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Stable multiple-layer stationary solutions of a semilinear parabolic equation in two-dimensional domains *

Arnaldo Simal do Nascimento

Dedicated to the memory of Ennio De Giorgi (1928-1996)

Abstract

We use Γ–convergence to prove existence of stable multiple–layer stationary solutions (stable patterns) to a reaction–diffusion equation. Given nested simple closed curves in $\mathbb{R}^2$, we give sufficient conditions on their curvature so that the reaction–diffusion problem possesses a family of stable patterns. In particular, we extend to two-dimensional domains and to a spatially inhomogeneous source term, a previous result by Yanagida and Miyata.

1 Introduction

This paper is a contribution to the investigation of some diffusion processes, wherein the interplay between diffusivity and a source term gives rise to stable multiple-layer stationary solutions. This kind of solution will be referred to as a multiple-layer pattern. At the same time, along with [23] and [24], it consolidates the use of the Γ–convergence technique to show the existence of multiple-layer patterns of some semilinear parabolic equations which involve a small parameter.

Specifically we will be focusing on processes governed by the evolution problem

$$\begin{aligned} \frac{\partial v_\varepsilon}{\partial t} &= \varepsilon^2 \operatorname{div}\left(k_1(x)\nabla v_\varepsilon\right) + k_2(x)(v_\varepsilon-\alpha)(\beta-v_\varepsilon)(v_\varepsilon-\gamma_\varepsilon(x))\,, \\ &\quad \text{for } (x,t)\in\Omega\times\mathbb{R}^+ \\ &\quad v_\varepsilon(x,0)=v_0 \quad \frac{\partial v_\varepsilon}{\partial \widehat{n}}=0\,, \quad \text{for } x\in\partial\Omega\,,\ t>0 \end{aligned} \tag{1.1}$$

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where $\widehat{n}$ is the inward normal to $\partial\Omega$; $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$, with ε a small positive parameter; $k_i(x)$, $i = 1, 2$, are positive functions in $C^1(\Omega)$; $\Omega \subset \mathbb{R}^2$ has smooth boundary $\partial\Omega$, say of class C^3; and

$$\begin{aligned} \gamma_\varepsilon(x) &= \left(\tfrac{\alpha+\beta}{2}\right) + g_\varepsilon(x),\ \alpha < \beta,\ \gamma_\varepsilon \in C(\overline{\Omega}) \\ g_\varepsilon(x) &= o(\varepsilon), \quad \text{uniformly in } \Omega \text{ as } \varepsilon \to 0\,. \end{aligned} \tag{1.2}$$

In [21], the existence of multiple-layer patterns has been proved for the one-dimensional case, i.e, $\Omega = [0, 1]$, with the hypotheses that $k_2(x) \equiv 1$, $\gamma_\varepsilon = 1/2 + a_1\varepsilon + o(\varepsilon^2)$,$a_1$ constant, $\alpha = 0$, and $\beta = 1$. Here, besides considering any two-dimensional domain Ω, we allow for spatially inhomogeneous perturbations of the state $v = [(\alpha + \beta)/2]$.

The procedure used in [21] was to construct super and sub-solutions by modifying a traveling wave solution of (1.1), with the aforementioned restrictions, to obtain a multiple-layer pattern. However, this procedure is not suitable for a generalization to two-dimensional domains. Herein a technique known as Γ–convergence devised by De Giorgi [11] and developed by many others is used.

When seeking stable stationary solutions to some classes of semilinear parabolic equations, the Γ–convergence approach turns out to be very useful. This approach replaces the original problem of minimizing a family of functionals by a more tractable problem in the space of functions of bounded variation, $BV(\Omega)$, which usually yields more precise information on the geometric structure of the minimizers.

The prototype for the source term in (1.1) is the case $\alpha = 0$, $\beta = 1$ and $0 < \gamma_\varepsilon < 1$, which stems from the theory of population genetics where α, β and γ_ε denote some probability measures. The case in which $\Omega = [0, 1]$, $\alpha = 0$, $\beta = 1$ and $\gamma_\varepsilon(x) = a(x)$, $0 < a(x) < 1$, is studied in [1].

There are many works dealing with the existence of multiple or double-layer patterns for equations similar to (1.1). Most of these deal with the unidimensional case. Among those which bear more resemblance to (1.1) are [7, 13, 14].

In [8], the generation and propagation of internal layers for a related problem are studied for the case $\Omega = \mathbb{R}$. See also [5]. For a physical background on Problem (1.1) the interested reader is referred to [1, 12].

2 Main Result

Let γ_i be smooth simple closed curves whose traces lie inside Ω and which are nested, in the sense that if O_i denotes the open region enclosed by γ_i, i.e. $\gamma_i = \partial O_i$, $i = 1, \ldots, p$, then $O_1 \subset O_2 \subset \cdots \subset O_{p+1} \stackrel{\text{def}}{=} \Omega$ and $\partial O_i \cap \partial O_{i+1} = \emptyset$, $i = 1, \ldots, p$. We abuse the notation and denote by γ_i the map as well as the trace of the curve itself and set throughout

$$\Omega_1 \stackrel{\text{def}}{=} O_1\,,\ \Omega_2 = O_2\backslash\overline{O}_1, \ldots, \Omega_p = O_p\backslash\overline{O}_{p-1}\,,\ \Omega_{p+1} = \Omega\backslash\overline{O}_p\ .$$

For future reference, we consider the following function

$$v_0 = \alpha \chi_{\Omega^0_\alpha} + \beta \chi_{\Omega^0_\beta} \tag{2.1}$$

where χ_A stands for the characteristic function of the set A and

$$\Omega^0_\alpha \stackrel{\text{def}}{=} \bigcup_{\substack{1\le j\le p+1 \\ j \text{ is odd}}} \Omega_j\,, \qquad \Omega^0_\beta \stackrel{\text{def}}{=} \bigcup_{\substack{1\le j\le p+1 \\ j \text{ is even}}} \Omega_j\,. \tag{2.2}$$

Let γ_i be arc–length parametrized, i.e., $\gamma_i(s)$, $0 \le s \le L_i$, where L_i is the total arc length of γ_i. Let $\widehat{n}_i$ be the unit inner normal to γ_i, and $\kappa_i(y)$, $y \in \gamma_i$, its signed curvature. Around each narrow enough tubular neighbourhood of γ_i, we set a principal coordinate system as follows. See [10] for more details.

If $d(x, \gamma_i)$ denotes the usual signed distance function which is positive inside Ω_i and negative outside Ω_i, we set

$$N_{\delta,i} \stackrel{\text{def}}{=} \{x \in \Omega : |d(x,\gamma_i)| < \delta\}\ .$$

For δ small enough, the change of coordinate map

$$\Sigma_i : \gamma_i \times (-\delta, \delta) \longrightarrow N_{\delta,i}$$

defined by $\Sigma_i(s,d) = \gamma_i(s) + d\,\widehat{n}_i(s)\,,\ 0 \le s < L_i\,,\ -\delta < d < \delta$, is a diffeomorphism. Moreover the Jacobian of $\Sigma_i(s,d)$ is given by

$$J_{\Sigma_i}(s,d) = (1 - d\,\kappa_i(s)),\ 0 \le s \le L_i\,,\ i = 1,\dots,p\ .$$

Note that for δ small enough $J_{\Sigma_i} > 0$ and $\Sigma_i(s,0) = \gamma_i(s)$.

Let us also set $\widetilde{k}_1(s,d) = k_1(\Sigma(s,d))$, $\widetilde{k}_2(s,d) = k_2(\Sigma(s,d))$ and regarding (1.1) put

$$k(x) \stackrel{\text{def}}{=} [k_1(x)k_2(x)]^{1/2}\ .$$

We now state our main result.

Theorem 2.1 *Suppose that γ_i, $i = 1,\dots,p$, is a ν_i–level curve of k, i.e.,*

$$k(x) = \nu_i\,, \quad \textit{for}\ \ x \in \gamma_i\,,\ i = 1,\dots,p\,.$$

Let $\widetilde{k}(s,d) = k(\Sigma(s,d))$ and $\Lambda_i(s,d) = \widetilde{k}(s,d)J_{\Sigma_i}(s,d)$, $(s,d) \in [0,L_i] \times (-\delta,\delta)$. Suppose that

$$\Lambda_i(s,d) > \Lambda_i(s,0) = \nu_i\,, \quad \textit{for}\ \ d \in (-\delta,\delta),\ d \ne 0,\ i = 1,\dots,p \tag{$H1$}$$

Then there is a family $\{v_\varepsilon\}_{0<\varepsilon\le\varepsilon_0}$, with ε_0 small, of stationary solutions of (1.1) such that for each $\varepsilon \in (0,\varepsilon_0)$:

(2.1.i) *$v_\varepsilon \in C^{2,\sigma}(\overline{\Omega})$, $0 < \sigma < 1$ and $\alpha < v_\varepsilon(x) < \beta$, $\forall\, x \in \overline{\Omega}$.*

(2.1.ii) $\|v_\varepsilon - v_0\|_{L^1(\Omega)} \longrightarrow 0$, *as* $\varepsilon \to 0$.

(2.1.iii) *For each* $\lambda \in (0, (\beta-\alpha)/2)$ *and* $\Omega_\varepsilon^\lambda = \{x \in \Omega : \alpha + \lambda < v_\varepsilon(x) < \beta - \lambda\}$ *it holds that* $\left|\Omega_\varepsilon^\lambda\right| \longrightarrow 0$, *as* $\varepsilon \to 0$, *where* $|\cdot|$ *is the 2-dimensional Lebesgue measure.*

(2.1.iv) v_ε *is a stable stationary solution of (1.1).*

Next we give necessary conditions for (H1) to be satisfied.

Lemma 2.1 *Suppose that each* γ_i, $i = 1, \dots, p$, *satisfies*

$$\begin{aligned} \frac{\partial \widetilde{k}(s,0)}{\partial d} &= \nu_i\, \kappa_i(s), \quad 0 \le s \le L_i \\ \frac{\partial^2 \widetilde{k}(s,0)}{\partial d^2} &> 2\, \nu_i\, \kappa_i^2(s), \quad 0 \le s \le L_i \end{aligned} \tag{$H2$}$$

Then hypothesis (H1) is satisfied.

Remark. An example of a function which satisfies (H2) can be easily constructed. For instance, let $\gamma_i \subset \Omega \subset \mathbb{R}^2$ with $\gamma_i = \{x \in \Omega :\ \|x\| = 1\}$. Then the function $\tilde{k}(s,t) = 1 + t + 2\,t^2$ with $t \in (-\delta, \delta)$, δ small, and $s \in [0, 2\pi)$, satisfies (H2) with $\nu_i = 1$.

Remark. The hypothesis on γ_i could have been slightly more general. It would suffice to have γ_i, $i = 1, \dots, p$, a subset of a ν_i^1–level set of k_1 and also a subset of a ν_i^2–level set of k_2. Then γ_i would be a ν_i–level curve of $k(x)$ where $\nu_i = \left(\nu_i^1 \nu_i^2\right)^{1/2}$.

Remark. The two conditions in (H2) dictate the behavior of k in a neighbourhood of the limiting transition-phase curve γ_i. The first condition requires that $k(x)$ increase or decrease as x crosses γ_i along the direction of $\widehat{n}_i$ according to the sign of the curvature κ_i, its slope being proportional to $|\kappa_i|$ at the crossing-point, provided $\nu_i \neq 0$. The second one, for a fixed s relates the concavity of $k(\Sigma(s,d))$ with the curvature $\kappa_i(s)$ of γ_i at s. Note that $\frac{\partial \widetilde{k}}{\partial d}$ changes sign whenever the curvature of γ_i does so.

Each limiting phase-transition curve γ_i acts through (H2) as a barrier which prevents a, say, diffusing substance whose initial concentration evolves in time according to (1.1), to spread homogeneously in space and eventually settling down in a uniform concentration.

Actually this would be the case if for instance we had Ω convex, $k_1(x) \equiv$ const., $k_2(x) \equiv$ const. and $g_\varepsilon(x) \equiv 0$. See [17, 3] for this matter. An example where this occurs, in despite of the fact that the diffusion function k_2 is not constant, can be found in [22].

Remark. The stability referred to in (2.1.iv) should be understood in the following sense: a stationary solution $v(x)$ of (1.1) is said to be stable in the $H^1(\Omega)$-norm, say, if for any $\mu > 0$ there exists $\delta > 0$ such that $T(t)\psi$ exists for all $t > 0$ (here $T(t)$ denotes the nonlinear semigroup generated by (1.1)) and

$$\|T(t)\psi - v\|_{H^1(\Omega)} < \mu\,, \quad 0 < t < \infty\,,$$

for any $\psi \in H^1(\Omega)$ which satisfies $\|\psi - v\|_{H^1(\Omega)} < \delta$. Here $T(0)\psi = \psi$. If in addition

$$\lim_{t\to\infty} \|T(t)\psi - v\|_{H^1(\Omega)} = 0$$

then v is said to be strongly stable.

Remark. It might be worthwhile to mention that each limiting phase-transition curve γ_i, $i = 1, \dots, p$, is a ν_i–level curve of $\widetilde{k}$ and a curve of minima of $\Lambda_i(s,d) = \widetilde{k}(s,d)(1 - d\,\kappa_i(s))$. Of course if γ_i contains a segment of straight line ℓ_i, say, then $\Lambda_i(s,d) = \widetilde{k}(s,d)$ on ℓ_i and the two notions coincide there.

3 Preliminaries on BV-functions

Before proving the main result we recall some notation on measures, and results on functions of bounded variation. The reader is referred to [26, 4, 9] for further background. The Lebesgue measure in $\mathbb{R}^n$ is denoted by $|\cdot|$ and the m–dimensional Hausdorff measure by $\mathcal{H}^m$, $m \in [0, 2]$.

If μ is a Borel measure on Ω with values in $[0, +\infty[$ or in $\mathbb{R}^k$, $k \geq 1$, its total variations is denoted by $|\mu|$. If F is a Borel subset of Ω, the measure $\mu \llcorner F$ is defined by $(\mu \llcorner F)(B) = \mu(B \cap F)$ for any Borel set $B \subset \Omega$. For every μ-integrable function f, the measure $f\mu$ is defined by $(f\mu)(B) = \int_B f d\mu$ for every Borel set $B \subset \Omega$. We shall use also the notation $\int_B f\mu$.

The space $BV(\Omega)$ of functions of bounded variation in Ω is defined as the set of all functions $u \in L^1(\Omega)$ whose distributional gradient Du is a Radon measure with bounded total variation in Ω. We denote by $BV(\Omega; \{\alpha, \beta\})$ the class of all $u \in BV(\Omega)$ which take values α, β only.

The essential boundary of a set $E \subset \mathbb{R}^n$ is the set $\partial_* E$ of all points in Ω where E has neither density 1 nor density 0. A set $E \subset \Omega$ has finite perimeter in Ω if its characteristic function χ_E belongs to $BV(\Omega)$. In this case $\partial_* E$ is rectifiable, and we may endow it with a measure theoretic normal ν_E so that the measure derivative $D\chi_E$ is represented as

$$D\chi_E(B) = \int\limits_{B\cap\partial_* E} \nu_E d\mathcal{H}^1 \quad \text{for every Borel set} \quad B \subset \Omega\,.$$

The following result is a simple generalization of the total variation of a BV function, and we omit the proof.

Lemma 3.1 *Let $A \subset \Omega$ be an open set and $v \in BV(\Omega)$. Let $\{v_j\}$ be a sequence of functions in $BV(\Omega)$ converging to v in $L^1(\Omega)$. Then*

$$\left|\frac{\partial v}{\partial x_i}\right|(A) \leq \liminf_{j\to+\infty}\left|\frac{\partial v_j}{\partial x_i}\right|(A) \quad \textit{for} \quad i=1,2.$$

The next lemma plays an important role in the proof of the main results. For simplicity, since we consider γ_i one of the curves introduced in Section 2, we drop the index i in the statement and in the proof.

Lemma 3.2 *Let $v \in BV(\Omega)$ and set $\mu = Dv$. Let Σ, δ, N_δ, J_Σ be defined as in Section 2. Let $\tilde{v} = v(\Sigma) : \gamma \times (-\delta,\delta) \to \mathbb{R}$ for δ small enough, and $\tilde{N}_\delta = \Sigma^{-1}(N_\delta)$. Then, for any Borel set $B \subset N_\delta$, we have*

$$|\mu|(B) \geq \left((|\widetilde{\mu}_1|(\widetilde{B}))^2 + (|\widetilde{\mu}_2|(\widetilde{B}))^2\right)^{1/2},$$

where $\tilde{B} = \Sigma^{-1}(B)$, $\tilde{\mu}_1 = \dfrac{\partial\tilde{v}}{\partial s}$, and $\tilde{\mu}_2 = J_\Sigma \dfrac{\partial \tilde{v}}{\partial d}$.

Proof: Let v_j be a sequence of functions of class $C^\infty(N_\delta)$ converging in $L^1(N_\delta)$ to v and such that $|\bigtriangledown v_j|(N_\delta) \to |Dv|(N_\delta)$ as $j \to +\infty$. Following the notation of Section 2, a direct computation yields

$$\left|M_\Sigma^{-1}\nabla\widetilde{v}_j\right| = \left[\left(\frac{\partial\widetilde{v}_j}{\partial s}\right)^2 J_\Sigma^{-2} + \left(\frac{\partial\widetilde{v}_j}{\partial d}\right)^2\right]^{1/2},$$

where M_Σ denotes the Jacobian matrix corresponding to the coordinate map Σ, and $\tilde{v}_j = v_j(\Sigma)$. If $B \subset N_\delta$ is a Borel set, we then have

$$\begin{aligned}
\int_B |Dv| &= \lim_{j\to+\infty}\int_B |\nabla v_j| = \lim_{j\to+\infty}\int_{\widetilde{B}} J_\Sigma \left|M_\Sigma^{-1}\nabla\,\widetilde{v}_j\right| \\
&= \lim_{j\to+\infty}\int_{\widetilde{B}}\left[\left(\frac{\partial\widetilde{v}_j}{\partial s}\right)^2 + J_\Sigma^2\left(\frac{\partial\widetilde{v}_j}{\partial d}\right)^2\right]^{1/2} \\
&\geq \lim_{j\to+\infty}\left[\left(\int_{\widetilde{B}}\left|\frac{\partial\widetilde{v}_j}{\partial s}\right|\right)^2 + \left(\int_{\widetilde{B}} J_\Sigma\left|\frac{\partial\widetilde{v}_j}{\partial d}\right|\right)^2\right]^{1/2}
\end{aligned}$$

where the last inequality follows by the inequality $\int |z| \geq |\int z|$, $z \in \mathbb{R}^2$. Then, using Lemma 3.1, we conclude this proof.

4 Local Minimizers Via Γ–convergence

A stationary solution of (1.1) satisfies the boundary value problem

$$\begin{aligned}
\varepsilon^2 \operatorname{div}\ [k_1(x)\nabla v] + k_2(x) f_\varepsilon(x,v) = 0\,, &\quad x \in \Omega \\
\nabla v(x)\cdot\widehat{n}(x) = 0\,, &\quad \text{for} \quad x \in \partial\Omega
\end{aligned} \tag{4.1}$$

where $f_\varepsilon(x,v) = (v-\alpha)(\beta-v)(v-(\theta+g_\varepsilon(x))$ and, for convenience, we set $\theta = (\alpha+\beta)/2$.

If $F_\varepsilon(x,v) = \int_\theta^v f_\varepsilon(x,\xi)d\xi$ then $F_\varepsilon(x,v) = F^0(v) - g_\varepsilon(x)F^1(v)$ where

$$F^0(v) = \int_\theta^v (\xi-\alpha)(\beta-\xi)(\xi-\theta)d\xi \quad\text{and}\quad F^1(v) = \int_\theta^v (\xi-\alpha)(\beta-\xi)d\xi\,.$$

Next we define a family of functionals $E_\varepsilon : L^1(\Omega) \to \mathbb{R}$,

$$E_\varepsilon(v) = \begin{cases} \displaystyle\int_\Omega \big\{\frac{\varepsilon\, k_1(x)}{2}\,|\nabla v|^2 + \varepsilon^{-1}k_2(x)\left[F(\alpha)-F^0(v)\right] \\ \quad +\varepsilon^{-1}g_\varepsilon(x)k_2(x)F^1(v)\big\}\,dx\,, \quad \text{if } v\in H^1(\Omega) \\ \infty\,, \quad \text{otherwise.} \end{cases} \tag{4.2}$$

The term $\varepsilon^{-1}k_2(x)F(\alpha)$ has been artificially added since it does not affect E_ε, as long as existence of minimizers is concerned and what is more important, the potential function

$$\widetilde{F}(v) \stackrel{\text{def}}{=} F(\alpha) - F^0(v)$$

satisfies

$$\begin{gathered} \widetilde{F}(\alpha) = \widetilde{F}(\beta) = 0\,, \quad \widetilde{F}\in C^2(\mathbb{R}) \\ \widetilde{F}(v) > 0 \ \text{ for any } \ v\in\mathbb{R}\,,\ v\neq\alpha\,,\ v\neq\beta \\ \widetilde{F}'(\alpha) = \widetilde{F}'(\beta) = 0\,,\ \widetilde{F}''(\alpha) > 0\,,\ \widetilde{F}''(\beta) > 0\,. \end{gathered} \tag{4.3}$$

It is easy to see that any local minimizer v_ε of E_ε will be a solution to (H1) and, by regularity, $v_\varepsilon \in C^{2,\nu}(\Omega)$, $0<\nu<1$.

Therefore, it is our aim now to find a family of local minimizers of E_ε. To that end we will need the concept of Γ–convergence and of a local isolated minimizer of E_ε. See for example [16].

Definition A family $\{E_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ of real-extended functionals defined in $L^1(\Omega)$ is said to Γ–converge, as $\varepsilon\to 0$, to a functional E_0, at v and we write

$$\Gamma(L^1(\Omega)^-) - \lim_{\varepsilon\to 0} E_\varepsilon(v) = E_0(v)$$

if

- For each $v\in L^1(\Omega)$ and for any sequence $\{v_\varepsilon\}$ in $L^1(\Omega)$ such that $v_\varepsilon \to v$ in $L^1(\Omega)$, as $\varepsilon\to 0$, it holds that $E_0(v) \leq \liminf\limits_{\varepsilon\to 0} E_\varepsilon(v_\varepsilon)$.
- For each $v\in L^1(\Omega)$ there is a sequence $\{w_\varepsilon\}$ in $L^1(\Omega)$ such that $w_\varepsilon \to v$ in $L^1(\Omega)$, as $\varepsilon\to 0$ and also $E_0(v) \geq \limsup\limits_{\varepsilon\to 0} E_\varepsilon(w_\varepsilon)$.

Definition. We say that $v_0 \in L^1(\Omega)$ is an L^1-local minimizer of E_0 if there is $\rho > 0$ such that

$$E_0(v_0) \leq E_0(v) \quad \text{whenever} \quad 0 < \|v - v_0\|_{L^1(\Omega)} < \rho\,.$$

Moreover, if $E_0(v_0) < E_0(v)$ for $0 < \|v - v_0\|_{L^1(\Omega)} < \rho$, then v_0 is called an isolated L^1-local minimizer of E_0.

De Giorgi's Γ–convergence provides, for equicoercive functionals, the convergence of global minimizers to a global minimizer of the Γ–limit. Concerning convergence of local minimizers, the following theorem extends an observation made by Kohn and Sternberg in [K,S]; the proof is the same as the one in [15].

Theorem 4.1 *Suppose that a family of extended–real functionals $\{E_\varepsilon\}$, Γ–converges, as $\varepsilon \to 0$, to a extended–real functional E_0 and the following hypotheses are satisfied:*

(4.1.i) *Any sequence $\{u_\varepsilon\}_{\varepsilon>0}$ such that $E_\varepsilon(u_\varepsilon) \leq const. < \infty$, $\varepsilon > 0$, is compact in $L^1(\Omega)$.*

(4.1.ii) *There exists an isolated L^1-local minimizer v_0 of E_0.*

Then there exists $\varepsilon_0 > 0$ and a family $\{v_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ such that:

- *v_ε is an L_1-local minimizer of E_ε*
- $\|v_\varepsilon - v_0\|_{L^1(\Omega)} \to 0$, *as* $\varepsilon \to 0$.

Let us consider the family of functionals defined by (4.2). For the case in which $g_\varepsilon(x) \equiv 0$ and $\widetilde{F}$ satisfies (4.3), the Γ–convergence of this type of functionals has been treated in [25, 19, 20].

In (4.2) the presence of the term $\varepsilon^{-1}g_\varepsilon(x)k_2(x)F^1(v)$ adds no additional difficulty because its smoothness makes of it a continuous perturbation with respect to L^1–convergence. Moreover by virtue of (1.2) the perturbation term $\int_\Omega \varepsilon^{-1}g_\varepsilon(x)k_2(x)F^1(v_\varepsilon)dx$ vanishes when one takes the Γ–limit. Hence the above results can be evoked thus yielding the following theorem.

Theorem 4.2 *Consider the family of functionals given by (4.2). Then*

$$\Gamma(L^1(\Omega)^-) - \lim_{\varepsilon\to 0} E_\varepsilon(v) = E_0(v)$$

where

$$E_0(v) = \begin{cases} C_0 \displaystyle\int_\Omega k\left|D\chi_{\{v=\beta\}}\right|, & \textit{if } v \in BV(\Omega,\{\alpha,\beta\})\,, \\ & 0 < |\{v=\beta\}| < |\Omega|; \\ \infty & \textit{otherwise.} \end{cases} \tag{4.4}$$

and $C_0 = (\beta - \alpha)\int_\alpha^\beta (\widetilde{F}(t))^{1/2}dt$.

Our next goal is to apply Theorem 4.1 to $\{E_\varepsilon\}$, $0<\varepsilon\le\varepsilon_0$ and to E_0, as defined in (4.2) and (4.4). We only have to worry about (4.1.ii) since (4.1.i) has essentially been proved in [15, 20]. Their proof can be easily adapted to fit our case.

As for (4.1.ii) it has been proved in [23] and [24] for the case of just one limiting transition–phase curve. We give here a neater and more geometric proof for this case and then consider the case of p limiting transition–phase curves.

For the single curve case the proof presented in the references above uses the approximation result mentioned at the beginning of the proof of Lemma 3.2. This approach is somehow cumbersome since it has to deal with sequences and subsequences throughout the proof and unnatural since it avoids to work with the geometry of BV–spaces which is the setting our problem is naturally casted into. This approximation approach can be avoided by resorting to Lemma 3.1.

In the next theorem we will be dealing with the single curve case, i.e., $p=1$, and therefore we set for simplicity $\gamma=\gamma_1$. Thus $N_\delta=N_\delta(\gamma)$ will denote its tubular neighbourhood, Σ the corresponding coordinate map and we drop the subindex 1 in all other definitions. The proof is a modification of a similar proof given in [15].

Theorem 4.3 *With the notation of Section 2, suppose that $\gamma\subset\Omega$ is a simple closed ν–level curve of k and that* (H1) *is satisfied with $\Omega=\Omega_1\cup\gamma\cup\Omega_2$. Then*

$$v_0(x)=\alpha\,\chi_{\Omega_1}(x)+\beta\,\chi_{\Omega_2}(x)\,,\quad x\in\Omega$$

is an L^1-local isolated minimizer of E_0.

Proof: It suffices to prove that, if $v\in BV(\Omega;\{\alpha,\beta\})$ and $0<\|v-v_0\|_{L^1(N_\delta)}<\rho$ for a suitable $\rho>0$, then

$$\int_{N_\delta(\gamma)}k|Dv|>\int_{N_\delta(\gamma)}k|Dv_0|.$$

Let us start by computing $E_0(v_0)$. By the coarea formula (see [6]) we obtain

$$\begin{aligned}E_0(v_0)&=\int_{N_\delta(\gamma)}k|Dv_0|=\int_\alpha^\beta\int_{N_\delta\cap\partial_*\{v_0>\xi\}}kd\mathcal{H}^1d\xi\\&=(\beta-\alpha)\int_\gamma kd\mathcal{H}^1=(\beta-\alpha)\nu\,L\,,\end{aligned}$$

where L is the total arc–length of γ. For a fixed $d\in(-\delta,\delta)$ define

$$\ell_d=\{(s,d)\,,\,0<s<L\}\ .$$

Then the trace of $\widetilde{v}(\cdot,d)$ is well-defined on ℓ_d, a.e. in $(-\delta,\delta)$; because each ℓ_d is C^1 (actually Lipschitz would suffice).

Suppose that

i) $\widetilde{v}=\widetilde{v}_0$, along $\ell_{\widetilde{d}}\cup\ell_{-\widetilde{d}}$, in the sense of traces, for some $\widetilde{d}\in(\delta/2,\delta)$.

Recall that $\Lambda(s,d)=\widetilde{k}(s,d)J_\Sigma(s,d)$. We say that v is an admissible function if $\|v-v_0\|_{L^1(N_\delta)}>0$ and $v\in BV(N_\delta(\gamma);\{\alpha,\beta\})$. For any such function v and with the use of Lemma 3.2 we have:

$$
\begin{aligned}
E_0(v) &= \int_{N_\delta} k|Dv| \\
&\geq \int_{\widetilde{N}_\delta} \widetilde{k}\left(|\widetilde{\mu}_1|^2+|\widetilde{\mu}_2|^2\right)^{1/2} \geq \int_{\widetilde{N}_\delta} \widetilde{k}|\widetilde{\mu}_2| \\
&\geq \nu\int_0^L\int_{-\delta}^{\delta}\left|\frac{\partial\widetilde{v}}{\partial d}\right| \text{ (by } (H_1) \text{ and definition of } \widetilde{\mu}_2) \\
&\geq \nu(\beta-\alpha)L\,,
\end{aligned}
$$

since i) implies that $\int_{-\delta}^{\delta}\left|\frac{\partial\widetilde{v}}{\partial d}\right|\geq(\beta-\alpha)$ for any $s\in[a,b]$.

We claim that $E_0(v_0)<E_0(v)$. If this were not the case then for any admissible function v, the coarea formula would yield

$$
\begin{aligned}
(\beta-\alpha)\nu L &= \int_{N_\delta} k|Dv| = \int_{-\infty}^{\infty}\left(\int_{N_\delta\cap\partial\{v>\xi\}} k|Dv|\right)d\xi \\
&= \int_{-\infty}^{\infty}\left(\int_{N_\delta\cap\partial_*\{v>\xi\}} kd\mathcal{H}^1\right)d\xi \\
&= \int_{-\infty}^{\infty}\left(\int_{\widetilde{N}_\delta\cap\partial_*\{\tilde{v}>\xi\}} \Lambda(s,t)d\mathcal{H}^1\right)d\xi \\
&= (\beta-\alpha)\int_{\widetilde{N}_\delta\cap\partial_*\{\widetilde{v}=\beta\}\cap\partial_*\{\widetilde{v}=\alpha\}} \Lambda(s,t)d\mathcal{H}^1 \qquad (4.5)
\end{aligned}
$$

To deduce (4.5) we use that

$$|D\widetilde{v}|=\mathcal{H}^1\llcorner\left(\partial_*\{\widetilde{v}=\alpha\}\cap\partial_*\{\widetilde{v}=\beta\}\right)$$

taking into account that $\partial\{\widetilde{v}=\alpha\}\cap\partial\{\widetilde{v}=\beta\}$ is a set of finite perimeter in $\widetilde{N}_\delta$.

Also our hypothesis implies

$$\int_{\widetilde{N}_\delta}\widetilde{k}\left(|\widetilde{\mu}_1|^2+|\widetilde{\mu}_2|^2\right)^{1/2}=\int_{\widetilde{N}_\delta}\widetilde{k}|\widetilde{\mu}_2|.$$

But the above equality holds if and only if $|\widetilde{\mu}_1|\equiv 0$ on $\widetilde{N}_\delta$, what is to say that $\widetilde{v}(\cdot,d)$ is $\mathcal{H}^1$-a.e. constant along each ℓ_d, for a.e. $d\in(-\delta,\delta)$. Hence,

$$\widetilde{N}_\delta\cap\partial_*\{\widetilde{v}=\alpha\}\cap\partial_*\{\widetilde{v}=\beta\}=\cup_{j=1}^m\ell_{d_j}\,,$$

for some $m \in \mathbb{N}$ and $-\delta < d_j < \delta$, $j = 1, \dots, m$. Note that

$$\widetilde{N}_\delta \cap \partial\{\widetilde{v}_0 = \alpha\} \cap \partial\{\widetilde{v}_0 = \beta\} = \ell_0 \,.$$

Therefore by virtue of (H1), (4.5) holds if and only if $m = 1$ and $d_1 = 0$, i.e., $\widetilde{v} = \widetilde{v}_0$ a.e. in $\widetilde{N}_\delta$. This is a contradiction since v is an admissible function and as such $\|\widetilde{v} - \widetilde{v}_0\|_{L^1(\widetilde{N}_\delta)} > 0$.

This takes care of our theorem in the case in which i-) holds. Now if i) does not hold then one of the following cases would occur, in the sense of traces of BV-functions:

ii) $\widetilde{v}$ is not const. $\mathcal{H}^1$–a.e. along $\ell_{\widetilde{d}}$, for a.e. $\widetilde{d} \in (\delta/2\,,\,\delta)$

iii) $\widetilde{v}$ is not const. $\mathcal{H}^1$–a.e. along $\ell_{-\widetilde{d}}$, for a.e. $\widetilde{d} \in (\delta/2\,,\,\delta)$

iv) $\widetilde{v} \equiv \alpha\; \mathcal{H}^1$-a.e. along $\ell_{\widetilde{d}}$ and $\widetilde{v} \equiv \beta\,, \mathcal{H}^1$-a.e. along $\ell_{-\widetilde{d}}$, for a. e. $\widetilde{d} \in (\delta/2, \delta)$.

Following ideas set forth in [15], we define a set $\Delta \subset (0, \delta)$,

$$\Delta \overset{\text{def}}{=} \left\{ \begin{array}{l} \widetilde{d} \in (0,\delta) : \displaystyle\int_0^L \left\{ |\widetilde{v} - \widetilde{v}_0|(s,\widetilde{d}) J_\Sigma(s,\widetilde{d}) + |\widetilde{v} - \widetilde{v}_0|(s,-\widetilde{d}) J_\Sigma(s,-\widetilde{d}) \right\} ds \\ \qquad > 4\rho/\delta, \text{ where } \displaystyle\int_{\widetilde{N}_\delta} |\widetilde{v} - \widetilde{v}_0| J_\Sigma < \rho \end{array} \right\}$$

Hence $|\Delta| < \delta/4$. Now if iv) does not hold, then by choosing $\rho < \delta L(\beta - \alpha)/2$ we obtain

$$\int_0^L \left\{ |\widetilde{v} - \widetilde{v}_0|(s,\widetilde{d}) J_\Sigma(s,\widetilde{d}) + |\widetilde{v} - \widetilde{v}_0|(s,-\widetilde{d}) J_\Sigma(s,-\widetilde{d}) \right\} = 2\,L\,(\beta - \alpha) > (4\rho/\delta)\,,$$

which by definition implies that $\widetilde{d} \in \Delta$. But then for a.e. $\widetilde{d} \in ((\delta/2, \delta)\backslash\Delta)$, either ii) or iii) holds.

For any admissible v, Lemma 3.1 allows us to conclude that

$$\begin{aligned} E_0(v) &= \int_{N_\delta} k\,|Dv| \ge \int_{N_\delta \backslash N_{\delta/2}} k\,|Dv| + \int_{N_{\delta/2}} k\,|Dv| \\ &= \int_{\widetilde{N}_\delta \backslash \widetilde{N}_{\delta/2}} \widetilde{k}\, \left(|\widetilde{\mu}_1|^2 + |\widetilde{\mu}_2|^2\right)^{1/2} + \int_{\widetilde{N}_{\delta/2}} \widetilde{k}\, \left(|\widetilde{\mu}_1|^2 + |\widetilde{\mu}_2|^2\right)^{1/2} \\ &\ge \int_{\widetilde{N}_\delta \backslash \widetilde{N}_{\delta/2}} \widetilde{k}\, \left|\frac{\partial \widetilde{v}}{\partial s}\right| + \int_{\widetilde{N}_{\delta/2}} \Lambda\, \left|\frac{\partial \widetilde{v}}{\partial d}\right| \\ &= I_1 + I_2 \end{aligned}$$

where I_1 and I_2 denote respectively the first and the second integrals in the last term of the above inequalities.

If $k_m = \min_{x\in\overline{\Omega}} k(x)$ then by virtue of ii) and iii), for a.e. $\widetilde{d} \in (\delta/2, \delta)\backslash\Delta$ it holds that

$$\begin{aligned}
&\int_0^L \left\{\widetilde{k}(s,d)\,|\widetilde{\mu}_1|(s,\widetilde{d}) + \widetilde{k}(s,-\widetilde{d})\,|\widetilde{\mu}_1|(s,-\widetilde{d})\right\} \\
&\quad\geq \quad k_m\left\{\text{ess } V_0^L\left[\widetilde{v}(\cdot,\widetilde{d})\right] + \text{ess } V_0^L\left[\widetilde{v}(\cdot,-\widetilde{d})\right]\right\} \\
&\quad\geq \quad k_m(\beta-\alpha),
\end{aligned}$$

where ess $V_0^L\left[\widetilde{v}(\cdot,\widetilde{d})\right]$ stands for the essential variation of $\widetilde{v}(\cdot,\widetilde{d})$ on $[0, L]$. Note that since $\tilde{v} \in BV(\tilde{N}_\delta, \{\alpha,\beta\})$, the function

$$\tilde{d} \longrightarrow \text{ ess } V_0^L[\tilde{v}(\cdot,\tilde{d})] = \left|\frac{\partial\tilde{v}(\cdot,\tilde{d})}{\partial s}\right|(0,L)$$

is integrable in $(\delta/2, \delta)$. See [26, 4]. Then by integrating over $(\delta/2, \delta)\backslash\Delta$,

$$\begin{aligned}
I_1 \quad &\geq \quad k_m\int_{\delta/2}^{\delta}\left\{\text{ess } V_0^L\left[\widetilde{v}(\cdot,\widetilde{d})\right] + \text{ess } V_0^L\left[\widetilde{v}(\cdot,-\widetilde{d})\right]\right\} \\
&\geq \quad (\delta/4)k_m(\beta-\alpha).
\end{aligned}$$

In order to obtain a lower estimate for I_2 we begin by remarking that since

$$|(0,\delta/2)\backslash\Delta| \geq \delta/4$$

and $\widetilde{v}$ is an admissible function then there is $\overline{d} \in (0, \delta/2)\backslash\Delta$ such that $(s, \overline{d})$ and $(s, -\overline{d})$ are points of approximate continuity of $\widetilde{v}(s,\widetilde{d})$, for a.e. $s \in [0, L]$. Also by the definition of $\widetilde{v}_0$,

$$|\widetilde{v}(s,\widetilde{d}) - \widetilde{v}(s,-\widetilde{d})| \geq (\beta-\alpha) - \left\{|\widetilde{v}_0 - \widetilde{v}|(s,\widetilde{d}) + |\widetilde{v}_0 - \widetilde{v}|(s,-\widetilde{d})\right\}$$

for any $(s,\widetilde{d}) \in \widetilde{N}_\delta$ such that $\widetilde{v}$ is approximately continuous at $(s,\widetilde{d})$. Hence

$$\begin{aligned}
I_2 \quad &= \quad \int_{\widetilde{N}_{\delta/2}} \Lambda\left|\frac{\partial\widetilde{v}}{\partial d}\right| \geq \nu\int_0^L\int_{-\delta/2}^{\delta/2}\left|\frac{\partial\widetilde{v}}{\partial d}\right| \\
&= \quad \nu\int_0^L \text{ess } V_{-\delta/2}^{\delta/2}\left[\widetilde{v}(s,\cdot)\right]ds \geq \nu\int_0^L |\widetilde{v}(s,\overline{d}) - \widetilde{v}(s,-\overline{d})|ds \\
&\geq \quad \nu\int_0^L\left\{(\beta-\alpha) - \left[|\widetilde{v}-\widetilde{v}_0|(s,\overline{d}) + |\widetilde{v}-\widetilde{v}_0|(s,-\overline{d})\right]\right\}ds \\
&\geq \quad \left\{\nu\, L(\beta-\alpha) - \frac{4\,\rho\,\nu}{\delta\, J_m(-\delta,\delta)}\right\},
\end{aligned}$$

where

$$J_m(-\delta,\delta) = \min_{-\delta\leq d\leq\delta}\left\{\min_{0\leq s\leq L} J_\Sigma(s,d)\,,\ \min_{0\leq s\leq L} J_\Sigma(s,-d)\right\}.$$

These estimates finally yield

$$\begin{aligned} E_0(v) \geq I_1 + I_2 &\geq \left[\frac{\delta}{4}\, k_m(\beta-\alpha) + \nu\, L(\beta-\alpha) - \frac{4\,\rho\,\nu}{\delta\, J_m(-\delta,\delta)}\right] \\ &> \nu(\beta-\alpha)L = E_0(v_0)\,, \end{aligned}$$

as long as we take

$$\rho < \min\left\{ \frac{k_m(\beta-\alpha)\delta^2 J_m(-\delta,\delta)}{16\nu}\,,\, \frac{\delta\, L(\beta-\alpha)}{2} \right\} \tag{4.6}$$

Now our claim follows by extending v_0 to be constant on each connected component of $\Omega\backslash\gamma$ and observing that $|Dv_0|\,(\Omega\backslash\gamma) = 0$. □

The next task is to generalize Theorem 3.1 to the case of p limiting phase-transition curves. Recall the notation set forth in Section 2 where $N_{\delta,i}$ is a δ-tubular neighbourhood around each limiting phase-transition curve γ_i, $i = 1, 2, \ldots, p$.

Corollary 4.1 *Let γ_i, $i = 1, \ldots, p$, Ω_i, k be as in Section 2. If (H1) is satisfied, then the function v_0 is a L^1–local minimizer of E_0.*

Proof: For any $v \in BV(\Omega, \{\alpha, \beta\})$, $||v - v_0||_{L^1(\Omega)} < \rho$, where ρ satisfies (4.6) with $\nu = \max\{\nu_i, i = 1, ..., p\}$, let

$$\Omega_\alpha \stackrel{\text{def}}{=} \{x \in \Omega : v(x) = \alpha\}\,, \quad \Omega_\beta \stackrel{\text{def}}{=} \{x \in \Omega : v(x) = \beta\}$$

and consider the sets Ω^0_α and Ω^0_β defined by (2.2).

Then the set $(\Omega_\alpha \cup \Omega_\beta) \cap \Omega$ has finite perimeter. By applying Theorem 4.3 to each γ_i, $i = 1, \ldots, p$ we obtain, using the coarea formula

$$\begin{aligned} E_0(v) &= \int_\Omega k|Dv| = (\beta-\alpha) \int_{\partial_*\Omega_\alpha \cap \partial_*\Omega_\beta \cap \Omega} k\, d\mathcal{H}^1 \\ &\geq (\beta-\alpha) \sum_{i=1}^{p} \int_{\partial_*\Omega_\alpha \cap \partial_*\Omega_\beta \ \ capN_{\delta,i}} k\, d\,\mathcal{H}^1 \\ &> (\beta-\alpha) \sum_{i=1}^{p} \int_{\partial_*\Omega^0_\alpha \cap \partial_*\Omega^0_\alpha \cap N_{\delta,i}} k\, d\,\mathcal{H}^1 \\ &= (\beta-\alpha) \int_{\bigcup_{i=1}^p \gamma_i} k\, d\,\mathcal{H}^1 = \int_\Omega k\,|Dv_0| = E_0(v_0)\,. \end{aligned}$$

5 Proof of Theorem 2.1

In this section we show how the previous sequence of lemmas can be used to accomplish the proof of Theorem 2.1.

In view of Corollary 4.1 and previous remarks, Theorem 4.1 yields a family $\{v_\varepsilon\}_{0<\varepsilon\leq\varepsilon_0}$ of L^1–local minimizers of E_ε, where E_ε is given by 4.2. Clearly any such minimizer v_ε is a weak solution (H^1-sense) of (4.1) and regularity theory implies $v_\varepsilon \in C^{2,\sigma}(\overline{\Omega})$, $0<\sigma<1$. An application of the maximum principle then gives $\alpha < v_\varepsilon(x) < \beta$, $\forall\, x \in \overline{\Omega}$ and (2.1.i) is proved.

As for (2.1.ii) it is obtained from Theorem 4.1. In order to prove (2.1.iii) we examine the potential function $\widetilde{F}(v) = F(\alpha) - F^0(v)$.

Suppose by contradiction that there is a sequence ε_j , $\varepsilon_j \to 0$ as $j\to\infty$, and $\tau > 0$ such that

$$\left|\Omega^\lambda_{\varepsilon_j}\right| \geq \tau > 0\,,$$

for a fixed λ and $\forall\, j$. Since $\|v_{\varepsilon_j} - v_0\|_{L^1(\Omega)} \stackrel{j\to\infty}{\longrightarrow} 0$ it follows that $\{\varepsilon_j\}$ has a subsequence, still denoted by $\{\varepsilon_j\}$ such that $v_{\varepsilon_j} \to v_0$, a.e. in Ω. Thus, (4.3) allows us to invoke the Lebesgue convergence theorem to conclude that

$$\lim_{j\to 0} \int_\Omega \widetilde{F}(v_{\varepsilon_j})dx = 0\,.$$

But this is a contradiction to

$$\int_\Omega \widetilde{F}(v_{\varepsilon_j})dx \geq \min\left\{\widetilde{F}(\alpha+\lambda),\, \widetilde{F}(\beta+\lambda)\right\}\left|\Omega^\lambda_{\varepsilon_j}\right| \geq \widetilde{F}(\alpha+\lambda)\,\tau > 0\,,$$

for a fixed λ and $\forall\, j$. Note that $\widetilde{F}(\alpha+\lambda) = \widetilde{F}(\beta-\lambda)$.

It remains to prove 2.1.iv), which concerns stability of the family $\{v_\varepsilon\}$, $0<\varepsilon\leq\varepsilon_0$. From the fact that v_ε is also a H^1-local minimizer of E_ε, $0<\varepsilon\leq\varepsilon_0$, it follows that

$$\langle E''_\varepsilon(v_\varepsilon)\psi\,,\,\psi\rangle_{H^1,H^*} \geq 0\,,\quad \forall\,\psi\in H^1(\Omega) \tag{5.1}$$

where H^* stands for the dual of $H^1(\Omega)$.

Consider now the linearized eigenvalue problem

$$\begin{gathered}\varepsilon^2 \operatorname{div}\left[k_1(x)\nabla\psi\right] + k_2(x) f'_\varepsilon(x,v_\varepsilon)\psi = \lambda\psi\,,\quad x\in\Omega\\ \tfrac{\partial\psi}{\partial n} = 0 \quad\text{on}\quad \partial\Omega\end{gathered} \tag{5.2}$$

where $f'_\varepsilon(x,v) = \dfrac{\partial f_\varepsilon(x,v)}{\partial v}$.

Denoting by $\{\lambda_n\}_n$, $n=1,2,\ldots$, the sequence of eigenvalues of (5.2) then, taking into account (5.1), and using the variational characterization of the eigenvalues, we infer that $\lambda_n \leq 0$, $n=1,2,\ldots$. By well-known results from linearized stability and semigroup theory we conclude that in the case that the first eigenvalue λ_1 is negative then v_ε is a strongly stable stationary solution of (1.1).

Now if $\lambda_1 = 0$ a classical application of the Krein-Rutman theorem gives that zero is a simple eigenvalue of (5.2). In this case there is a local one dimensional critical manifold $M_c(v_\varepsilon)$, tangent to $[\psi_1]$ (the eigenspace spanned by the principal eigenfunction ψ_1), at v_ε, such that if v_ε is stable in $M_c(v_\varepsilon)$ then it is also stable in $H^1(\Omega)$.

For this matter we refer to Theorem 6.2.1 in [12], which proof can be adapted to fit our case. But now the stability of v_ε in $M_c(v_\varepsilon)$ follows from the fact that the semigroup $\{T(t)\}_{t\geq 0}$ generated by (1.1) defines a gradient flow in $H^1(\Omega)$. To be more specific the functionals $E_\varepsilon[v_\varepsilon(x,t)]$ defines a Lyapunov function and along each solution $v_\varepsilon(x,t)$ it holds that

$$\frac{d}{dt}\, E_\varepsilon[v_\varepsilon(x,t)] \leq 0, \quad t \geq 0\ .$$

This concludes the proof of Theorem 2.1. □

Remark. Condition (2.1.iii) actually shows the multiple–layer profile of v_ε, for small ε, and it should hold that $v_\varepsilon \to v_0$, as $\varepsilon \to 0$, uniformly on compact sets of $\Omega \backslash \cup_{i=1}^p \gamma_i$. This should be done following ideas set forth in [2].

Remark. The difficulties when trying to generalize our results to higher space dimensions are of technical nature and inherent to the proof of Theorem 4.3. For instance, in this case one would no longer have a global parametrization of the limiting phase-transition hypersurface.

Remark. Other types of patterns can be considered rather than just the case of nested limiting phase–transition curves. For instance, let γ_i, $i = 1, \ldots, p$, be smooth simple closed curves in Ω with O_i denoting the open region enclosed by γ_i, and $\overline{O}_i$ its closure, then $\cap_{i=1}^p \overline{O}_i$ is empty. Then Theorem 4.3 and Corollary 4.1 still hold with

$$v_0 = \alpha\chi_{\cup_{i=1}^p O_i} + \beta\chi_{\Omega\backslash\cup_{i=1}^p \overline{O}_i}\ .$$

A suitable combination of the two patterns referred to above could also be considered.

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Arnaldo Simal do Nascimento
Universidade Federal de São Carlos, D.M.
13565-905 - São Carlos, S.P. Brazil
E-mail address: dasn@power.ufscar.br