for the embedding $\mathcal{D}^{1,2} \to L^{2^*}$, that is

$$S = \inf\left\{ \frac{\int |\nabla u|^2}{\left(\int |u|^{2^*}\right)^{2/2^*}} \;\;|\;\; u \in \mathcal{D}^{1,2},\; u \not\equiv 0 \right\}.$$

Letting

$$Q(t) \equiv \frac{1}{2} t^{2-(q+1)} - \frac{S^{-2^*/2}}{2^*} t^{2^*-(q+1)}, \quad t \ge 0\,,$$

there is $\rho > 0$ such that

$$\max_{t \ge 0} Q(t) = Q(\rho) > 0.$$

Taking $\|u\| = \rho$ and $\lambda^* = \frac{q+1}{c_{q+1}} Q(\rho)$ we get (i).

Verification of (ii)**.** Taking $\phi \not\equiv 0$, $\phi \ge 0$, $\phi \in E$ we have

$$I(t\phi) = \frac{t^2}{2}\|\phi\|^2 - \frac{t^{q+1}}{q+1}\lambda \int \phi^{q+1} - \frac{t^{2^*}}{2^*}\int \phi^{2^*}$$

which gives (ii).

Verification of (iii)**.** It is clear from the expression of $I(t\phi)$ above taking into account that $0 < q < 1$.

Proof of Theorem 1. By the proof of lemma 4, I is bounded from below on $\overline{B_\rho}$. By the Ekeland Principle [8], there exists $u_\epsilon \in \overline{B_\rho}$ such that

$$I(u_\epsilon) \leq \inf_{\overline{B_\rho}} I + \epsilon$$

and

$$I(u_\epsilon) < I(u) + \epsilon \|u - u_\epsilon\|, \quad u \not\equiv u_\epsilon.$$

Now since $0 < q < 1$ it follows that

$$I(t\phi) < 0, \text{ for small } t > 0, \quad \phi \not\equiv 0, \text{ and } \phi \in \mathcal{C}_o^\infty .$$

Again by Lemma 4

$$\inf_{\partial B_\rho} I \geq r > 0 \quad \text{and} \quad \inf_{\overline{B_\rho}} I < 0.$$

Choose $\epsilon > 0$ such that

$$0 < \epsilon < \inf_{\partial B_\rho} I - \inf_{\overline{B_\rho}} I.$$

Hence

$$I(u_\epsilon) < \inf_{\partial B_\rho} I$$

so that

$$u_\epsilon \in B_\rho.$$

Hence letting

$$F(u) \equiv I(u) + \epsilon \|u - u_\epsilon\|$$

we notice that u_ϵ is a point of minimum of F on $\overline{B_\rho}$ and so

$$\frac{I(u_\epsilon + \delta v) - I(u_\epsilon)}{\delta} + \epsilon \|v\| \quad \geq \quad 0$$

which by passing to the limit as $\delta \to 0$ gives that

$$\langle I'(u_\epsilon), v\rangle + \epsilon \|v\| \geq 0$$

and hence $\|I'(u_\epsilon)\| \leq \epsilon$. Therefore, there is a sequence $u_n \in \overline{B_\rho}$ such that

$$I(u_n) \to c^* \equiv \inf_{\overline{B_\rho}} I < 0 \quad \text{and} \quad I'(u_n) \to 0.$$

Since of course u_n is bounded,

$$u_n \rightharpoonup u^* \quad \text{in} \quad E$$

and

$$u_n \to u^* \text{ a.e. in } \mathbb{R}^N.$$

Now passing to the limit in

$$o(1) = \int (\nabla u_n \nabla \phi + a u_n \phi) - \lambda \int u_{n+}^q \phi - \int u_{n+}^{2^*-1} \phi, \quad \phi \in E$$

we infer that $I'(u^*) = 0$ showing that u^* is a solution of problem $(*)$.

In order to show that $u^* \not\equiv 0$, we follow the arguments in [6]. Assume that $u^* \equiv 0$ and that

$$\|u_n\|^2 \to \ell \geq 0.$$

Using $I'(u_n) \to 0$ we have

$$\|u_n\|^2 - \int u_{n+}^{2^*} = o(1)$$

so that $\int u_{n+}^{2^*} \to \ell$ and from the expression

$$c^* + o(1) = \lambda \left(\frac{1}{2} - \frac{1}{q+1} \right) \int u_{n+}^{q+1} + \frac{1}{N} \int u_{n+}^{2^*}$$

we infer that

$$c^* = \frac{\ell}{N}$$

which is impossible.

Proof of Theorem 2. By Lemma 4 and the Mountain Pass Theorem, there exists a sequence u_n *in* E such that

$$I(u_n) \to c \text{ and } I'(u_n) \to 0$$

where

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I(\gamma(t)), \quad c \geq r$$

and

$$\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = 0, \ \gamma(1) = e\}$$

where $e \in E$ satisfies $I(e) \leq 0$.

Claim. *There is* $e \equiv e_\lambda$ *such that* $0 < c < \frac{1}{N} S^{\frac{N}{2}}$, $0 < \lambda < \lambda^*$.

From the expression

$$\langle I'(u_n), u_n \rangle - 2^* I(u_n) = \left(1 - \frac{2^*}{2} \right) \|u_n\|^2 + \lambda \left(\frac{2^*}{2} - 1 \right) \int u_{n+}^2$$

one shows, by taking $\lambda^* > 0$ smaller than the one found in lemma 4, that u_n is bounded. So that, passing to subsequences,

$$u_n \rightharpoonup u \text{ in } E \text{ and } u_n \to u \text{ a.e. in } \mathbb{R}^N$$

for some $u \in E$.

Remark. $I(u_{n+}) \to c$ and $I'(u_{n+}) \to 0$.
Indeed, u_{n-} is also bounded so that

$$o(1) = \langle I'(u_n), u_{n-}\rangle = \int (|\nabla u_{n-}|^2 + a u_{n-}^2)\,.$$

Moreover, if $\phi \in E$ then

$$\begin{aligned}\langle I'(u_{n+}), \phi\rangle &= \langle u_{n+}, \phi\rangle - \lambda \int u_{n+}\phi - \int u_{n+}^{2^*-1}\phi \\ &= \langle I'(u_n), \phi\rangle - \langle u_{n-}, \phi\rangle\end{aligned}$$

so that, $I'(u_{n+}) \to 0$. On the other hand,

$$\begin{aligned}&I(u_n) - \frac{1}{2}\int (|\nabla u_{n-}|^2 + a u_{n-}^2) \\ &= \frac{1}{2}\int (|\nabla u_{n+}|^2 + a u_{n+}^2) - \frac{\lambda}{2}\int u_{n+}^2 - \frac{1}{2^*}\int u_{n+}^{2^*} \\ &= I(u_{n+})\,,\end{aligned}$$

which gives $I(u_{n+}) \to c$. So we may assume that $u_n \geq 0$ and thus $u \geq 0$.
Now as in the proof of Theorem 1 one shows that u satisfies the equation in $(*)$. Again arguing as in [6] we assume that $u \equiv 0$. Then

$$\|u_n\|^2 \to \ell \;\text{ for some } \ell \geq 0$$

and using the facts that

$$I(u_n) \to c, \quad I'(u_n) \to 0$$

we infer that $c \geq \frac{1}{N}S^{N/2}$, contradicting $0 < c < \frac{1}{N}S^{N/2}$ given by the Claim.

Proof of the Claim. (Arguments adapted from [6].)
Consider the cut-off function $\phi \in \mathcal{C}_o^\infty$ such that

$$\phi \equiv 1 \;\text{ on }\; B_{r_0}, \quad \phi \equiv 0 \;\text{ on }\; \mathbb{R}^N \backslash B_{2r_0}.$$

Now consider the function

$$w_\epsilon(x) = \frac{[N(N-2)\epsilon]^{(N-2)/4}}{(\epsilon + |x|^2)^{(N-2)/2}}, \qquad x \in \mathbb{R}^N,\ \epsilon > 0$$

which satisfies

$$-\Delta w_\epsilon = w_\epsilon^{2^*-1} \;\text{ in } \mathbb{R}^N.$$

It is well known (see e.g. Talenti [5], Aubin [11]) that

$$\|w_\epsilon\|_1^2 = |w_\epsilon|_{2^*}^{2^*} = S^{N/2}.$$

Let

$$\psi_\epsilon = \phi w_\epsilon$$

and let $v_\epsilon \in \mathcal{C}_o^\infty$ given by

$$v_\epsilon = \frac{\psi_\epsilon}{\left(\int {\psi_\epsilon}^{2^*}\right)^{1/2^*}}.$$

Now it can be shown (see e.g. [6], [10]) that $X_\epsilon \equiv \int |\nabla v_\epsilon|^2$ satisfies

$$X_\epsilon \le S + O(\epsilon^{(N-2)/2}).$$

Moreover there is some $t_\epsilon > 0$ such that

$$\max_{t\ge 0} I(tv_\epsilon) = I(t_\epsilon v_\epsilon)$$

and

$$\frac{d}{dt} I(tv_\epsilon)|_{t=t_\epsilon} = 0.$$

which gives

$$0 < t_\epsilon < X_\epsilon^{1/(2^*-2)} \equiv t_0.$$

Notice that $a = 0$ on B_{2r_0} and $v_\epsilon = 0$ on $\mathbb{R}^N \backslash B_{2r_0}$. Moreover $t_\epsilon \ge d_0 \equiv d_0(r_0)$ for some $d_0 > 0$. Otherwise since X_ϵ is bounded, if $t_\epsilon \to 0$, then $I(t_\epsilon v_\epsilon) \to 0$ contradicting

$$I(t_\epsilon v_\epsilon) = \max_{t\ge 0} I(tv_\epsilon) \ge r > 0$$

given by lemma 4 (i). On the other hand

$$\begin{aligned} I(tv_\epsilon) &= \frac{t^2}{2}\int |\nabla v_\epsilon|^2 - \frac{t^{2^*}}{2^*} - \frac{\lambda t^2}{2}\int v_\epsilon^2 \\ &\le \left(\frac{t^2}{2} t_0^{2^*-2} - \frac{t^{2^*}}{2^*}\right) - \frac{\lambda t^2}{2}\int_{B_{2r_0}} v_\epsilon^2 \,. \end{aligned}$$

Now recalling that as a function of t,

$$\left(\frac{t^2}{2} t_0^{2^*-2} - \frac{t^{2^*}}{2^*}\right)$$

increases on the interval $(0, t_0)$ we get

$$\begin{aligned} I(t_\epsilon v_\epsilon) &\le t_0^{2^*}\left(\frac{1}{2} - \frac{1}{2^*}\right) - \frac{\lambda t_\epsilon^2}{2}\int_{B_{2r_0}} v_\epsilon^2 \\ &\le \frac{1}{N} t_0^{2^*} - \frac{\lambda d_0^2}{2}\int_{B_{2r_0}} v_\epsilon^2 \\ &\le \frac{1}{N}\left[S + O\left(\epsilon^{(N-2)/2}\right)\right]^{2^*/(2^*-2)} - \frac{\lambda d_0^2}{2}\int_{B_{2r_0}} v_\epsilon^2 \\ &= \frac{1}{N}\left[S + O\left(\epsilon^{(N-2)/2}\right)\right]^{N/2} - \frac{\lambda d_0^2}{2}\int_{B_{2r_0}} v_\epsilon^2. \end{aligned}$$

Using the inequality

$$(b+c)^\alpha \leq b^\alpha + \alpha(b+c)^{\alpha-1}c \quad b, c \geq 0, \alpha \geq 1$$

with $b = S$, $c = O\left(\epsilon^{(N-2)/2}\right)$ and $\alpha = N/2$ we get

$$I(t_\epsilon v_\epsilon) \leq \frac{1}{N} S^{N/2} + O(\epsilon^{(N-2)/2}) - c_0\lambda \int_{B_{2r_0}} v_\epsilon^2.$$

Therefore,

$$I(t_\epsilon v_\epsilon) \leq \frac{1}{N} S^{N/2} + \epsilon^{(N-2)/2} \left\{ M - c_0\lambda\epsilon^{(2-N)/2} \int_{B_{2r_0}} v_\epsilon^2 \right\},$$

where $c_0 = d_0^2/2$ and M is a positive constant.

We shall show that

$$\epsilon^{(N-2)/2} \left\{ M - c_0\lambda\epsilon^{(N-2)/2} \int_{B_{2r_0}} v_\epsilon^2 \right\} < 0, \quad \text{for small } \epsilon > 0\,.$$

So that

$$I(t_\epsilon v_\epsilon) < \frac{1}{N} S^{N/2}$$

and hence

$$0 < c < \frac{1}{N} S^{N/2}.$$

Noticing that

$$d_1 \leq \int_{B_{2r_0}} \psi_\epsilon^{2^*} \leq d_2, \quad \text{for some} \quad d_1, d_2 > 0,$$

(see [6]), it follows by a change of variables that

$$I(t_\epsilon v_\epsilon) \leq \frac{1}{N} S^{N/2} + \epsilon^{(N-2)/2} \left\{ M - c_0\lambda\epsilon^{(4-N)/2} \int_0^{r_0\epsilon^{-1/2}} \frac{s^{N-1}ds}{(1+s^2)^{N-2}} \right\}.$$

We are going to consider separately the cases $N = 4$ and $N \geq 5$.

Case $N = 4$**.** We have

$$\begin{aligned} I(t_\epsilon v_\epsilon) &\leq \frac{1}{4} S^2 + \epsilon \left\{ M - c_0\lambda \int_0^{r_0\epsilon^{-1/2}} \frac{s^3 ds}{(1+s^2)^2} \right\} \\ &\leq \frac{1}{4} S^2 + \epsilon \left\{ M - c_0\lambda \ln(r_0\epsilon^{-1/2}) \right\}. \end{aligned}$$

Now since

$$c_0\lambda \ln(r_0\epsilon^{-1/2}) \to \infty \text{ as } \epsilon \to 0$$

we infer that

$$I(t_\epsilon v_\epsilon) < \frac{1}{4} S^2, \text{ for small } \epsilon > 0.$$

Case $N \geq 5$. Noticing that

$$c_0\lambda\epsilon^{(4-N)/2}\int_0^{r_0\epsilon^{-1/2}} \frac{s^{N-1}ds}{(1+s^2)^{N-2}} \to \infty \text{ as } \epsilon \to 0$$

we infer that

$$I(t_\epsilon v_\epsilon) < \frac{1}{N}S^{N/2} \text{ for small } \epsilon > 0\,,$$

which concludes the proof of this claim.

References

[1] A. Ambrosetti, H. Brézis & G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, Preprint.

[2] D. Costa, *On a nonlinear elliptic problem in* $\mathbb{R}^N$, Preprint.

[3] E. S. Noussair, C. A. Swanson & Y. Jianfu, *Positive finite energy solutions of critical semilinear elliptic problems*, Canadian J. Math. 44(1992) 1014-1029.

[4] J. G. Azorero & I. P. Alonzo, *Multiplicity of solutions for elliptic problems with critical exponents with a nonsymmetric term*, Trans. Amer. Math. Society 323(1991) 877-895.

[5] G. Talenti, *Best constant in Sobolev inequality*, Ann. Math. 110 (1976) 353-372.

[6] H. Brézis & L.Nirenberg, *Some Variational problems with lack of compactness.* Proc. Symp. Pure Math. (AMS) 45(1986) 165-201.

[7] H. Egnell, *Existence and nonexistence results for m-Laplace equations involving critical Sobolev exponents*, Arch. Rat. Mech. Anal. 104(1988) 57-77.

[8] I. Ekeland, *On the variational principle.* J. Math. Anal. App. 47(1974) 324-353.

[9] M. Guedda & L. Veron, *Quasilinear elliptic equations involving critical Sobolev exponents*, Non. Anal. 12(1989) 879-902.

[10] O. Miyagaki, *On a class of semilinear elliptic problems in* $I\!R^n$ *with critical growth*, CMS Technical Report, Univ. Wisconsin (1994).

[11] T. Aubin, *Problemes isoperimetriques et espaces de Sobolev*, J. Diff. Geometry 11(1976) 573-598.

[12] V. Benci & G. Cerami, *Existence of positive solutions of the equation* $-\Delta u + a(x)u = u^{(n+2)/(n-2)}$ *in* $\mathbb{R}^n$, J. Funct. Anal. 88(1990) 90-117.

[13] W. Omana & M. Willem, *Homoclinic orbits for a class of Hamiltonian systems*, Diff. Int. Equations 5(1992) 1115-1120.

[14] Y. Jianfu & Z. Xiping, *On the existence of nontrivial solution of a quasi-linear elliptic boundary value problem for unbounded domains*, Acta Math. Sci. 7(1987) 341-359.

C.O. Alves
Dep. Mat. Univ. Fed. Paraiba
58109-970 - Campina Grande(PB), Brasil
E-mail coalves@dme.ufpb.br

J.V. Goncalves
Dep. Mat. Univ. Brasília
70.910-900 Brasilia(DF), Brasil
E-mail jv@mat.unb.br

O.H. Miyagaki
Dep. Mat. Univ. Fed. Viçosa
36570-000 Viçosa(MG), Brasil
E-mail olimpio@mail.ufv.br