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ON THE NUMBER OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS INVOLVING THE P–LAPLACIAN

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Abstract. This paper is concerned with multiplicity questions for solutions of the boundary value problem

$$(\varphi(u'))' + \lambda f(t,u) = 0,\ a < t < b$$
$$u(a) = 0 = u(b)$$

where φ is an odd, increasing homeomorphism on $\mathbb{R}$, and λ is a positive parameter. The tools employed are fixed point and continuation methods.

1. Introduction

In this paper, we are interested in the existence and multiplicities of positive solutions of the boundary value problem.

$$\begin{aligned}(\varphi(u'))' + \lambda f(t,u) = 0,\ a < t < b \\ u(a) = 0 = u(b)\end{aligned} \tag{1.1}$$

with f continuous (but not necessarily locally Lipschitz continuous). We make the following assumptions:
(A.1) φ is an odd, increasing homeomorphism on $\mathbb{R}$ and

$$\limsup_{x\to\infty} \frac{\varphi(\sigma x)}{\varphi(x)} < \infty$$

for every $\sigma > 0$.

(A.2) $f : [a,b] \times [0,\infty) \to (0,\infty)$ is continuous and there exists an interval $[c,d] \subset (a,b),\ c < d$ such that

$$\lim_{u\to\infty} \frac{f(t,u)}{\varphi(u)} = \infty$$

uniformly for $t \in [c,d]$.

Our main result is:

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Theorem 1. *Let* (A.1) *and* (A.2) *hold. Then there exists a positive number* λ^* *such that the problem* $(1.1)_\lambda$ *has at least two positive solutions for* $0 < \lambda < \lambda^*$*, at least one for* $\lambda = \lambda^*$ *and none for* $\lambda > \lambda^*$*.*

Note that in the special case where $\varphi(u') = u'$, theorem 1 was proved in [2,6] under the additional assumption that f is of class C^2. Related results for the case $\varphi(u') = |u'|^{p-2}u'$ can be found in [4,5,7] and the references in these papers. As we shall see, a more specific result, which establishes the existence of solution continua of $(1.1)_\lambda$ which satisfy the above conditions, may be obtained also.

One of the primary motivations for studying problems of the above types are boundary value problems for partial differential equations for perturbations of the p-Laplacian of the form

$$\begin{gathered} \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \lambda g(|x|, u) = 0, a < |x| < b, x \in \mathbb{R}^N \\ u = 0 \text{ at } |x| \in \{a, b\} \end{gathered} \tag{1.2}$$

where $N \neq p$ and λ is a positive parameter.

Seeking the existence of radial solutions, one is led to the equation

$$(|u'|^{p-2}u')' + \lambda f(t, u) = 0$$

via the change of variables

$$t = \left(\frac{|N-p|}{(p-1)r}\right)^{\frac{N-p}{p-1}} \quad , r = |x|,$$

and

$$f(t, u) = \left(\frac{(p-1)r}{|N-p|}\right)^{\frac{p(N-1)}{p-1}} g\left(\frac{|N-p|}{p-1} t^{-\frac{p-1}{N-p}}, u\right).$$

In proving theorem 1, we shall employ upper and lower solution methods. These methods are, of course, standard for semilinear equations; they are also applicable in the nonlinear case and we present the type of theorem needed for the nonlinear case. Such a theorem is established in section 2. Section 3 is then devoted to the proof of theorem 1, and to some more specific information about the solution structure.

2. Upper and lower solutions

Consider the problem

$$\begin{gathered} (\varphi(u'))' + g(t, u) = 0, \; a < t < b \\ u(a) = 0 = u(b), \end{gathered} \tag{2.1}$$

where φ is an odd, increasing homeomorphism on $\mathbb{R}$ and $g : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous. We say that a function $\alpha \in C^0[a, b]$ is a lower solution of (2.1), if for each $t_0 \in [a,b]$ there exists a neighborhood I_{t_0} and a finite set of functions $\{\alpha_k\}_{1\le k\le n} \subset C^1(I_{t_0})$ with $\varphi(\alpha_k') \in C^1(I_{t_0})$ such that

$$\begin{aligned} &\alpha(t) = \max_{1\le k\le n}(\alpha_k(t)),\ t\in(I_{t_0}) \\ &(\varphi(\alpha_k'))' + g(t,\alpha_k) \ge 0,\ t\in(I_{t_0}) \end{aligned} \tag{2.2}$$

and

$$\alpha(a) \le 0,\ \alpha(b) \le 0. \tag{2.3}$$

An upper solution is defined similarly, replacing max by min and reversing the inequalities in (2.2), (2.3).

Theorem 2. *Let α and β be a pair of lower and an upper solutions to* (2.1) *respectively with $\alpha \le \beta$. Then* (2.1) *has a minimum solution $u_{\min}$ such that $\alpha \le u_{\min} \le \beta$, and if u is any solution to* (2.1) *with $u \ge \alpha$, then $u \ge u_{\min}$. An analogous result holds for the existence of a maximal solution.*

Proof. We first prove that there exists a solution u to (2.1) with $\alpha \le u \le \beta$.
Define

$$\tilde g(t,v) = \begin{cases} g(t,\beta(t)) + \frac{\beta - v}{1+v^2} & \text{if} \quad v(t) \ge \beta(t) \\ g(t,v(t)) & \text{if} \quad \alpha(t) \le v(t) \le \beta(t) \\ g(t,\alpha(t)) + \frac{\alpha - v}{1+v^2} & \text{if} \quad v(t) \le \alpha(t) \end{cases}$$

For each $v \in C^0[a,b]$, let $u = Av$ be the solution of

$$\begin{aligned} &(\varphi(u'))' = -\tilde g(t,v), \\ &u(a) = 0 = u(b) \end{aligned}$$

Note that

$$u(t) = \int_a^t \varphi^{-1}[c - \int_a^s \tilde g(\tau,v)d\tau]ds$$

where c is such that $u(b) = 0$. Then $A : C^0[a,b] \to C^0[a,b]$ is a completely continuous mapping. Since A is bounded, it follows from the Schauder fixed point theorem that A has a fixed point u. We verify that $\alpha(t) \le u(t) \le \beta(t),\ a \le t \le b$. Indeed, if there exists $t_0 \in [a,b]$ such that $u(t_0) < \alpha(t_0)$, then the continuous function $v = u - \alpha$ will have a negative minimum, say $d = v(\tilde t)$ and there exists an interval (t_1,t_2) containing $\tilde t$ with $v(t_1) = v(t_2) = 0,\ v(t) < 0,\ t \in (t_1,t_2)$ Furthermore v is left and right differentiable at every point in (a,b). Thus $v_+'(\tilde t) \ge v_-'(\tilde t)$, where $v_\pm'$ denote the right and left derivatives. On the other hand, since u is differentiable, the conditions on α imply that $v_+'(\tilde t) = v_-'(\tilde t)$, and v is differentiable at $\tilde t$. We next employ the differential equation satisfied by u and the fact that α is a lower solution to conclude that v has a local maximum at $\tilde t$, which is in contradiction to the fact that v assumes a global minimum at $\tilde t$. Similarly, we have $u(t) \le \beta(t),\ a \le t \le b$ and thus u is a solution of (2.1).

Let $U = \{v \in C^0[a,b] : v \ge \alpha$ and v is an upper solution to (2.1)$\}$. Define $u_{\min}(t) =\ \inf\{v(t) : v \in U\}$. Using the argument in [8, p.279] (also [1]), it can be verified that $u_{\min}$ is the minimum solution of (2.1) whose existence was asserted.

3 Existence results

In this section, we prove theorem 1. Since we are interested in nonnegative solutions we shall make the convention that $f(t,u) = f(t,0)$ if $u < 0$. We shall denote by $|\cdot|_k$ the norm in the space $C^k[a,b]$.

Lemma 3. *Let $v \in C^0[a,b]$ with $v \le 0$ and let u satisfy*

$$(\varphi(u'))' = v$$

$$u(a) = 0 = u(b).$$

Then

$$u(t) \ge |u|_0 p(t), \quad t \in [a,b]$$

where

$$p(t) = \frac{\min(t-a, b-t)}{b-a}$$

Proof. Since $\varphi(u')$ is nonincreasing and φ^{-1} is increasing it follows that u' is nonincreasing. Hence, lemma 3 follows from lemma 2.2 in [3]. For convenience to the readers, we give a direct proof. Let $|u|_0 = |u(T)|,\ T \in [a,b]$. Since u is concave and $u(a) = 0$, it follows that

$$\begin{aligned} u(t) &= u(cT + (1-c)a) \ge cu(T) \\ &\ge \frac{t-a}{b-a}|u|_0, \quad t \in [a,T] \end{aligned}$$

where $c = \frac{t-a}{T-a}$. Similarly,

$$u(t) \ge \frac{b-t}{b-a}|u|_0, \quad t \in [T,b]$$

completing the proof of lemma 3.

Lemma 4. *Suppose that $g : [a,b] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and there exists a positive number M and an interval $[a_1,b_1] \subset (a,b)$ such that*

$$g(t,u) \ge M(\varphi(u)+1),\ t \in [a_1,b_1],\ u \ge 0. \tag{3.1}$$

There exists a positive number $M_0 = M_0(\varphi, a_1, b_1)$ such that the problem

$$\begin{aligned} &(\varphi(u'))' = -g(t,u) \\ &u(a) = 0 = u(b) \end{aligned} \tag{3.2}$$

has no solution whenever $M \ge M_0$.

Proof. Let u be a solution of (3.2). Then

$$u(t) = \int_a^t \varphi^{-1}[c - \int_a^s g(\tau,u)d\tau]ds \tag{3.3}$$

where $c = \varphi(u'(a))$. Let $|u|_0 = u(t_0),\ t_0 \in [a,b]$. Then $u'(t_0) = 0$ and by (3.3),

$$u(t) = \int_a^t \varphi^{-1}[\int_s^{t_0} g(\tau,u)d\tau]ds \tag{3.4}$$

If $t_0 \geq \frac{a_1+b_1}{2}$, then

$$\begin{aligned}|u|_0 \geq u(a_1) &> \int_a^{a_1} \varphi^{-1}[M\int_{a_1}^{\frac{a_1+b_1}{2}} (\varphi(u)+1)] \\ &> (a_1-a)\varphi^{-1}[M\frac{(b_1-a_1)}{2}[\varphi(|u|_0\delta)+1]],\end{aligned}$$

where

$$\delta = \min_{a_1 \leq t \leq b_1} p(t).$$

This implies

$$\varphi(\frac{|u|_0}{a_1-a}) > M\frac{(b_1-a_1)}{2}[\varphi(|u|_0\delta)+1] \tag{3.5}$$

If $t_0 \leq \frac{a_1+b_1}{2}$, then by rewriting (3.4) as

$$u(t) = \int_t^b \varphi^{-1}[\int_{t_0}^s g(\tau,u)d\tau]ds$$

we deduce

$$\varphi\left(\frac{|u|_0}{b-b_1}\right) > \frac{M(b_1-a_1)}{2}[\varphi(|u|_0\delta)+1] \tag{3.6}$$

Combining (3.5) and (3.6), we obtain

$$\varphi(\gamma|u|_0) > \frac{M(b_1-a_1)}{2}[\varphi(|u|_0\delta)+1]$$

where $\gamma = \max\left(\frac{1}{b-b_1}, \frac{1}{a_1-a}\right)$.
Consequently,

$$\frac{\varphi(\gamma|u|_0)}{\varphi(\delta|u|_0)} > \frac{M}{2}(b_1-a_1)$$

a contradiction to (A.1) if M is sufficiently large.

Remark 5. *It follows from the proof, that problem* (3.2) *has no solution* u *satisfying*

$$g(t,u(t)) \geq M(\varphi(u(t))+1), \quad t \in [a_1,a_2],$$

if $M \geq M_0$.

These considerations further imply the following result:

Theorem 6. *There exists a positive number $\bar{\lambda}$ such that problem* $(1.1)_\lambda$ *has no solution for $\lambda > \bar{\lambda}$.*

Proof. It follows immediately from (A.2) that there exists a constant $\mu > 0$ such that

$$f(t,u) \geq \mu(\varphi(u)+1),\ u \in \mathbb{R}^+,\ c \leq t \leq d.$$

Hence the result follows from the previous lemma.

Lemma 7. *For each $\mu > 0$, there exists a positive constant C_μ such that the problem*

$$\begin{aligned}(\varphi(u'))' = -\lambda\theta f(t,u) - (1-\theta)M_0(|\varphi(u)|+1)\\ u(a) = 0 = u(b)\end{aligned} \tag{3.7}$$

with $\lambda \geq \mu$, $\theta \in [0,1]$ and M_0 given by remark 5, has no solution satisfying $|u|_0 > C_\mu$.

Proof. Let u be a solution of (3.7) with $\lambda \geq \mu$ and $\theta \in [0,1]$. Then $u \geq 0$. By (A.2), there exists $M_1 > 0$ such that

$$f(t,u) > \frac{M_0}{\mu}(\varphi(u)+1) \tag{3.8}$$

for $t \in [c,d]$ and $u \geq M_1$. Let $\delta = \min_{c \leq t \leq d} p(t)$. Then if $|u|_0 > \frac{M_1}{\delta}$ we have by lemma 3

$$u(t) \geq |u|_0\delta > M_1, \quad t \in [c,d]$$

which implies by (3.8) that

$$\begin{aligned}&\lambda\theta f(t,u(t)) + (1-\theta)M_0(\varphi(u(t))+1)\\ \geq&\frac{\lambda\theta M_0}{\mu}(\varphi(u(t))+1) + (1-\theta)M_0\varphi(u(t))+1)\\ \geq&M_0(\varphi(u(t))+1), \quad t \in [c,d]\end{aligned}$$

a contradiction with remark 5, and the lemma is proved.

Now, let Λ be the set of all $\lambda > 0$ such that $(1.1)_\lambda$ has a solution and let $\lambda^* = \sup \Lambda$. Note that by lemma 3, every solution of $(1.1)_\lambda$ is positive.

Lemma 8. $0 < \lambda^* < \infty$ *and* $\lambda^* \in \Lambda$.

Proof. $u \in C^0[a,b]$ is a solution of $(1.1)_\lambda$ if and only if $u = F(\lambda, u)$, where

$$F : [0,\infty) \times C^0[a,b] \to C^0[a,b]$$

is the completely continuous mapping given by

$$u = F(\lambda, v),$$

with u the solution of

$$\begin{aligned}&(\varphi(u'))' = -\lambda f(t, v),\\&u(a) = 0 = u(b).\end{aligned}$$

We note that $F(0, v) = 0,\ v \in C^0[a, b]$. Hence it follows from the continuation theorem of Leray-Schauder that there exists a solution continuum $\mathcal{C} \subset [0, \infty) \times C^0[a, b]$ of solutions of $(1.1)_\lambda$ which is unbounded in $[0, \infty) \times C^0[a, b]$, and thus, $(1.1)_\lambda$ has a solution for $\lambda > 0$ sufficiently small, and hence $\lambda^* > 0$. By theorem 6 , $\lambda^* < \infty$. We verify that $\lambda^* \in \Lambda$. Let $\{\lambda_n\}_n \subset \Lambda$ be such that $\lambda_n \to \lambda^*$ and let $\{u_n\}$ be the corresponding solutions of $(1.1)_{\lambda_n}$. By lemma 7, $\{u_n\}$ is bounded in $C^1[a, b]$ and hence $\{u_n\}$ has a subsequence converging to $u \in C^0[a, b]$. By standard limiting procedures, it follows that u is a solution of $(1.1)_{\lambda^*}$.

Lemma 9. *Let $0 < \lambda < \lambda^*$ and let u_{λ^*} be a solution of $(1.1)_{\lambda^*}$. Then there exists $\epsilon_0 > 0$ such that $u_{\lambda^*} + \epsilon,\ 0 \le \epsilon \le \epsilon_0$ is an upper solution of $(1.1)_\lambda$.*

Proof. Let $c_1 > 0$ be such that $f(t, u_{\lambda^*}(t)) \ge c_1$ for every $t \in [a, b]$ and let $\epsilon_0 > 0$ be such that

$$|f(t, u_{\lambda^*}(t) + \epsilon) - f(t, u_{\lambda^*}(t))| < \frac{c_1(\lambda^* - \lambda)}{\lambda}, \quad t \in [a, b],\ 0 \le \epsilon \le \epsilon_0.$$

Then we have

$$\begin{aligned}(\varphi(u'_{\lambda^*}))' = -\lambda^* f(t, u_{\lambda^*}) &= -\lambda f(t, u_{\lambda^*} + \epsilon) +\\&+ \lambda[f(t, u_{\lambda^*} + \epsilon) - f(t, u_{\lambda^*})] + (\lambda - \lambda^*) f(t, u_{\lambda^*})\\&\le -\lambda f(t, u_{\lambda^*} + \epsilon)\end{aligned}$$

i.e. $u^*_\lambda + \epsilon$ is an upper solution of $(1.1)_\lambda$.

Proof of theorem 1. Let $0 < \lambda < \lambda^*$. Since 0 is a lower solution and u_{λ^*} is an upper solution, there exists a minimum solution u_λ of $(1.1)_\lambda$ with $0 \le u_\lambda \le u_{\lambda^*}$. We next establish the existence of a second solution to $(1.1)_\lambda$.

Let $F(\lambda, u)$ be defined as in the proof of lemma 8. Further define

$$\tilde{f}(t, v(t)) = \begin{cases} f(t, u_{\lambda^*}(t) + \epsilon) & \text{if} \quad v(t) \ge u_{\lambda^*}(t) + \epsilon \\ f(t, v(t)) & \text{if} \quad -\epsilon \le v(t) \le u_{\lambda^*}(t) + \epsilon \\ f(t, -\epsilon) & \text{if} \quad v(t) \le -\epsilon \end{cases}$$

where ϵ is given in lemma 9, and let $\tilde{F}(\lambda, u)$ be the operator analogous to F defined by $\tilde{f}$. Consider

$$B = \left\{u \in C^0[a, b] : -\epsilon < u(t) < u_{\lambda^*}(t) + \epsilon, \quad t \in [a, b]\right\}.$$

Then B is open and $u_\lambda \in B,\ 0 \le \lambda \le \lambda^*$. Since $\tilde{F}$ is bounded for λ in compact intervals,

$$\deg\,(I - \tilde{F}(\lambda, \cdot), B(u_\lambda, R), 0) = 1$$

if R is sufficiently large. Here $B(u_\lambda, R)$ is the ball centered at u_λ with radius R in $C^0[a,b]$. If there exists $u \in \partial B$ such that $u = \tilde{F}(\lambda, u)$ then u is a second solution of $(1.1)_\lambda$. Suppose that $u \neq \tilde{F}(\lambda, u)$ for every $u \in \partial B$. Then deg $(I - \tilde{F}(\lambda, \cdot), B, 0)$ is well defined and since $\tilde{F}(\lambda, \cdot)$ has no fixed point in $B(u_\lambda, R) \setminus B$ (see e.g [9]), we have by the excision property

$$\deg\ (I - F(\lambda, \cdot), B, 0) = \deg\ (I - \tilde{F}(\lambda, \cdot), B, 0) = 1,\ 0 \leq \lambda \leq \lambda^*.$$

On the other hand, it follows from lemma 7 that for $\mu > 0$ there exists $M > 0$ such that for $\lambda \geq \mu$, λ in compact intervals,

$$\deg\ (I - F(\lambda, \cdot), B(0, M), 0) = \text{constant},$$

where $B(0, M)$ is the ball centered at 0 of radius M in $C^0[a,b]$. The latter degree, on the other hand, must equal 0, since for $\lambda > \lambda^*$ no solutions exist. Thus the existence of a second solution follows from the excision principle of the Leray-Schauder degree.

We remark that theorem 2, together with lemma 9 (appropriately interpreted), implies that the mapping

$$\lambda \mapsto u_\lambda,\ 0 \leq \lambda \leq \lambda^*,$$

where u_λ is the minimal solution of $(1.1)_\lambda$ is a continuous mapping $[0, \lambda^*] \to C^0[a,b]$. For it is the case that for any $\tilde{\lambda} \in (0, \lambda^*]$ the minimal solutions $\{u_\lambda$ satisfy $u_\lambda \leq u_{\tilde{\lambda}},\ \lambda \leq \tilde{\lambda}$. Furthermore the limit $u = \lim_{\lambda \to \tilde{\lambda}} u_\lambda$ exists and is a solution of $(1.1)_{\tilde{\lambda}}$. Hence $u_{\tilde{\lambda}} = \lim_{\lambda \to \tilde{\lambda}} u_\lambda$. It therefore follows that

$$\{(\lambda, u_\lambda),\ 0 \leq \lambda \leq \lambda^*\} \subset \mathcal{C},$$

where $\mathcal{C}$ is the continuum in the proof of lemma 8. Using separation results on closed sets in compact metric spaces (Whyburn's lemma), one may use the arguments used in the above proof to verify that for each $\lambda \in (0, \lambda^*)$ there are at least two solutions on the continuum $\mathcal{C}$.

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