Electronic Journal of Differential Equations, Vol. **1996**(1996), No. 3, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu (147.26.103.110)
telnet (login: ejde), ftp, and gopher access: ejde.math.swt.edu or ejde.math.unt.edu

Radial and Nonradial Minimizers for Some Radially Symmetric Functionals *

Orlando Lopes

Abstract

In a previous paper we have considered the functional

$$V(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\operatorname{grad} u(x)|^2\,dx + \int_{\mathbb{R}^N} F(u(x))\,dx$$

subject to

$$\int_{\mathbb{R}^N} G(u(x))\,dx = \lambda > 0\,,$$

where $u(x) = (u_1(x), \dots, u_K(x))$ belongs to $H^1_K(\mathbb{R}^N) = H^1(\mathbb{R}^N) \times \cdots \times H^1(\mathbb{R}^N)$ (K times) and $|\operatorname{grad} u(x)|^2$ means $\sum_{i=1}^{K} |\operatorname{grad} u_i(x)|^2$. We have shown that, under some technical assumptions and except for a translation in the space variable x, any global minimizer is radially symmetric.

In this paper we consider a similar question except that the integrals in the definition of the functionals are taken on some set Ω which is invariant under rotations but not under translations, that is, Ω is either a ball, an annulus or the exterior of a ball. In this case we show that for the minimization problem without constraint, global minimizers are radially symmetric. However, for the constrained problem, in general, the minimizers are not radially symmetric. For instance, in the case of Neumann boundary conditions, even local minimizers are not radially symmetric (unless they are constant). In any case, we show that the global minimizers have a symmetry of codimension at most one.

We use our method to extend a very well known result of Casten and Holland to the case of gradient parabolic systems. The unique continuation principle for elliptic systems plays a crucial role in our method.

I. Introduction

In a previous paper ([1]) we have shown that, under some technical assumptions and except for a translation in the space variable, any global minimizer of the

*1991 *Mathematics Subject Classifications:* 35J20, 49J10.
Key words and phrases: Variational Problems, Radial and Nonradial Minimizers.

Submitted: December 6, 1995. Published February 29, 1996.

functional

$$V(u)=\frac{1}{2}\int_{\mathbb{R}^N}|\operatorname{grad}u(x)|^2\,dx+\int_{\mathbb{R}^N}F(u(x))\,dx \tag{I.1}$$

subject to

$$\int_{\mathbb{R}^N}G(u(x))\,dx=\lambda>0 \tag{I.2}$$

is radially symmetric. In (I.1) and (I.2) $u(x)=(u_1(x),\dots,u_K(x))$ belongs to the space $H^1_K(\mathbb{R}^N)=H^1(\mathbb{R}^N)\times\cdots\times H^1(\mathbb{R}^N)$ (K times) and $|\operatorname{grad}u(x)|^2$ means $\sum_{i=1}^K|\operatorname{grad}u_i(x)|^2$.

In this paper we consider a similar problem except that the integrals in the definition of the functionals are taken on a set Ω which is rotation invariant but not translation invariant (that is, Ω is either a ball or an annulus or the exterior of a ball).

In this case we show that, for the unconstrained minimization problem with either Dirichlet or Neumann boundary conditions,any global minimizer is radially symmetric. However, for the constrained minimization problem with Neumann boundary conditions, even a local minimizer is not radially symmetric (unless it is a constant function).

In the case of a constrained minimization problem with Dirichlet boundary condition, there are examples for which the global minimizer is radially symmetric and examples for which the global minimizer is not radially symmetric.

In all cases we show that the global minimizers have a symmetry of codimension at most one.

II. The Unconstrained Problem

We consider the functional

$$W(u)=\frac{1}{2}\int_{\Omega}|\operatorname{grad}u(x)|^2\,dx+\int_{\Omega}F(x,u(x))\,dx \tag{II.1}$$

Here $u(x)=(u_1(x),u_2(x),\dots,u_K(x))$ is a K-vector valued function defined for $x\in\Omega\subset\mathbb{R}^N$. The function $u(\cdot)$ will be taken in the space $H(\Omega)$ which is the Cartesian product of K factors each of which is either $H^1_0(\Omega)$ or $H^1(\Omega)$.

Our assumptions are the following

H_1) Ω is a C^2 (bounded or unbounded) open connected set of $\mathbb{R}^N$ which is symmetric with respect to the hyperplane $x_1=0$.

H_2) $F(x,u)$ is a real valued function defined for $(x,u)\in\Omega\times\mathbb{R}^K$, which is continuous with respect to (x,u) together with their first and second derivatives with respect to u.

H_3) $F(-x_1,x_2,\dots,x_N,u)=F(x_1,x_2,\dots,x_N,u)$ for $x\in\Omega$ and $u\in\mathbb{R}^K$.

H_4) $W(u)$ is well defined for $u \in H(\Omega)$.

H_5) If $u \in H(\Omega)$ minimizes W then it satisfies the Euler system

$$-\Delta u(x) + \operatorname{grad} F(x, u(x)) = 0 \tag{II.2}$$

together with boundary conditions and $u \in L^\infty(\Omega)$.

Remarks.

II.3. Assumptions H_4 and H_5 are satisfied if $F(x, u)$ and its derivatives with respect to u satisfy certain growth conditions on u and depend on x in a convenient way (see [2] for a discussion of them).

II.4. The boundary condition for the i-component $u_i(x)$ of the minimizer $u(x)$ is either Dirichlet or Neumann boundary condition, depending on whether the i-component of the space $H(\Omega)$ is $H_0^1(\Omega)$ or $H^1(\Omega)$.

Theorem II.5. Under assumptions $H_1 - H_5$, if $u(\cdot)$ minimizes W (given by II.1) in the space $H(\Omega)$ then $u(-x_1, x_2, \ldots, x_N) = u(x_1, x_2, \ldots, x_N)$ for $x \in \Omega$.

Proof. We define the sets $\Omega_l = \{x \in \Omega : x_1 \leq 0\}$ and $\Omega_r = \{x \in \Omega : x_1 \geq 0\}$. Let $u \in H(\Omega)$ be a minimizer of W. We claim that

$$\begin{aligned}\frac{1}{2}\int_{\Omega_l} |\operatorname{grad} u(x)|^2\,dx + \int_{\Omega_l} F(x, u(x))\,dx = \\ \frac{1}{2}\int_{\Omega_r} |\operatorname{grad} u(x)|^2\,dx + \int_{\Omega_r} F(x, u(x))\,dx.\end{aligned} \tag{II.6}$$

Let us denote by A and B the left and the right side of II.6, respectively, and, by contradiction, suppose $A < B$. We define $U(x)$ in the following way:

$$U(x) = \begin{cases} u(x) & \text{if } x \in \Omega_l \\ u(x') & \text{if } x \in \Omega_r i \end{cases} \tag{II.7}$$

where x' denotes the reflection of x with respect to the hyperplane $x_1 = 0$. Clearly $U \in H(\Omega)$; moreover we have $W(U) = A + A < A + B = W(u)$, a contradiction. This proves II.6.

Keeping the definition II.7 for $U(\cdot)$, from II.6 we conclude that $W(U) = A + A = A + B = W(u)$ and this means that both u and U are minimizers and so, by assumption H_5, they satisfy the Euler systems:

$$-\Delta u(x) + \operatorname{grad} F(u(x)) = 0 \tag{II.8}$$

$$-\Delta U(x) + \operatorname{grad} F(U(x)) = 0\,. \tag{II.9}$$

If we define $z(x) = u(x) - U(x)$ and we subtract II.9 from II.8 then, from assumptions H_2 and H_5, we see that $z(x)$ satisfies a linear system of the form

$$-\Delta z + A(x)z = 0\,,$$

where $A(x)$ is a $K \times K$ matrix whose entries belong to $L_\infty(\Omega)$ and, since $z = 0$ in Ω_l, from the unique continuation principle ([3]), we conclude that $z(x)$ vanishes in Ω and this proves the theorem. ♠

Corollary II.10. If Ω is either a ball or an annulus or the exterior of a ball centered at the origin and $F(x,u) = F(|x|,u)$, where $|x| = (x_1^2 + \cdots + x_N^2)^{1/2}$, then any minimizer for W is radially symmetric.

Proof. We apply theorem II.5 for any hyperplane passing through the origin.

Corollary II.10. has been proved in [1] for the case of Dirichlet boundary conditions assuming that F does not depend on $|x|$. Theorem II.5 has been proved in [4] in the case of Dirichlet boundary condition and assuming that F does not depend on x and that Ω is convex in the x_1-direction (that is, if two points A and B belong to Ω and the segment AB is parallel to x_1, then the segment AB is contained in Ω). The method of the proof of theorem II.5 is, basically, the method used in [1] and [4].

Remarks.

II.11. In the scalar case ($K = 1$), if Ω and F are as in corollary II.10, then any *local* minimizer for W is radially symmetric ([5]). The proof of this statement depends on the maximum principle and so it can be extended to the system case provided $\operatorname{grad} F(u)$ satisfies a cooperative condition.

II.12. Corollary II.10 holds for functionals given by $\int_\Omega \varphi(|x|, u(x), |\operatorname{grad} u(x)|)\,dx$ provided that the corresponding Euler system is elliptic nondegenerate and the minimizer is regular enough.

II.13. If we add to II.1 a term $\int_{\partial\Omega} H(u(x))\,dS$ involving an integral on the boundary, then we can prove corollary II.10 for other boundary conditions.

III. Nonradial Minimizers

In this part we consider the functional

$$V(u) = \frac{1}{2}\int_\Omega |\operatorname{grad} u(x)|^2\,dx + \int_\Omega F(u(x))\,dx \qquad \text{(III.1)}$$

subject to

$$\int_\Omega G(u(x))\,dx = \lambda. \tag{III.2}$$

The function $u(x) = (u_1(x), \dots, u_K(x))$ will be taken in the space $H^1_K(\Omega) = H^1(\Omega)\times\cdots\times H^1(\Omega)$ (K times) and by $|\operatorname{grad} u(x)|^2$ we mean $\sum_{i=1}^K |\operatorname{grad} u_i(x)|^2$.

The first results about the break of symmetry of minimizers for the problem III.1 – III.2 are due to M. Esteban ([6], [7]) and V. Coti-Zelati-M. Esteban ([8]) for the scalar case. If Ω is either the exterior of a ball centered at the origin or the ball itself, $F(u) = u^2$, $G(u) = |u|^p$, $2 < p < \frac{2N}{N-2}$ and $\lambda > 0$, then the global minimizers for III.1 – III.2 are not radially symmetric ([6], [7]). A similar statement holds for the annulus ([8]).

In this paper the set $\Omega \subset \mathbb{R}^N$ will be either a ball with radius R centered at the origin or the annulus $\{x \in \mathbb{R}^N : 0 < R < |x| < R_1\}$ (R_1 is allowed to be $+\infty$) and our assumptions are the following:

H_1) $F, G : \mathbb{R}^K \to \mathbb{R}$ are C^2 functions.

H_2) $|F'(u)|,\ |G'(u)| \leq$ const. $|u|^{p-1}$ for $|u|$ large with $p < \frac{2N}{N-2}$ (p finite if $N = 2$).

H_3) In the case $R_1 = +\infty$ we also assume that $F(0) = G(0) = 0$ and $F'(0) = G'(0) = 0$.

Theorem III.3. Suppose $\Omega, F(u)$ and $G(u)$ satisfy the assumptions above and let $u \in H^2_K(\Omega)$ be a radially symmetric non-constant solution of the system

$$-\Delta u + \operatorname{grad} H(u) = 0 \tag{III.4}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega\,, \tag{III.5}$$

where $H(u) = F(u) + \alpha G(u)$, $\alpha =$ constant. Suppose also that grad $G(u(\cdot)) \not\equiv 0$ (a manifold condition) and that $u(\cdot)$ satisfies III.2. Then $u(\cdot)$ is not a local minimizer for the problem III.1 - III.2.

Proof. Due to the growth and regularity assumptions on $F(u)$ and $G(u)$ we know that $u \in W^{3,q}(\Omega)$, $2 \leq q < \infty$.

We consider the quadratic functional

$$Q(h) = \int_\Omega |\operatorname{grad} h(x)|^2\,dx + \int_\Omega \langle h(x), H''(u(x))h(x)\rangle\,dx \tag{III.6}$$

subject to the linear constraint

$$\int_\Omega \langle \operatorname{grad} G(u(x)), h(x)\rangle\,dx = 0, \tag{III.7}$$

for $h \in H^1_K(\Omega)$, where $H''(u(x))$ denotes the Hessian matrix of $H(u)$ at $u = u(x)$.

Denoting by $\mathcal{F}(\Omega)$ the set of the C^∞ functions defined in Ω with values in R^K and having support contained in a ball centered at the origin (of course this last demand makes sense only if Ω is the exterior of a ball),our first claim is that there is an element $k \in \mathcal{F}(\Omega)$ satisfying III.7 such that $Q(k) < 0$.

We start by showing that there is an element $k \in H^1_K(\Omega)$ satisfying III.7 such that $Q(k) < 0$. In fact, if we define $h_0(x) = \frac{\partial u}{\partial x_1} = \frac{x_1}{r}u'(r)$ and we differentiate III.4 with respect to x_1 we get

$$-\Delta h_0 + H''(u(x))h_0 = 0 \tag{III.8}$$

and taking the scalar product of III.8 with $h_0(x)$ and integrating we get $Q(h_0) = 0$ (we have used that h_0 vanishes on $\partial\Omega$, due to III.5).

Moreover, since the function $\frac{1}{r}\langle \operatorname{grad} G(u(r)), u'(r)\rangle$ depends just on r and $x = (x_1, x_2, \ldots, x_N) \to x_1$ is an odd function, we have

$$\int_\Omega \langle \operatorname{grad} G(u(r)), h_0(x)\rangle\, dx = 0$$

and this means that h_0 satisfies III.7.

Next we claim that, in despite of III.8, h_0 is not a critical point for the problem III.6-III.7 for h in the space $H^1_K(\Omega)$. In fact, if it was, then for some real number β and any $\varphi \in H^1_K(\Omega)$ we would have

$$\begin{aligned}\int_\Omega \langle \operatorname{grad} h_0(x), \varphi(x)\rangle\, dx + \int_\Omega \langle \varphi(x), H''(u(x))h_0(x)\rangle\, dx \\ +\beta \int_\Omega \langle \operatorname{grad}\, G(u(x)), \varphi(x)\rangle\, dx \quad = \quad 0,\end{aligned}$$

and, since $h_0(x)$ is regular enough, an integration by parts would give

$$-\Delta h_0(x) + H''(u(x))h_0(x) + \beta \operatorname{grad}\, G(u(x)) = 0$$

(which holds with $\beta = 0$ in view of III.8) and $\frac{\partial h_0}{\partial n} = 0$ on $\partial\Omega$. But, from the definition of $h_0(x)$ and the boundary condition $u'(R) = 0$, we see that $\frac{\partial h_0}{\partial n} = 0$ on $\partial\Omega$ implies $u''(R) = 0$. But since III.4 is a second order ordinary differential system, if we have $u'(R) = 0 = u''(R)$ then, by uniqueness, $u(r)$ is constant and this contradiction shows that h_0 is not a critical point for III.6-III.7; in particular, it is not a minimizer and this implies that there is a $k \in H^1_K(\Omega)$ satisfying III.7 such that $Q(k) < 0$.

The fact that k can be taken in $\mathcal{F}(\Omega)$ follows from the following remark: if $g \in L^2_K(\Omega)$ and $g \not\equiv 0$ (in our case $g(x) = \operatorname{grad} G(u(x))$) then the set of the elements $\psi \in \mathcal{F}(\Omega)$ such that $\int_\Omega \langle g(x), \psi(x)\rangle\, dx = 0$ is dense in the set of the elements $\psi \in H^1_K(\Omega)$ such that $\int_\Omega \langle g(x), \psi(x)\rangle\, dx = 0$. In order to prove the remark we fix an element $\varphi \in \mathcal{F}(\Omega)$ such that $\int_\Omega \langle g(x), \varphi(x)\rangle\, dx \neq 0$; if

$f \in H^1_K(\Omega)$ is such that $\int_\Omega \langle g(x), f(x)\rangle = 0$ and $f_n \in \mathcal{F}(\Omega)$ is a sequence converging to f in $H^1_K(\Omega)$ and we define $\hat{f}_n = f_n + \epsilon_n \varphi$, where ϵ_n is chosen in such way that $\int_\Omega \langle g(x), \hat{f}_n(x)\rangle \, dx = 0$, then $\hat{f}_n$ converges to f in $H^1_K(\Omega)$ because ϵ_n tends to zero.

Now, if $k \in \mathcal{F}(\Omega)$ is as above, we can construct a smooth admissible curve that is tangent to k at u. In fact, if $\varphi \in \mathcal{F}(\Omega)$ is a fixed element such that $\int_\Omega \langle \text{grad } G(u(x)), \varphi(x)\rangle \, dx \neq 0$ and we define the function

$$S(s,t) = \int_\Omega G(u(x) + s\varphi(x) + tk(x)) \, dx - \lambda \,,$$

we have $S(0,0) = 0$ and

$$\frac{\partial S}{\partial s}(0,0) = \int_\Omega \langle \text{grad}\, G(u(x)), \varphi(x)\rangle \, dx \neq 0 \,.$$

Hence, by the implicit function theorem, there is C^2 function $s(t)$ defined for t in some open interval J containing $t = 0$ such that $s(0) = 0$ and

$$\int_\Omega G(u(x) + s(t)\varphi(x) + tk(x)) \, dx = \lambda, \tag{III.9}$$

for t in the interval J. So, if we define $h(t,x) = u(x) + s(t)\varphi(x) + tk(x)$ we have $\dfrac{\partial h}{\partial t}(0,x) = k(x)$ and differentiating III.9 twice with respect to t and setting $t = 0$ we get

$$\int_\Omega (\langle \text{grad}\, G(u(x)), \frac{\partial^2 h}{\partial t^2}(0,x)\rangle + \langle k(x), G''(u(x))k(x)\rangle) \, dx = 0. \tag{III.10}$$

Now, a short computation shows that $\frac{d}{dt} V(h(t,x))\Big|_{t=0} = 0$ (of course this is a consequence of the fact that u is a critical point for III.1-III.2 and that the curve $h(t,.)$ is admissible in the sense of III.9). Furthermore,

$$\begin{aligned}
\frac{d^2}{dt^2} V(h(t,x))\Big|_{t=0} &= \int_\Omega (\langle \text{grad}\, k(x), \text{grad } k(x)\rangle + \langle \text{grad } u(x), \frac{\partial^2 h(0,x)}{\partial t^2}\rangle \\
&\quad + \langle \text{grad}\, F(u(x)), \frac{\partial^2 h(0,x)}{\partial t^2}\rangle + \langle k(x), F''(u(x))k(x)\rangle) \, dx \\
&= \int_\Omega (\langle \text{grad } k(x), \text{grad } k(x)\rangle - \langle \Delta u(x), \frac{\partial^2 h(0,x)}{\partial t^2}\rangle \\
&\quad + \langle \text{grad}\, F(u(x)), \frac{\partial^2 h(0,x)}{\partial t^2}\rangle + \langle k(x), F''(u(x))k(x)\rangle) \, dx \\
&= \int_\Omega |\text{ grad } k(x)|^2 \, dx + \int_\Omega \langle k(x), H''(u(x))k(x)\rangle \, dx \\
&= Q(k) < 0 \,.
\end{aligned}$$

(we have performed an integration by parts and have used that $\frac{\partial u}{\partial n} = 0$ in $\partial\Omega$;we have also used III.4 and III.10).

The conclusion is this: under the assumptions of the theorem, we were able to find a curve $h(t,x)$ such that $h(0,x) = u(x)$,

$$\int_\Omega G(h(t,x))\,dx = \lambda, \quad \frac{dV(h(t))}{dt}\bigg|_{t=0} = 0 \text{ and } \frac{d^2V(h(t))}{dt^2}\bigg|_{t=0} < 0\,.$$

Clearly this implies that u is not a local minimizer for III.1-III.2 and the theorem is proved. ♠

Remarks.

III.11. For bounded Ω the existence of global minimizer for III.1 - III.2 in the subcritical case follows from standard arguments. In a forthcoming paper we will discuss the same question for the exterior domain.

III.12. Theorem III.3 with the same proof also holds in the case of several constraints.

III.13. The stability and the un-stability of standing waves for the Schrodinger equation in the exterior of a ball, with Neumann boundary conditions, have been studied in [9]. For particular nonlinearities it has been shown that radially symmetric standing waves are unstable. Theorem III.3 is, perhaps, an indication that those waves are unstable for more general nonlinearities.

If Ω is the exterior of a ball then the case $u =$ constant cannot occur. If Ω is bounded and $F(u)$ as a function from $\mathbb{R}^K$ into $\mathbb{R}$ has a global minimum at $u = u_0$ and $\lambda =$ (meas. Ω) $G(u_0)$, then $u = u_0$ is a global minimum for III.1 - III.2. This means that if Ω is bounded the case $u =$ constant may occur.

Next we give an example for which the global minimizer is not a constant. If we take $F(u,v) = auv^2 + (a+1)u^3 - (4+a)u$, $G(u,v) = u^2 + v^2$, $\lambda = 1$ and Ω is either a ball or an annulus with, say, measure one, then the minimum of III.1 - III.2 on the set of the constant functions is achieved at $u = 1$ and $v = 0$. In order to analyze whether this function is a local minimizer for III.1 - III.2 we have to consider the quadratic form

$$Q(h,k) = \int_\Omega (|\operatorname{grad} h(x)|^2 + |\operatorname{grad} k(x)|^2)\,dx + \int_\Omega ((8a-1)h^2(x) + k^2(x))\,dx$$

subject to $\int_\Omega h(x)\,dx = 0$. If we take $h = \varphi_2$ and $k = 0$, where φ_2 is the eigenfunction corresponding to the second eigenvalue of $-\Delta$ on Ω with Neumann boundary condition, we have $Q(\varphi_2, 0) = 8a - 1 + \lambda_2$ and so, if we choose a in such way that $8a - 1 + \lambda_2 < 0$ then $u = 1$ and $v = 0$ is not a local minimum for III.1

- III.2. If $N \leq 3$ and $F(u,v)$ and $G(u,v)$ are as above then $V(u,v)$ is bounded below on the admissible set and then, by the classical methods, $V(u,v)$ has minimizer and, according with the previous argument, this minimizer cannot be a constant and so, by theorem III.3, it is not radially symmetric.

If we consider the unconstrained problem

$$V(u) = \frac{1}{2}\int_\Omega |\operatorname{grad} u(x)|^2\,dx + \int_\Omega F(u(x))\,dx \tag{III.14}$$

where Ω is either a ball or an annulus centered at the origin then, the same argument presented in the proof of theorem III.3, shows that a local minimizer for $V(u)$ in $H^1_K(\Omega) = H^1(\Omega)\times\cdots\times H^1(\Omega)(K$ times) is either a constant or not radially symmetric.

If $u \in H^1_K(\Omega)$ is a *global* minimizer for III.14 then, according with corollary II.10, u has to be radially symmetric and so it has to be constant.

In the scalar case ($K = 1$) it has been known since 1978 ([10]) that local minimizers for III.12 in $H^1(\Omega)$ are constant if either Ω is bounded and convex (with smooth boundary) or Ω is an annulus. This result is false if, for instance, Ω consists of two balls joined by a thin channel ([11]) or it is shaped like a dumbbell ([12]).

Next we show that in the case Ω is bounded and convex, then the same result holds for the system case. The proof is basically the same as in the scalar case but a slight change is required to avoid the maximum principle. If Ω is an annulus we do not know how to do that (unless $\operatorname{grad} F(u)$ satisfies a cooperative condition).

Theorem III.15. Let $F : \mathbb{R}^K \to \mathbb{R}$ be a C^2 function whose first derivatives are bounded by $c|u|^{p-1}$ for u large and $p < \frac{2N}{N-2}$. If Ω is bounded, convex and has a smooth boundary then any local minimizer of V given by III.14 in the space $H^1_K(\Omega)$ is constant.

Proof. Let u be a local minimizer of V in the space $H^1_K(\Omega)$; then u satisfies the elliptic system

$$-\Delta u + \operatorname{grad} F(u) = 0, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega. \tag{III.16}$$

In view of the assumptions on $F(u)$ we see that $u \in W^{3,q}_K(\Omega),\ 1 \leq q < \infty$. If we differentiate III.16 with respect to x_i, take the scalar product of the resulting equation with $\frac{\partial u}{\partial x_i}$ and integrate we get:

$$\sum_{i=1}^N V''(u)\left(\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_i}\right) = \sum_{j=1}^K \int_{\partial\Omega} \langle \operatorname{grad} u_j(x), \frac{\partial}{\partial n}\operatorname{grad} u_j(x)\rangle\,dS \tag{III.17}$$

If x is an arbitrary point in $\partial\Omega$ then, without loss we generality, we can assume that x is the origin of the coordinate system. We suppose that $x_N = g(x_1, \ldots, x_{N-1})$ is a C^2 function whose graph describes the boundary of Ω in some neighborhood of the origin and it is such that at the origin the $-x_N$ axis in the outward normal direction. Then according with [10], page 269, equation 12, we have

$$\langle \operatorname{grad} u_j(0), \frac{\partial}{\partial n} \operatorname{grad} u_j(0)\rangle = -\sum_{i=1}^{N-1}\sum_{k=1}^{N-1} \frac{\partial^2 g(0)}{\partial x_i \partial x_k}\frac{\partial u_j(0)}{\partial x_i}\frac{\partial u_j(0)}{\partial x_k}. \qquad \text{(III.18)}$$

For sake of completeness we reproduce here the argument presented in [10]. Since $\dfrac{\partial u_j(0)}{\partial x_N} = \dfrac{\partial u_j(0)}{\partial n} = 0$ we have

$$\langle \operatorname{grad} u_j(0), \frac{\partial \operatorname{grad} u_j(0)}{\partial n}\rangle = -\sum_{i=1}^{N-1} \frac{\partial u_j(0)}{\partial x_i}\frac{\partial^2 u_j(0)}{\partial x_i x_N}. \qquad \text{(III.19)}$$

Now $\dfrac{\partial u_j}{\partial n} = 0$ on $\partial\Omega$ is equivalent to

$$\begin{aligned}\sum_{i=1}^{N-1} \frac{\partial u_j}{\partial x_i}(x_1,...,x_{N-1},g(x_1,...,x_{N-1}))\frac{\partial g}{\partial x_i}(x_1,...,x_{N-1}) - \qquad \text{(III.20)}\\ \frac{\partial u_j}{\partial x_N}(x_1,...,x_{N-1},g(x_1,...,x_{N-1})) = 0 \quad j = 1,...,K.,\end{aligned}$$

If we differentiate III.20 with respect to x_k, $k = 1, \ldots, N-1$ and take in account that $\frac{\partial g(0)}{\partial x_i} = 0$, $i = 1, \ldots, N-1$ (because $x_N = 0$ is tangent to the graph of $g(x_1, \ldots, x_{N-1})$ at the origin) we get

$$\sum_{i=1}^{N-1} \frac{\partial u_j(0)}{\partial x_i}\frac{\partial^2 u_j(0)}{\partial x_k \partial x_N} - \frac{\partial^2 u_j(0)}{\partial x_k \partial x_N} = 0 \qquad \text{(III.21)}$$

$k = 1, \ldots, N-1$.Substituting expressions III.21 for $\dfrac{\partial^2 u_j(0)}{\partial x_k \partial x_N}$ into III.19 we get III.18. Since Ω is convex the right hand side of III.18 is nonpositive.

In order to prove the theorem we have to show that there is an element $h \in H^1_K(\Omega)$ such that $V''(u)(h,h) < 0$, unless u is constant. So, assume $V''(u)(h,h) \geq 0$ for any $h \in H^1_K(\Omega)$. Since $\displaystyle\sum_{i=1}^{N} V''(u)\left(\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_i}\right) \leq 0$ we must have $V''(u)\left(\dfrac{\partial u}{\partial x_i}, \dfrac{\partial u}{\partial x_i}\right) = 0, i = 1, \ldots, N$, and $\dfrac{\partial}{\partial n}\dfrac{\partial u}{\partial x_i} = 0,\ i = 1, \ldots, N$ because the quadratic functional $V''(u)(h,h)$ has a minimum at $h = \dfrac{\partial u}{\partial x_i}$.

For a compact smooth hypersurface we know that there is a point where the Gauss-Kronecker curvature is strictly positive; this means that for an open set of the boundary the Hessian matrix of the function $g(x_1, \ldots, x_{N-1})$ is positive definite and then, from III.18, we conclude that $\frac{\partial u}{\partial x_i} = 0$, $i = 1, \ldots, N$, in an open set of the boundary. But, since

$$-\Delta \frac{\partial u}{\partial x_i} + F''(u(x)) \frac{\partial u}{\partial x_i} = 0 \quad \text{and} \quad \frac{\partial}{\partial n}\left(\frac{\partial u}{\partial x_i}\right) = 0\,,$$

from the unique continuation principle ([3]) we get $\frac{\partial u}{\partial x_i} = 0$ everywhere in $\Omega, i = 1, \ldots, N$, and then u is constant and the theorem is proved. ♠

Next we make some remarks about the global minimizer in the case of Dirichlet boundary condition. So we take $K = 1$ (the scalar case) and we consider III.1 - III.2 in the space $H_0^1(\Omega)$.

If Ω is the exterior of a bounded domain then there is no global minimizer ([6]).

If Ω is a ball centered at the origin and $F(u)$ and $G(u)$ are even functions, then a global minimizer cannot change sign and so, thanks to a theorem of Gidas, Ni and Nirenberg ([13]), a global minimizer has to be radially symmetric.

If Ω is the annulus $\{0 < R_2 \leq |x| \leq R_1 < \infty\}$ and we take $F(u) \equiv 0$ and $G(u) = |u|^p$, then for $p = 2$ the global minimizer in $H_0^1(\Omega)$ is the first eigenfunction which, by uniqueness, is radially symmetric; however, for $N \geq 3$, there is a $p_0, 2 < p_0 < \frac{2N}{N-2}$, such that for $p_0 < p < \frac{2N}{N-2}$ the global minimizer is not radially symmetric ([14]).

IV. Codimension One Symmetry of Minimizers

In this final section we consider the functional

$$E(u) = \frac{1}{2} \int_\Omega |\operatorname{grad} u(x)|^2 dx + \int_\Omega F(r, u(x)) dx \tag{IV.1}$$

subject to

$$\int_\Omega G(r, u(x)) dx = \lambda > 0\,. \tag{IV.2}$$

As before, $u(x) = (u_1(x), \ldots, u_K(x))$ is a vector-valued function, $|\operatorname{grad} u(x)|^2$ means $|\operatorname{grad} u_1(x)|^2 + \ldots + |\operatorname{grad} u_K(x)|^2$ and $r = (x_1^2 + \ldots + x_N^2)^{1/2}$.

Problem IV.1 - IV.2 will considered in the space $H(\Omega)$ as in section II, that is, $H(\Omega)$ is the Cartesian product of K factors, each of which is either $H^1(\Omega)$ or $H_0^1(\Omega)$ and our assumptions are the following:

H_1) Ω is either a ball or an annulus or the exterior of a ball centered at the origin.

H_2) $F(r,u)$ and $G(r,u)$ are real valued functions defined for $(x,u) \in \Omega \times \mathbb{R}^K$ which are continuous with respect to (x,u) together with their first and second derivatives with respect to u.

H_3) $E(u)$ is well defined for $u \in H(\Omega)$ in the admissible set.

H_4) if $u \in H(\Omega)$ minimizes $E(u)$ on the admissible set, then it satisfies the Euler system

$$-\Delta u(x) + \operatorname{grad} F(r,u(x)) + \alpha \operatorname{grad} G(r,u(x)) = 0,$$

for some constant α, together with boundary conditions (see remark II.4) and $u \in L^\infty(\Omega)$.

Theorem IV.3. Under assumptions $H_1 - H_4$, if $u \in H(\Omega)$ is a global minimizes for IV.1 - IV.2, then there is line L through the origin such that $u(\cdot)$ is symmetric with respect to any hyperplane containing L.

Remark IV.4. The motivation for theorem IV.3 is that for real valued positive functions $u(x)$ defined on the exterior of a ball, it is possible to define a function $u^*(x)$ that has the symmetry mentioned in the theorem and behaves like the symmetrization for positive functions defined in $\mathbb{R}^N$ (see [9], proposition I.4). For the proof of theorem IV.3 we need the following

Lemma IV.5. Let Ω be a rotation invariant subset of $\mathbb{R}^N$ and let h be an element of $L^1(\Omega)$. Then for any subspace $S \subset \mathbb{R}^N$ of codimension 2 there is a hyperplane P containing S such that

$$\int_{\Omega+} h(x)dx = \int_{\Omega-} h(x)dx\,,$$

where Ω_+ and Ω_- are the intersections of Ω with the half-spaces determined by P (we will say that the hyperplane P splits the integral $\int_\Omega h(x)\,dx$ in the middle).

Proof. We assume that assume the orthogonal subspace to S is spanned by $e_1 = (1,0,\dots,0)$ and $e_2 = (0,1,\dots,0)$. We define $e(\theta) = (\cos\theta, \sin\theta, \dots, 0)$, $0 \leq \theta \leq \pi$, we denote by $P(\theta)$ the hyperplane spanned by $\{e(\theta)\} \cup S$ and by $P_+(\theta)$ and $P_-(\theta)$ the half spaces of the vectors x such that $\langle x, e(\theta)\rangle \geq 0$ and $\langle x, e(\theta)\rangle \leq 0$, respectively. Next we let

$$g(\theta) = \int_{\Omega+(\theta)} h(x)dx - \int_{\Omega_-(\theta)} h(x)dx$$

where $\Omega_+(\theta) = \Omega \cap P_+(\Omega)$ and $\Omega_-(\theta) = \Omega \cap P_-(\theta)$. Then $g(\theta)$ is a single valued continuous function for θ in $[0,\pi]$ and $g(\pi) = -g(0)$; this implies that $g(\theta)$ vanishes somewhere and the lemma is proved.

Remark IV.6. Clearly lemma IV.5 holds also if S has codimension greater than two.

Proof of Theorem IV.3. Let $u \in H(\Omega)$ be as in the theorem; we define $h(x) = G(r, u(x))$ and we start with any line L_0 through the origin. According with lemma IV.5, we know that there is a hyperplane P_1 containing L_0 that splits the constraint in the middle. We denote by L_1 the line orthogonal to P_1 through the origin and by N_1 its corresponding unit vector. Using lemma IV.5 for L_1, we construct a hyperplane P_2 containing L_1 that splits the constraint in the middle. We denote by L_2 the line orthogonal to P_2 through the origin and by N_2 the corresponding unit vector. Next we use lemma IV.5 for the subspace S_2 spanned by N_1 and N_2. With this procedure we construct mutually orthogonal hyperplanes $P_1, P_2, \ldots, P_{N-1}$ containing the the origin such that each one splits the constraint in the middle. We may assume that the coordinate system $(x_1, x_2, \ldots, x_N)$ is such that P_i is the hyperplane $x_i = 0$.

Now, arguing exactly as in the proof of theorem II.4, we see that any global minimizer of IV.1-IV.2 is symmetric with respect to any hyperplane containing the origin that splits the constraint in the middle and so, $u(-x_1, x_2, \ldots, x_N) = u(x_1, x_2, \ldots, x_N)$, $u(x_1, -x_2, \ldots, x_N) = u(x_1, x_2, \ldots, x_N)$ and then $u(-x_1, -x_2, \ldots, x_N) = u(x_1, x_2, \ldots, x_N)$. From this last equality we see that any hyperplane containing the x_N-axis splits the constraint in the middle and this proves the theorem.

Acknowledgment. We thank Prof. F. Mercury for having taught Gauss-Kronecker curvature to us and Prof. O. Kavian for reference [14].

References

[1] O. Lopes, Radial Symmetry of Minimizers for Some Translation and Rotation Invariant Functionals, J. Diff. Eqns. (to appear).

[2] O. Kavian, Introduction à la Theorie des Points Critiques, Mathematiques et Applications (SMAI), Springer-Verlag, 1993.

[3] L. Hormander, The Analysis of Linear Partial Differential Operators, vol. III, Springer-Verlag.

[4] K. Benmlik, Etude Qualitative de Certain Problemes Semi-Lineaires Elliptique, Thèse, Université Henry Poincaré, Nancy I, 1994.

[5] G. Sweers, A sign-changing global minimizer on a convex domain, C. Bandle et allii, Progress in PDE's, Pitman Research Notes in Mathematics Series, 266(1992), 251-258.

[6] M. Esteban, Nonsymmetric ground states of symmetric variational problems, Comm. Pure and Appl. Math., vol. XLIV(1991), 259-274.

[7] M. Esteban, Rupture de symétrie pour des problèmes de Neumann surlinéaires dans les ouverts extérieur, C.R.A.S. 308, Série I, (1989), 281-286.

[8] V. Coti-Zelati-M. Esteban, Symmetry breaking and multiple solutions for a Neumann problem in an exterior domain Proc. Royal Soc. Edinburgh, vol. 116A(1990), 327-339.

[9] M. Esteban and W. Strauss, Nonlinear bound states outside an insulated sphere, Comm. Partial Diff. Eqns., 19(1994), No. 1-2, 177-197.

[10] R. Casten and C. Holland, Instability results for a reaction-diffusion equation with Neumann boundary conditions, J. Diff. Eqns., 27(1978), 266-273.

[11] J. Hale and J. Vegas, A nonlinear parabolic equation with varying domain, Arch. Rat. Mech. Anal., 86(1984), No. 2, 99-123.

[12] H. Matano, Asymptotic behavior and stability of solutions of semilinear diffusion equations, Publ. Res. Inst. Math. Sci. Kyoto Univ., 15(1979), 401-454.

[13] B. Gidas, W.N. Ni and L. Nirenberg, Symmetry of Positive Solutions of Nonlinear Elliptic Equations in $\mathbb{R}^N$, in Math. Anal. Appl., part I, L. Nachbin (ed.) Academic Press, 1981.

[14] O. Kavian, Personal Communication, 1995.

Orlando Lopes
IMECC-UNICAMP - CP 6065
13081-970, Campinas, SP, Brazil
E-mail: lopes@ime.unicamp.br