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EXISTENCE OF AXISYMMETRIC WEAK SOLUTIONS OF THE 3-D EULER EQUATIONS FOR NEAR-VORTEX-SHEET INITIAL DATA

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Abstract. We study the initial value problem for the 3-D Euler equation when the fluid is inviscid and incompressible, and flows with axisymmetry and without swirl. On the initial vorticity ω_0, we assumed that ω_0/r belongs to $L(\log L(\mathbb{R}^3))^\alpha$ with $\alpha > 1/2$, where r is the distance to an axis of symmetry. To prove the existence of weak global solutions, we prove first a new *a priori* estimate for the solution.

Introduction

We consider the Euler equations for homogeneous inviscid incompressible fluid flow in $\mathbb{R}^3$

$$\frac{\partial v}{\partial t} + (v\cdot\nabla)v = -\nabla p\,, \qquad \operatorname{div} v = 0 \quad \text{ in } \mathbb{R}_+ \times \mathbb{R}^3\,, \tag{1}$$

$$v(0,\cdot) = v_0\,, \tag{2}$$

where $v(t,x) = (v_1(t,x), v_2(t,x), v_3(t,x))$ is the velocity of the fluid flow and $p(t,x)$ is the pressure. The problem of finite-time breakdown of smooth solutions to (1)-(2) for smooth initial data is a longstanding open problem in mathematical fluid mechanics. (See [6,13,14] for a detailed discussion of this problem.) The situation is similar even for the case of axisymmetry (see e.g.[11], [4]). In the case of axisymmetry without swirl velocity (θ-component of velocity), however, we have a global unique smooth solution for smooth initial data [14,17]. In this case a crucial role is played by the fact that $\omega_\theta(t,x)/r$ (where $\omega = \operatorname{curl} v$, $r = \sqrt{x_1^2 + x_2^2}$) is preserved along the flow, and the problem looks similar to that of the 2-D Euler equations.

This apparent similarity between the axisymmetric 3-D flow without swirl and the 2-D flow for smooth initial data breaks down for nonsmooth initial data. In particular, Delort [8] found the very interesting phenomenon that for a sequence of approximate solutions to the axisymmetric 3-D Euler equations with nonnegative vortex-sheet initial data, either the sequence converges strongly in $L^2_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^3)$, or the weak limit of the sequence is not a weak solution of the equations. This is in contrast with Delort's proof of the existence of weak solutions for the 2-D Euler equations with the single-signed vortex-sheet initial data, where we have weak

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convergence for the approximate solution sequence. Due to the subtle concentration cancellation type of phenomena in the nonlinear term, the weak limit itself becomes a weak solution [7,10,15]. We refer to [13, Section 4.3] for an illuminating discussion on the differences between the the quasi 2-D Euler equations and the "pure" 2-D Euler equations for weak initial data.

In this paper we prove existence of weak solutions to (1)-(2) for the axisymmetric initial data without swirl in which the vorticity satisfies

$$\Big|\frac{\omega_0}{r}\Big|\Big[1+\Big(\log^+\Big|\frac{\omega_0}{r}\Big|\Big)^\alpha\Big]\in L^1(\mathbb{R}^3),\quad \alpha>\frac{1}{2}\,,$$

where $\log^+ t=\max\{0,\log t\}$. The idea of proof is as follows. We divide $\mathbb{R}^3$ into two parts: the region near the axis of symmetry, and the region away from the axis. For the latter region, using the 2-D structure of the equations expressed in cylindrical coordinate system, we obtain strong compactness for the approximate solution sequence using arguments previously used in the 2-D problem in [3]. For the region near axis, we could not adapt the previous 2-D arguments. See the next section for explicit comparison between the nonlinear terms in the pure 2-D Euler case and our case. Here we use a new *a priori* estimate for the axisymmetric flow, combined with Delort's argument in [8] to overcome these difficulties.

To the authors' knowledge this *a priori* estimate (See Lemma 2.1) is completely new for the 3-D Euler equations with axisymmetry. On the other hand, the results obtained in this paper improve substantially the results in [5], where the authors proved existence of weak solutions for

$$\Big|\frac{\omega_0}{r}\Big|\in L^1(\mathbb{R}^3)\cap L^p(\mathbb{R}^3),\quad p>\frac{6}{5}\,.$$

It would be very interesting to study (1)-(2) with initial data in $L^1(\mathbb{R}^3)$.

1. Preliminaries

By a weak solution of the Euler equations with an initial data v_0, we mean the vector field $v\in L^\infty([0,T];(L^2_{\rm loc}(\mathbb{R}^3))^3)$ with $\operatorname{div} v=0$ such that

$$\int_0^T\int_{\mathbb{R}^3}[v\cdot\varphi_t+v\otimes v:\nabla\varphi]\,dx\,dt+\int_{\mathbb{R}^3}v_0\cdot\varphi(0,x)\,dx=0\,,$$

for all $\varphi\in C^\infty([0,T];[C_0^\infty(\mathbb{R}^3)]^3)$ with $\operatorname{div}\varphi\equiv 0$ and $\varphi(T,x)\equiv 0$ Here we have used the notation $v\otimes v:\nabla\varphi=\sum_{i,j=1}^3 v_iv_j(\varphi_i)_{x_j}$.

We are concerned with the axisymmetric solutions to the Euler equations. By an axisymmetric solution of equations (1)-(2) we mean a solution of the form

$$v(t,x)=v_r(r,x_3,t)e_r+v_\theta(r,x_3,t)e_\theta+v_3(r,x_3,t)e_3$$

in the cylindrical coordinate system, using the canonical basis

$$e_r=(\frac{x_1}{r},\frac{x_2}{r},0),\quad e_\theta=(\frac{x_2}{r},-\frac{x_1}{r},0),\quad e_3=(0,0,1),r=\sqrt{x_1^2+x_2^2}\,.$$

For such flows the first equation in (1) can be written as

$$\frac{\tilde{D}v_r}{Dt} - \frac{(v_\theta)^2}{r} = -\frac{\partial p}{\partial r}\,, \tag{3}$$

$$\frac{\tilde{D}}{Dt}(rv_\theta) = 0\,, \tag{4}$$

$$\frac{\tilde{D}v_3}{\partial t} = -\frac{\partial p}{\partial x_3}\,, \tag{5}$$

for each component of velocity in the cylindrical coordinate system, where

$$\frac{\tilde{D}}{Dt} = \frac{\partial}{\partial t} + v_r\frac{\partial}{\partial r} + v_3\frac{\partial}{\partial x_3}\,.$$

On the other hand, the second equation of (1) becomes

$$\frac{\partial}{\partial r}(rv_r) + \frac{\partial}{\partial x_3}(rv_3) = 0\,. \tag{6}$$

We observe that θ-component of the vorticity equation is written as

$$\frac{\tilde{D}}{Dt}\left(\frac{\omega_\theta}{r}\right) = \frac{1}{r^4}\frac{\partial}{\partial x_3}(rv_\theta)^2\,, \tag{7}$$

where

$$\omega_\theta = \frac{\partial v_r}{\partial x_3} - \frac{\partial v_3}{\partial r} \tag{8}$$

is the $\theta-$component of the vorticity vector ω. If we assume that the initial velocity

$$v_0 \in V^m = \{v \in [\,\mathrm{H}^m(\mathbb{R}^3)]^3 : \ \mathrm{div}\ v = 0\}$$

with $m \geq 4$ is axisymmetric, then due to the symmetry properties of the Euler equations, and by the existence of local unique classical solutions [12], the solution remains axisymmetric during its existence. Here we used the standard Sobolev space

$$\mathrm{H}^m(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3)\ :\ D^\alpha u \in L^2(\mathbb{R}^3),\ |\alpha| \leq m\}\,.$$

Furthermore, if v_0 has no "swirl" component, i.e. $v_{0,\theta}$=0, then (4) and (7) imply that

$$\frac{\tilde{D}}{Dt}\left(\frac{\omega_\theta}{r}\right) = 0 \quad \forall t > 0\,. \tag{9}$$

We observe that in this case the vorticity becomes $\omega(t,x) = \omega_\theta(t,r,x_3)e_\theta$. Thus, we have, in particular,

$$|\omega(t,x)| = |\omega_\theta(t,r,x_3)|\,,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^3$ in the left hand side, and the absolute value in the right hand side of the equation. In [17] Saint-Raymond proved existence of a global unique smooth solution for smooth v_0 without swirl.

Below we show explicitly the difference between the nonlinear terms for the 2-D Euler equations and those for 3-D Euler equations with axisymmetry and without swirl. In the weak formulation of the 2-D Euler equations, if we use a test function of the form $\varphi = (-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1})$ in order to satisfy $\operatorname{div}\varphi = 0$, then

$$\int_0^T\int_{\mathbb{R}^2}[v\otimes v : \nabla\varphi]\,dx\,dt = \int_0^T\int_{\mathbb{R}^2}\Big[(v_1^2 - v_2^2)\frac{\partial^2\psi}{\partial x_1\partial x_2} - v_1v_2\Big(\frac{\partial^2\psi}{\partial x_1^2} - \frac{\partial^2\psi}{\partial x_2^2}\Big)\Big]\,dx\,dt\,.$$

On the other hand, in the axisymmetric 3-D Euler equation without swirl, if we use as a test function $\varphi(t,x) = \varphi_r(t,r,x_3)e_r + \varphi_3(t,r,x_3)e_3$ with

$$\varphi_r = \frac{1}{r}\frac{\partial\psi}{\partial x_3},\quad \varphi_3 = -\frac{1}{r}\frac{\partial\psi}{\partial r}$$

to satisfy $\frac{\partial(r\varphi_r)}{\partial r} + \frac{\partial(r\varphi_3)}{\partial x_3} = 0$, then

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^3}[v\otimes v : \nabla\varphi]\,dx\,dt = 2\pi\int_0^T\int_{\mathbb{R}\times\mathbb{R}_+}\Big[(v_r^2 - v_3^2)\frac{\partial^2\psi}{\partial r\partial x_3} - v_rv_3\Big(\frac{\partial^2\psi}{\partial r^2} - \frac{\partial^2\psi}{\partial x_3^2}\Big) \\ + \frac{v_rv_3}{r}\frac{\partial\psi}{\partial r} - \frac{v_r^2}{r}\frac{\partial\psi}{\partial x_3}\Big]\,dr\,dx_3\,dt\,.\end{aligned}$$

Here we have extra two nonlinear terms compared to the 2-D case, which have apparent singularities on the axis of symmetry.

Before closing this section, we provide a brief introduction to the Orlicz spaces. For more details see [1,9], and for applications to the 2-D Euler equations, see [3,16]. By an N-function we mean a real valued function $A(t)$, $t \geq 0$ which is continuous, increasing, convex, and satisfies

$$\lim_{t\to 0}\frac{A(t)}{t} = 0,\quad \lim_{t\to\infty}\frac{A(t)}{t} = +\infty\,.$$

We say that $A(t)$ satisfies Δ_2-condition near infinity if there exist $k > 0$, $t_0 \geq 0$ such that

$$A(2t) \leq kA(t)\quad \forall t \geq t_0\,.$$

We denote $A(t) \succ B(t)$ if for every $k > 0$

$$\lim_{t\to\infty}\frac{A(kt)}{B(t)} = \infty\,.$$

Let Ω be a domain in $\mathbb{R}^n$. Then the Orlicz class $K_A(\Omega)$ is defined as the set all functions u such that $\int_\Omega A(|u(x)|)\,dx < \infty$. On the other hand, the Orlicz space $L_A(\Omega)$ is defined as the linear hull of the Orlicz class $K_A(\Omega)$. The set $L_A(\Omega)$ is a Banach space equipped with the Luxembourg norm

$$\|u\|_A = \inf\big\{k : \int_\Omega A(\frac{u}{k})\,dx \leq 1\big\}\,.$$

In general $K_A(\Omega) \subset L_A(\Omega)$, but in case the domain Ω is bounded in $\mathbb{R}^n$, and the N-function A satisfies the Δ_2-condition near infinity we have $K_A(\Omega) = L_A(\Omega)$ (see [1]). For example $L^p(\Omega)$, $1 < p < \infty$ is an Orlicz space with N-functions given by $A(t) = t^p$.

Recall that for a bounded domain Ω we have the continuous imbedding, [1],

$$L_A(\Omega) \hookrightarrow L_B(\Omega)\quad \text{if}\quad A(t) \succ B(t).$$

Also recall the following duality relations [3, Lemma 4]. (Below X^* denotes the dual of X)

Lemma 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, and $\alpha > 0$. Let $A(\cdot), B(\cdot)$ be N-functions given by $A(t) = t(\log^+ t)^\alpha$, $B(t) = \exp(t^{q/\alpha}) - 1$, where $t \geq 0$. Then, we have*

$$L_B(\Omega) = L_A^*(\Omega)\,.$$

By the Orlicz-Sobolev space $W^m L_A(\Omega)$ we mean a subspace of the Orlicz space $L_A(\Omega)$ consisting of functions u such that the distributional derivatives $D^\alpha u$ are contained in $L_A(\Omega)$ for all multi-index α' with $|\alpha| \leq m$, equipped with a Banach space norm

$$\|u\|_{m,A} = \max_{|\alpha|\leq m} \|D^\alpha u\|_A\,.$$

The following lemma corresponds to a special case of the general result by Donaldson and Trudinger [9].

Lemma 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and $B(t) = \exp(t^2) - 1$, then we have a continuous imbedding*

$$H_0^1(\Omega) \hookrightarrow L_B(\Omega).$$

Moreover, for any N-function $A(t)$ with $A(t) \prec B(t)$ we have a compact imbedding

$$H_0^1(\Omega) \hookrightarrow\hookrightarrow L_A(\Omega).$$

Combining dual of the compact imbedding in Lemma 1.2, and Lemma 1.1 we have

Corollary 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $A(t) = t(\log^+ t)^\alpha$ with $\alpha > \frac{1}{2}$. Then we have the compact imbedding*

$$L_A(\Omega) \hookrightarrow\hookrightarrow H^{-1}(\Omega).$$

2. Main Results

Our main result is as follows:

Theorem 2.1. *Suppose $\alpha > \frac{1}{2}$ is given. Let $v_0 \in V^0$ be an axisymmetric initial data with $v_{0,\theta} \equiv 0$, and $|\frac{\omega_0}{r}|\left[1 + (\log^+ |\frac{\omega_0}{r}|)^\alpha\right] \in L^1(\mathbb{R}^3)$. Then there exists a weak solution of problem (1)-(2). Moreover, the solution satisfies*

$$\|v(t,\cdot)\|_{V^0} \leq \|v_0\|_{V^0},$$

and

$$\int_{\mathbb{R}^3} \left|\frac{\omega(t,\cdot)}{r}\right| \left[1 + \left(\log^+ \left|\frac{\omega(t,\cdot)}{r}\right|\right)^\alpha\right] dx \leq \int_{\mathbb{R}^3} \left|\frac{\omega_0}{r}\right| \left[1 + \left(\log^+ \left|\frac{\omega_0}{r}\right|\right)^\alpha\right] dx$$

for almost every $t \in [0, \infty)$.

In this section our aim is to prove the above theorem. Below we denote

$$Q = [0,T] \times \mathbb{R}^3, \quad G = \{(r, x_3) \in \mathbb{R}^2 \,|\, r > 0, x_3 \in \mathbb{R}\}.$$

We start from establishment of the following a priori estimate.

Lemma 2.1. *Let $v(t,x) \in C([0,T];[C^1(\overline{\mathbb{R}^3}) \bigcap H^1(\mathbb{R}^3)]^3) \bigcap C([0,T];V^0)$ be the classical solution of the Euler equations for the axisymmetric initial data v_0 without the swirl component, and with the vorticity satisfying $\frac{\omega_0}{r} \in L^1(\mathbb{R}^3)$. Then the following estimate holds:*

$$\int_0^T \int_{\mathbb{R}^3} \frac{1}{1+x_3^2} \left(\frac{v_r}{r}\right)^2 dx\,dt \le C\left(\|v_0\|_{V^0}^2 + \left\|\frac{\omega_0}{r}\right\|_{L^1(\mathbb{R}^3)}\right). \tag{10}$$

Proof. The velocity conservation law for the Euler equations implies the estimate

$$\|\sqrt{r}v_r\|_{L^\infty(0,T;L^2(G))} + \|\sqrt{r}v_3\|_{L^\infty(0,T;L^2(G))} \le C\|v_0\|_{V^0}. \tag{11}$$

Moreover, (9) immediately yields the estimate for L^1-norm of vorticity

$$\|\omega(t,\cdot)\|_{L^1(G)} \le \|\omega_0\|_{L^1(G)}.$$

We set $\rho(x_3) = \int_{-\infty}^{x_3} 1/(1+\tau^2)\,d\tau$. Multiplying (9) by $2\pi r\rho(x_3)$ scalarly in $L^2(0,T;L^2(G))$ and integrating by parts, we obtain

$$\begin{aligned}
0 &= \int_{\mathbb{R}^3} \frac{\rho\omega_\theta}{r}\,dx\Big|_0^T - \int_0^T\int_G 2\pi\rho' v_3\omega_\theta\,dr\,dx_3\,dt \\
&= \int_{\mathbb{R}^3} \frac{\rho\omega_\theta}{r}dx\Big|_0^T + \int_0^T\int_G 2\pi\rho' v_3\left(\frac{\partial v_3}{\partial r} - \frac{\partial v_r}{\partial x_3}\right) dr\,dx_3\,dt \\
&= \int_{\mathbb{R}^3} \frac{\rho\omega_\theta}{r}dx\Big|_0^T - \int_0^T\int_{-\infty}^{+\infty} \pi\rho' v_3^2(t,0,x_3)\,dx_3\,dt \\
&\quad + \int_0^T\int_G 2\pi\left(\rho'' v_3 v_r + \rho' v_r\frac{\partial v_3}{\partial x_3}\right) dr\,dx_3\,dt\,,
\end{aligned} \tag{12}$$

where we used the regularity assumption of solution v, and the integration by parts used above can be justified easily. Indeed,

$$\begin{aligned}
&\int_0^T\int_G 2\pi\rho' v_3\left(\frac{\partial v_3}{\partial r} - \frac{\partial v_r}{\partial x_3}\right) dr\,dx_3\,dt \\
&\quad= \lim_{r_k\to+\infty} 2\pi\int_0^T\int_{-\infty}^{+\infty}\int_0^{r_k} \rho' v_3\frac{\partial v_3}{\partial r}\,dr\,dx_3\,dt \\
&\qquad - \lim_{b_k\to+\infty} 2\pi\int_0^T\int_{-b_k}^{b_k}\int_0^\infty \rho' v_3\frac{\partial v_r}{\partial x_3}\,dr\,dx_3\,dt \\
&\quad= -\int_0^T\int_{-\infty}^{+\infty} \pi\rho' v_3^2(t,0,x_3)\,dx\,dt + \lim_{r_k\to+\infty}\int_0^T\int_{-\infty}^{+\infty} \pi\rho' v_3^2(t,r_k,x_3)\,dx_3\,dt \\
&\qquad - \lim_{b_k\to+\infty}\int_0^T\int_0^\infty 2\pi\rho' v_3 v_r dr dt\Big|_{-b_k}^{b_k} \\
&\qquad + \lim_{b_k\to+\infty}\int_0^T\int_{-b_k}^{b_k}\int_0^\infty 2\pi\left(\rho'' v_3 v_r + \rho' v_r\frac{\partial v_3}{\partial x_3}\right) dr\,dx_3\,dt\,.
\end{aligned}$$

for all sequence $r_k \to +\infty$. Since $v \in C([0,T];(C^1(\overline{R^3}))^3)$,

$$\begin{aligned}\int_{(0,T)\times\mathbb{R}^3} |v|^2\,dx\,dt =& 2\pi\int_0^{+\infty}\left(\int_0^T\int_{-\infty}^{+\infty}|v|^2\,dx_3\,dt\right) r\,dr\\ =& 2\pi\int_{-\infty}^{\infty}\left(\int_0^T\int_0^{+\infty}|v|^2 r\,dr\,dt\right)dx_3<\infty\,,\end{aligned}$$

and $\lim_{x_3\to\infty}\rho'(x_3)=0$ one can find a sequence $r_k\to+\infty$ and $b_k\to+\infty$ such that

$$\int_0^T\int_{-\infty}^{\infty}\rho' v_3^2(t,r_k,x_3)\,dx_3\,dt\to 0,\quad \lim_{b_k\to+\infty}\int_0^T\int_0^{\infty}2\pi\rho v_3 v_r\,dt\,dr\Big|_{-b_k}^{b_k}\to 0\,.$$

From (6) we have

$$\frac{\partial v_3}{\partial x_3}=-\frac{v_r}{r}-\frac{\partial v_r}{\partial r}\,. \tag{13}$$

Therefore (12) and (13) imply

$$\begin{aligned}0=\int_{\mathbb{R}^3}\rho\frac{\omega_\theta}{r}dx\Big|_0^T &-\int_0^T\int_{-\infty}^{+\infty}\pi\rho' v_3^2(t,0,x_3)\,dx_3\,dt\\ &+\int_0^T\int_G 2\pi\left(\rho'' v_3v_r-\rho'\frac{(v_r)^2}{r}-\rho' v_r\frac{\partial v_r}{\partial r}\right)dr\,dx_3\,dt\,.\end{aligned}\tag{14}$$

Since, by assumption, $v(t,x)$ is a smooth and axisymmetric vector field

$$v_r(t,0,x_3)=0\quad\forall\, t\in\mathbb{R}^1_+,\ x_3\in\mathbb{R}^1.$$

Thus integration by parts in (14), which can be justified similarly to the above, implies

$$\begin{aligned}\int_0^T\int_{-\infty}^{+\infty}\pi\rho' v_3^2(t,0,x_3)\,dx_3\,dt&+\int_0^T\int_G 2\pi\rho'(x_3)\frac{(v_r)^2}{r}\,dr\,dx_3\,dt\\ &=\int_0^T\int_G 2\pi\rho'' v_3v_r\,dr\,dx_3\,dt+\int_{\mathbb{R}^3}\rho\frac{\omega_\theta}{r}dx\Big|_0^T\,.\end{aligned}$$

Since $\rho'(x_3)>0, |\rho(x_3)|<C$ for all $x_3\in\mathbb{R}^1$ we obtain the inequality

$$\begin{aligned}&2\pi\int_0^T\int_G\left|\rho'\frac{(v_r)^2}{r}\right|dr dx_3 dt\le 2\pi\int_0^T\int_G|\rho'' v_3v_r|\,dr\,dx_3\,dt+C\left\|\frac{\omega_0}{r}\right\|_{L^1(\mathbb{R}^3)}\\ &\le 2\pi\left(\int_0^T\int_G\rho'\frac{(v_r)^2}{r}\,dr\,dx_3\,dt\right)^{\frac12}\left(\int_0^T\int_G r v_3^2\frac{(\rho'')^2}{\rho'}\,dr\,dx_3\,dt\right)^{\frac12}+C\left\|\frac{\omega_0}{r}\right\|_{L^1(\mathbb{R}^3)}\,.\end{aligned}$$

Hence by the Cauchy-Bunyakovskii inequality we have

$$\int_0^T\int_G|\rho'|\frac{(v_r)^2}{r}\,dr\,dx_3\,dt\le C\left(\int_0^T\int_{\mathbb{R}^3}v_3^2\frac{|\rho''|^2}{\rho'}\,dx\,dt+\left\|\frac{\omega_{0,\theta}}{r}\right\|_{L^1(\mathbb{R}^3)}\right). \tag{15}$$

Since $\sup_{x_3\in\mathbb{R}}|\rho''(x_3)|^2/|\rho'(x_3)|\le C$, inequalities (11) and (15) imply the estimate (10). ■

Now, let v_0^ε be an axisymmetric initial datum without the swirl component such that

$$v_0^\varepsilon \to v_0 \text{ in } V^0, \quad v_0^\varepsilon \in (C^\infty(\mathbb{R}^3))^3\,, \quad \frac{\omega_{0,\theta}^\varepsilon}{r} \to \frac{\omega_{0,\theta}}{r} \text{ in } L^1(\mathbb{R}^3)\,. \tag{16}$$

Such an approximation v_0^ε for any axisymmetric function, $v_0 \in V^0$ without swirl was constructed in [17] for example. In [17] also, it was proved that in this case there exists a unique solution of the problem (1)-(2), $v_\varepsilon(t,\cdot)\in C([0,T];[C^2(\mathbb{R}^3)]^3)\cap L^2(0,T;H^1(\mathbb{R}^3))$. Without loss of generality, passing to a subsequence if it is necessary, we may assume that

$$v^\varepsilon \to v \text{ weakly in } [L^2((0,T)\times\mathbb{R}^3)]^3\,. \tag{17}$$

We have

Lemma 2.2. *Let $\{v^\varepsilon(x,t)\}_{\varepsilon\in(0,1)}$ be a sequence of smooth solutions of (1)-(2) associated with the initial datum $\{v_0^\varepsilon\}$ with axisymmetry and without swirl, and satisfying (16) and (17). Then, for each $\varphi\in C([0,T];C_0(\mathbb{R}^3))$, we have*

$$\int_Q [(v_r^\varepsilon)^2-(v_3^\varepsilon)^2]\varphi\,dx\,dt \to \int_Q [(v_r)^2-(v_3)^2]\varphi\,dx\,dt \ \textit{as}\ \varepsilon\to+0\,. \tag{18}$$

Remark. The above lemma is very similar to Delort's in [8], where he proved it in particular under the assumptions on the sequence $\{v^\varepsilon(x,t)\}$ that

$$\begin{gathered}\omega_\theta^\varepsilon(x,t)\ge 0 \ \text{ almost everywhere in } (0,T)\times\mathbb{R}^3, \text{ and}\\ \{\omega_\theta^\varepsilon\} \text{ is uniformly bounded in } L^\infty(0,\infty;L^1(\overline{\mathbb{R}}_+\times\mathbb{R},(1+r^2)\,dr\,dx_3))\,.\end{gathered}$$

In our case, however, we only need to assume $\frac{\omega_0}{r}\in L^1(\mathbb{R}^3)$, and $\{v^\varepsilon\}$ is the associated sequence of approximated solutions.

Proof of Lemma 2.2. We follow Delort's arguments. Denote

$$(\Delta^{-1}f)(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{f(y)}{|x-y|}\,dy\,.$$

Relation $v^\varepsilon = -\Delta^{-1}\mathbf{curl}\,\omega^\varepsilon$ implies

$$\begin{aligned} v_1^\varepsilon &= \Delta^{-1}\partial_3\omega_2^\varepsilon\,,\\ v_2^\varepsilon &= -\Delta^{-1}\partial_3\omega_1^\varepsilon\,,\\ v_3^\varepsilon &= \Delta^{-1}(\partial_2\omega_1^\varepsilon-\partial_1\omega_2^\varepsilon)\,. \end{aligned}$$

Let $\Phi\in C_0^\infty(\mathbb{R}^3)$, in $\Phi\ge 0, \Phi\equiv 1$ for all $(t,x)\in\operatorname{supp}\varphi$. Set

$$\begin{aligned} \varphi v_1^\varepsilon &= \varphi\Delta^{-1}\partial_3(\Phi\omega_2^\varepsilon)+w_1^\varepsilon\,,\\ \varphi v_2^\varepsilon &= -\varphi\Delta^{-1}\partial_3(\Phi\omega_2^\varepsilon)+w_2^\varepsilon\,,\\ \varphi v_3^\varepsilon &= \varphi\Delta^{-1}(\partial_2(\Phi\omega_1^\varepsilon)-\partial_1(\Phi\omega_2^\varepsilon))+w_3^\varepsilon\,, \end{aligned}$$

where

$$w_1^\varepsilon = \varphi[\Phi, \Delta^{-1}\frac{\partial}{\partial x_3}]\omega_2^\varepsilon\,,$$
$$w_2^\varepsilon = -\varphi[\Phi, \Delta^{-1}\frac{\partial}{\partial x_3}]\omega_1^\varepsilon\,,$$
$$w_3^\varepsilon = \varphi([\Phi, \Delta^{-1}\frac{\partial}{\partial x_2}]\omega_1^\varepsilon - [\Phi, \Delta^{-1}\frac{\partial}{\partial x_1}]\omega_2^\varepsilon)\,,$$

where $[A,B] = AB - BA$ is the commutator of operators A, B, and $\partial_i = \partial/\partial_{x_i}$. Note that w_i^ε are uniformly bounded in $L^\infty(0,T;\mathrm{H}^1_{\rm loc}(\mathbb{R}^3)) \cap H^1(0,T;H^{-4}_{\rm loc}(\mathbb{R}^3))$ for each $i = 1,2,3$. Really let us prove this claim for example for w_1^ε. Denote $z_\varepsilon = \Delta^{-1}\frac{\partial}{\partial x_3}(\Phi\omega_2^\varepsilon)$. Then $\Delta z_\varepsilon = \frac{\partial}{\partial x_3}(\Phi\omega_2^\varepsilon)$, and

$$\Delta(\Phi v_1^\varepsilon) = \partial_3(\Phi\omega_2^\varepsilon) - \partial_3\Phi\omega_2^\varepsilon + 2\sum_{i=1}^3 \frac{\partial}{\partial x_i}(v_1^\varepsilon \frac{\partial\Phi}{\partial x_i}) - v_1^\varepsilon\Delta\Phi\,.$$

Denote $u_\varepsilon = \Phi v_1^\varepsilon - z_\varepsilon$. Then

$$\Delta u_\varepsilon = -\frac{\partial(\partial_3\Phi v_1^\varepsilon)}{\partial x_3} + \frac{\partial^2\Phi}{\partial x_3^2}v_1^\varepsilon + \frac{\partial(\partial_3\Phi v_3^\varepsilon)}{\partial x_1} - \frac{\partial^2\Phi}{\partial x_1\partial x_3}v_3^\varepsilon + 2\sum_{i=1}^3 \frac{\partial}{\partial x_i}(v_1^\varepsilon\frac{\partial\Phi}{\partial x_i}) - v_1^\varepsilon\Delta\Phi$$

and

$$\begin{aligned} u_\varepsilon = \Delta^{-1}\left(-\frac{\partial(\partial_3\Phi v_1^\varepsilon)}{\partial x_3} + \frac{\partial(\partial_3\Phi v_3^\varepsilon)}{\partial x_1} + 2\sum_{i=1}^3 \frac{\partial}{\partial x_i}(v_1^\varepsilon\frac{\partial\Phi}{\partial x_i})\right) \\ + \Delta^{-1}\left(\frac{\partial^2\Phi}{\partial x_3^2}v_1^\varepsilon - \frac{\partial^2\Phi}{\partial x_1\partial x_3}v_3^\varepsilon - v_1^\varepsilon\Delta\Phi\right), \end{aligned}$$

where the first component of u_ε is bounded in $L^\infty(0,T;H^1(\mathbb{R}^3))$, and the second one is bounded in $L^\infty(0,T;H^1_{\rm loc}(\mathbb{R}^3))$ due to compactness of $\operatorname{supp}\Phi$ in $[0,T]\times\mathbb{R}^3$. Since the function φ also has a compact support in $[0,T]\times\mathbb{R}^3$, the first part of our statement is proved. To prove the uniform boundness of $\frac{\partial w_1^\varepsilon}{\partial t}$ in $L^2(0,T;H^{-4}_{\rm loc}(\mathbb{R}^3))$ we first recall that for any smooth solution v of the 3-D Euler equations with initial data v_0, we have in general

$$\|v(t_1) - v(t_2)\|_{H^{-3}(B_r)} \le C(r)\|v_0\|^2_{V^0}\ |t_1 - t_2|$$

for all t_1, t_2 with $0 < t_1 \le t_2 < T$, where B_r is a ball with the center 0 and radius r (see e.g. [5]). This estimate implies immediately that

$$\left\|\frac{\partial v^\varepsilon}{\partial t}\right\|_{L^\infty(0,T;H^{-3}(B_r))} \le C(r), \tag{19}$$

where C is independent of ε. Taking the time derivative of u_ε we have

$$\left\|\frac{\partial w^\varepsilon}{\partial t}\right\|_{L^2(0,T;H^{-4}(B_r))} \le C(r)\left(\left\|\frac{\partial v^\varepsilon}{\partial t}\right\|_{L^2(0,T;H^{-3}(B_r))} + \|v^\varepsilon\|_{L^2(0,T;V^0)}\right) \le C(r).$$

Hence to prove (18) we need only to pass to the limit in the following equation.

$$A^\varepsilon = \int_Q [(\Delta^{-1}\partial_3(\Phi\omega_2^\varepsilon))^2 + (\Delta^{-1}\partial_3(\Phi\omega_1^\varepsilon))^2 - (\Delta^{-1}\partial_2(\Phi\omega_1^\varepsilon))^2 - (\Delta^{-1}\partial_1(\Phi\omega_2^\varepsilon))^2 + 2(\Delta^{-1}\partial_2(\Phi\omega_1^\varepsilon))(\Delta^{-1}\partial_1(\Phi\omega_2^\varepsilon))]\varphi\, dx\, dt.$$

After simplifications we have:

$$A^\varepsilon = (\Phi\omega_2^\varepsilon, \varphi\Delta^{-2}(\partial_1^2 - \partial_3^2)(\Phi\omega_2^\varepsilon))_{L^2(Q)} + (\Phi\omega_1^\varepsilon, \varphi\Delta^{-2}(\partial_2^2 - \partial_3^2)(\Phi\omega_1^\varepsilon))_{L^2(Q)} - 2(\Phi\omega_1^\varepsilon, \varphi\Delta^{-2}\partial_1\partial_2(\Phi\omega_2^\varepsilon))_{L^2(Q)} + A_0^\varepsilon = A_1^\varepsilon + A_0^\varepsilon,$$

where

$$A_0^\varepsilon = (\Phi\omega_2^\varepsilon, [\Delta^{-1}\partial_3, \varphi](\Delta^{-1}\partial_3(\Phi\omega_2^\varepsilon)))_{L^2(Q)} + (\Phi\omega_1^\varepsilon, [\Delta^{-1}\partial_3, \varphi](\Delta^{-1}\partial_3(\Phi\omega_1^\varepsilon)))_{L^2(Q)} - (\Phi\omega_1^\varepsilon, [\Delta^{-1}\partial_2, \varphi](\Delta^{-1}\partial_2(\Phi\omega_1^\varepsilon)))_{L^2(Q)} - (\Phi\omega_2^\varepsilon, [\Delta^{-1}\partial_1, \varphi](\Delta^{-1}\partial_1(\Phi\omega_2^\varepsilon)))_{L^2(Q)} + 2(\Phi\omega_1^\varepsilon, [\Delta^{-1}\partial_2, \varphi](\Delta^{-1}\partial_1(\Phi\omega_2^\varepsilon)))_{L^2(Q)}.$$

Since each sequence $[\Delta^{-1}\partial_j, \varphi](\Delta^{-1}\partial_k(\Phi\omega_\ell^\varepsilon))$ belongs to a compact set in $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$, we obtain

$$A_0^\varepsilon \to A_0 \text{ as } \varepsilon \to 0$$

for a subsequence. In [8] Delort proved that the term A_1^ε can be rewritten as follows

$$A_1^\varepsilon = \int_0^T \int_G \int_G K(t, r, x_3, r', x_3')(\Phi\omega_\theta^\varepsilon)(t, r, x_3)(\Phi\omega_\theta^\varepsilon)(t, r', x_3')\, dr\, dx_3\, dr'\, dx_3'\, dt\,,$$

where the function $K(t, r, x_3, r', x_3')$ satisfies

$$K \in C^\infty \text{ on } \{(r, x_3, r', x_3') \in \overline{\mathbb{R}}_+ \times \mathbb{R} \times \overline{\mathbb{R}}_+ \times \mathbb{R}; (r, x_3) \neq (r', x_3')\}$$

and K is locally bounded on $\overline{\mathbb{R}}_+ \times \mathbb{R} \times \overline{\mathbb{R}}_+ \times \mathbb{R}$.

Let $\eta(\tau) \in C_0^\infty(\mathbb{R}^1)$, $\eta(\tau) \geq 0$ for any $\tau \in \mathbb{R}^1$ and $\eta \equiv 1$ in some neighborhood of 0. Set

$$\begin{aligned} A_1^\varepsilon = I_1^{\varepsilon,\delta} + I_2^{\varepsilon,\delta} = &\int_0^T \int_G \int_G K(t, r, x_3, r', x_3') \left(1 - \eta\left(\frac{r}{\delta}\right)\right)\left(1 - \eta\left(\frac{r'}{\delta}\right)\right) \times \\ &\left(1 - \eta\left(\frac{|r' - r| + |x_3 - x_3'|}{\delta}\right)\right)(\Phi\omega_\theta^\varepsilon)(t, r, x_3)(\Phi\omega_\theta^\varepsilon)(t, r', x_3')\, dr\, dx_3 dr' dx_3' dt \\ + &\int_0^T \int_G \int_G K(t, r, x_3, r', x_3') \Big[\eta\left(\frac{r}{\delta}\right)\left(1 - \eta\left(\frac{r'}{\delta}\right)\right)\left(1 - \eta\left(\frac{|r' - r| + |x_3 - x_3'|}{\delta}\right)\right) \\ &+ \eta\left(\frac{r'}{\delta}\right)\left(1 - \eta\left(\frac{|r - r'| + |x_3 - x_3'|}{\delta}\right)\right) \\ &+ \eta\left(\frac{|r - r'| + |x_3 - x_3'|}{\delta}\right)\Big](\Phi\omega_\theta^\varepsilon)(t, r, x_3)(\Phi\omega_\theta^\varepsilon)(t, r', x_3')\, dr\, dx_3\, dr'\, dx_3' dt. \end{aligned} \tag{20}$$

Our aim is to prove that for any $\kappa > 0$ there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that

$$|I_2^{\varepsilon,\delta}| \leq \kappa \quad \forall \varepsilon \in (0, \varepsilon_0),\ \delta \in (0, \delta_0). \tag{21}$$

We start from the following estimate

$$|I_2^{\varepsilon,\delta}| \le \hat{C}\Big[\int_0^T \int_{|r|\le c\delta} \int_{-\infty}^{+\infty} |\Phi\omega_\theta^\varepsilon| dx_3 dr dt \left(\Big\| \frac{\omega_0}{r} \Big\|_{L^1(\mathbb{R}^3)} + 1 \right) + \int_0^T \int_{(G\times G)\cap\{|r-r'|+|x_3-x_3'|<c\delta\}} |(\Phi\omega_\theta^\varepsilon)(t,r,x_3)(\Phi\omega_\theta^\varepsilon)(t,r',x_3')|\, dr\, dx_3\, dr'\, dx_3' dt \Big]. \tag{22}$$

Let us consider the system of ordinary differential equations

$$\frac{dX_\varepsilon(t,\alpha)}{dt} = v^\varepsilon(t, X_\varepsilon(t,\alpha)), \quad X_\varepsilon(t,\alpha)|_{t=0} = \alpha\,. \tag{23}$$

Using (23), one can write out the solution of (9) as

$$\left(\frac{\omega_\theta^\varepsilon}{r}\right)(t, X_\varepsilon(t,\alpha)) = \left(\frac{\omega_{0,\theta}^\varepsilon}{r}\right)(\alpha), \quad \alpha\in\mathbb{R}^3\,.$$

Or, equivalently

$$\frac{\omega_\theta^\varepsilon}{r}(t,\alpha) = \left(\frac{\omega_{0,\theta}^\varepsilon}{r}\right)(X_\varepsilon^{-1}(t,\alpha))\,. \tag{24}$$

Let us denote

$$\mathfrak{O}_{\delta,\varepsilon}^{(t)} = X_\varepsilon\left(t, \{(r,x_3)\in\overline{G}\,|\, r\le\delta,\ (r,x_3)\in\operatorname{supp}\Phi(t,\cdot)\}\right)\,.$$

Since, by assumption, $\operatorname{supp}\Phi$ is compact in $[0,T]\times\mathbb{R}^3$ and the mapping $X_\varepsilon^{-1}(t,\cdot)$ conserves a volumes, we have

$$\sup_{t\in[0,T]} \mu(\mathfrak{O}_{\delta,\varepsilon}^{(t)}) \to 0 \quad \text{as } \delta\to+0\,, \tag{25}$$

uniformly in ε.

Taking into account that $\det(\nabla X_\varepsilon(t,\alpha))\equiv 1$, one can estimate the first term of the right hand side of (22) as follows:

$$\int_0^T \int_{|r|\le c\delta} \int_{-\infty}^{+\infty} 2\pi |\Phi\omega_\theta^\varepsilon|\, dr\, dx_3\, dt \le C\int_0^T \int_{\mathfrak{O}_{\delta,\varepsilon}^{(t)}} \left| \left(\frac{\omega_{0,\theta}^\varepsilon}{r}\right)(t,x)\right| dx\, dt = B_{\varepsilon,\delta}\,.$$

Note that

$$B_{\varepsilon,\delta} \le C \sup_{t\in(0,T)} \int_{\mathfrak{O}_{\delta,\varepsilon}^{(t)}} \left(\left|\frac{\omega_{0,\theta}}{r}\right| + \left|\frac{\omega_{0,\theta}}{r} - \frac{\omega_{0,\theta}^\varepsilon}{r}\right| \right) dx\,.$$

Hence, by (16),(25) for any $\kappa>0$ there exists $\varepsilon_0>0, \delta_0>0$ such that

$$|B_{\varepsilon,\delta}| \le \frac{\kappa}{4a} \quad \forall\varepsilon\in(0,\varepsilon_0),\ \delta\in(0,\delta_0)\,, \tag{26}$$

where $a = \hat{C}(\|\frac{\omega_0}{r}\|_{L^1(\mathbb{R}^3)}+1)$. On other hand, from

$$\int_{(G\times G)\cap\{|r-r'|+|x_3-x_3'|\le c\delta\}} |(\Phi\omega_\theta^\varepsilon)(t,r,x_3)(\Phi\omega_\theta^\varepsilon)(t,r',x_3')|\, dr\, dx_3\, dr'\, dx_3'\, dt$$

after the change of variables we obtain

$$\begin{aligned}
&\int_0^T\int_G |(\Phi\omega_\theta^\varepsilon)(t,r',x_3')|(\int_{\{|\tilde r|+|\tilde x_3|\le c\delta\}} |(\Phi\omega_\theta^\varepsilon)(t,\tilde r+r',\tilde x_3+x_3')|d\tilde r d\tilde x_3)\,dr'\,dx_3'dt\\
&\le C\left\|\frac{\omega_0^\varepsilon}{r}\right\|_{L^1(\mathbb{R}^3)}\int_0^T(\sup_{(r',x_3')\in G}\int_{\{|\tilde r|+|\tilde x_3|\le c\delta\}}|(\Phi\omega_\theta^\varepsilon)(t,\tilde r+r',\tilde x_3+x_3')|\,d\tilde r\,d\tilde x_3)\,dt\\
&\le C\left\|\frac{\omega_0^\varepsilon}{r}\right\|_{L^1(\mathbb{R}^3)}\int_0^T(\sup_{(r',x_3')\in G}\int_{\{|\tilde r|+|\tilde x_3|\le c\delta\}}|\omega_\theta^\varepsilon(t,\tilde r+r',\tilde x_3+x_3')|\,d\tilde r\,d\tilde x_3)\,dt\\
&\le C\left\|\frac{\omega_0^\varepsilon}{r}\right\|_{L^1(\mathbb{R}^3)}\int_0^T\sup_{(r',x_3')\in G}\int_{\{(r,x_3)\in G||r-r'|+|x_3-x_3'|\le c\delta\}}\left|\frac{\omega_{0,\theta}^\varepsilon}{r}(X_\varepsilon^{-1}(t,x))\right|\,dx\,dt\,. \qquad (27)
\end{aligned}$$

Set $\mu(\{x\in\mathbb{R}^3||r-r'|+|x_3-x_3'|\le c\delta\})=\gamma(\delta)$. Then

$$\begin{aligned}
&\int_{(G\times G)\cap\{|r-r'|+|x_3-x_3'|\le c\delta\}}|(\Phi\omega_\theta^\varepsilon)(t,r,x_3)||(\Phi\omega_\theta^\varepsilon)(t,r',x_3')|\,dr\,dx_3\,dr'\,dx_3'\,dt\\
&\qquad\le \hat{C}T\left\|\frac{\omega_0^\varepsilon}{r}\right\|_{L^1(\mathbb{R}^3)}\sup_{\substack{\mathfrak{B}\\ \mu(\mathfrak{B})\le\gamma(\delta)}}\int_{\mathfrak{B}}\left|\frac{\omega_0^\varepsilon}{r}\right|\,dx\,. \qquad (28)
\end{aligned}$$

Since $\gamma(\delta)\to 0$ as $\delta\to 0$ for any $\kappa>0$, one can find $\delta_0>0$ and $\varepsilon_0>0$ such that right hand side of (28) is less than or equal to $\frac{\kappa}{4}$ for all $\delta\in(0,\delta_0)$ and $\varepsilon\in(0,\varepsilon_0)$. Then, taking into account (28) , we obtain (21).

On the other hand, we have

$$K(t,r,x_3,r'x_3')\left(1-\eta\left(\frac{r}{\delta}\right)\right)\left(1-\eta\left(\frac{r'}{\delta}\right)\right)\left(1-\eta\left(\frac{|r-r'|+|x_3-x_3'|}{\delta}\right)\right)
\in C^\infty([0,T]\times\overline{G}\times\overline{G}).$$

Hence

$$I_1^{\varepsilon,\delta}\to I_1^\delta \quad \text{as } \varepsilon\to 0. \qquad (29)$$

Thus by (21) and (29),

$$A_1^\varepsilon\to A_1.$$

The proof of the lemma is complete. ■

Let us introduce a class of axisymmetric vector fields without a swirl component, $L_a^2(\mathbb{R}^3)=\{v\in(L^2(\mathbb{R}^3))^3\,|\,v=v(r,x_3),v_\theta=0\}$. For a given N-function $A(t)$, following [3], we introduce

$$\mathcal{Q}_A(\mathbb{R}^3)=\{v\in L_a^2(\mathbb{R}^3)\cap W^1L_A(\mathbb{R}^3)\,|\,\mathbf{div}\,v=0,\mathbf{curl}\,v\in L_A(\mathbb{R}^3)\}$$

equipped with the Banach space norm $\|v\|_{\mathcal{Q}_A(\mathbb{R}^3)}=(\|v\|^2_{L^2(\mathbb{R}^3)}+\|\mathbf{curl}\,v\|^2_{L^2_A(\mathbb{R}^3)})^{1/2}$. Here the derivatives are in the distribution sense. We can extend our definition to $\mathcal{Q}_A(\Omega)$ for any axisymmetric domain Ω in $\mathbb{R}^3$.

Now, we establish the following compactness lemma, which is an axisymmetric analogue of Lemma 6. of [3].

Lemma 2.3. *Let $A(t)$ be an N-function satisfying the Δ_2-condition, and satisfies $A(t) \succ t(\log^+ t)^{\frac{1}{2}}$. Then for any bounded sequence $\{v^\varepsilon\}$ in $\mathcal{Q}_A(\mathbb{R}^3)$ there exists a subsequence, denoted by the same notation, $\{v^\varepsilon\}$ and $v \in \mathcal{Q}_A(\mathbb{R}^3)$ such that*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3} \rho|v^\varepsilon|^2\,dx = \int_{\mathbb{R}^3}\rho|v|^2\,dx$$

for any given axisymmetric test function $\rho \in C_0^\infty(\mathbb{R}^3)$ with $\operatorname{supp}\rho \subset \{(r,x_3) \in \mathbb{R}^2 \,|\, r > 0\}$.

Proof. Let $\{v^\varepsilon\}$ be a uniformly bounded sequence in $\mathcal{Q}_A(\mathbb{R}^3)$. Then, there exists a subsequence, denoted by $\{v^\varepsilon\}$, and v in $\mathcal{Q}_A(\mathbb{R}^3)$ such that

$$v^\varepsilon \to v \quad \text{weakly in } L^2(\mathbb{R}^3)\,. \tag{30}$$

For such $v^{(\varepsilon)}$ we introduce stream functions $\psi^{(\varepsilon)} = \psi^{(\varepsilon)}(r,x_3)$ such that

$$v_r^{(\varepsilon)} = -\frac{1}{r}\frac{\partial\psi^{(\varepsilon)}}{\partial x_3}, \quad v_3^{(\varepsilon)} = \frac{1}{r}\frac{\partial\psi^{(\varepsilon)}}{\partial r}.$$

Let a function $\rho \in C_0^\infty(\mathbb{R}^3)$ and a bounded domain W with $\operatorname{supp}\rho \subset \overline{W} \subset G$ be given. Then, by integration by part we obtain

$$\begin{aligned}
\int_{\mathbb{R}^3}\rho|v^{(\varepsilon)}|^2\,dx &= \int_{\mathbb{R}^3}\rho((v_r^{(\varepsilon)})^2 + (v_3^{(\varepsilon)})^2)dx \\
&= 2\pi\int_{\mathbb{R}^3}\left(-\rho v_r^{(\varepsilon)}\frac{1}{r}\frac{\partial\psi^{(\varepsilon)}}{\partial x_3} + \rho v_3^{(\varepsilon)}\frac{1}{r}\frac{\partial\psi^{(\varepsilon)}}{\partial r}\right) r\,dr\,dx_3 \\
&= 2\pi\int_{\mathbb{R}^3}\bigg(\frac{\partial\rho}{\partial x_3}v_r^{(\varepsilon)}\psi^{(\varepsilon)} - \frac{\partial\rho}{\partial r}v_3^{(\varepsilon)}\psi^{(\varepsilon)} \\
&\qquad + \rho\frac{\partial v_r^{(\varepsilon)}}{\partial x_3}\psi^{(\varepsilon)} - \rho\frac{\partial v_3^{(\varepsilon)}}{\partial r}\psi^{(\varepsilon)}\bigg)\,dr\,dx_3 \\
&= 2\pi\int_G\left(\frac{\partial\rho}{\partial x_3}v_r^{(\varepsilon)}\psi^{(\varepsilon)} - \frac{\partial\rho}{\partial r}v_3^{(\varepsilon)}\psi^{(\varepsilon)}\right)\,dr\,dx_3 \\
&\qquad + 2\pi\int_G \omega_\theta^{(\varepsilon)}\psi^{(\varepsilon)}\rho\,dr\,dx_3 = \{1\}^{(\varepsilon)} + \{2\}^{(\varepsilon)}\,.
\end{aligned} \tag{31}$$

Since

$$\|\nabla\psi^\varepsilon\|_{L^2(W)} \le C(W)\left\|\frac{\nabla\psi^\varepsilon}{r}\right\|_{L^2(W)} = C(W)\|v^\varepsilon\|_{L^2(W)} \le C,$$

we obtain by Rellich's compact imbedding lemma that

$$\rho_1\psi^\varepsilon \to \rho_1\psi \quad \text{strongly in} \quad L^2(W) \quad \forall\rho_1 \in C_0^\infty(W)$$

after choosing a subsequence. This, combined with (30), provides easily that $\{1\}^\varepsilon \to \{1\}$ in (31) as $\varepsilon \to 0$.

To prove $\{2\}^\varepsilon \to \{2\}$ we observe that

$$\rho\psi^\varepsilon \to \rho\psi\,, \tag{32}$$

and

$$\|\omega_\theta^\varepsilon\|_{L_{t(\log^+ t)^{\frac12}}(W)} \le C\|\omega_\theta^\varepsilon\|_{L_A(W)} \le C_2\,, \tag{33}$$

where $B(t) = \exp(t^2) - 1$. Since $A(t) = t(\log^+ t)^\alpha \succ t(\log^+ t)^{\frac12}$ by hypothesis, applying Corollary 1.1, we find that there exists a subsequence $\{\omega_\theta^\varepsilon\}$ and ω_θ in $\mathrm{H}^{-1}(W) \hookleftarrow\hookleftarrow L_A(W)$ such that

$$\omega_\theta^\varepsilon \to \omega_\theta \quad \text{in} \quad \mathrm{H}^{-1}(W)\,. \tag{34}$$

We decompose our estimate

$$\begin{aligned}\left|\int_G (\omega_\theta^\varepsilon\psi^\varepsilon - \omega_\theta\psi)\rho\,dr\,dx_3\right| &\le \left|\int_W (\omega_\theta^\varepsilon - \omega_\theta)\psi^\varepsilon\rho\,dr\,dx_3\right| + \left|\int_W (\psi^\varepsilon - \psi)\omega_\theta\rho\,dr\,dx_3\right| \\ &= J_1^\varepsilon + J_2^\varepsilon.\end{aligned}$$

From (32) and (34) we obtain

$$J_1^\varepsilon \le C\|\psi^\varepsilon\|_{H^1(W)}\|\omega_\theta^\varepsilon - \omega_\theta\|_{\mathrm{H}^{-1}(W)} \to 0$$

after choosing a subsequence, if necessary. On the other hand, the convergence $J_2^\varepsilon \to 0$ for another subsequence, if necessary, follows from (32). This completes the proof of the lemma. ■

Using Lemma 2.3 we establish the following

Lemma 2.4. *Suppose a sequence $\{v^\varepsilon\}$ and v be given as in (1.7), Lemma 2.2. Let $\eta(r) \in C^\infty(\mathbb{R}_+)$, $\eta(r) \ge 0$, $\eta(r) = 1$, $r \in [1,\infty]$ and $\eta(r) = 0$ for $r < \frac12$. Then for any $\delta > 0$ and $\varphi \in C^\infty([0,T]; C_0^\infty(\mathbb{R}^3))$ we have*

$$\int_Q \eta\left(\frac{r}{\delta}\right)|v^\varepsilon - v|^2\varphi\,dx\,dt \to 0 \;\; \text{as } \varepsilon \to +0\,, \tag{35}$$

after choosing a subsequence.

Proof. Let W be any given bounded domain in G whose closure does not intersect with the axis of symmetry. By conservation of $L^2(\mathbb{R}^3)$ norm of velocity we have

$$\|v^\varepsilon(t,\cdot)\|^2_{L^2(W)} \le C(W)\|v^\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} = C(W)\|v_0^\varepsilon\|^2_{L^2(\mathbb{R}^3)} \le C(W, v_0)\,. \tag{36}$$

On the other hand, the conservation of $\frac{\omega_\theta^\varepsilon(t,x)}{r}$ along the flow, (9), implies

$$\|\omega^\varepsilon(t,\cdot)\|_{L_A(W)} \le C(W)\left\|\frac{\omega^\varepsilon}{r}(t,\cdot)\right\|_{L_A(\mathbb{R}^3)} = C(W)\left\|\frac{\omega_0^\varepsilon}{r}\right\|_{L_A(\mathbb{R}^3)} \le C(W,v_0), \tag{37}$$

where $A = A(t) = t(\log^+ t)^\alpha$. Combining (36) and (37), we find that

$$\sup_{t\in[0,T]} \|v^\varepsilon(t,\cdot)\|_{\mathcal{Q}_A(W)} \le C. \tag{38}$$

From the estimate (38), combined with (19), together with Lemma 2.3, we deduce by using the standard compactness lemma that there is a subsequence $\{v^\varepsilon(t,r,x_3)\}$ such that

$$v^\varepsilon \to v \text{ strongly in } L^2([0,T]\times W)\,.$$

Now (35) follows from this immediately. The lemma is proved. ■

Proof of Theorem 1.1 To prove the theorem we have only to show that

$$I^\varepsilon = \int_Q v_i^\varepsilon v_j^\varepsilon \varphi \, dx\, dt \to I = \int_Q v_i v_j \varphi \, dx\, dt\,, \tag{39}$$

for all $i, j \in \{1,2,3\}$, and $\varphi \in C^\infty([0,T]; C_0^\infty(\mathbb{R}^3))$. Let $\eta(\tau) \in C^\infty(\mathbb{R}^1_+), 0 \le \eta \le 1, \eta(\tau) = 1$ for all $\tau \in [1, +\infty)$ and $\eta(\tau) = 0$ for $\tau \in [0, \frac{1}{2}]$. For any $\delta > 0$ we set

$$I^\varepsilon = I_1^{\varepsilon,\delta} + I_2^{\varepsilon,\delta} = \int_Q \eta\left(\frac{r}{\delta}\right) v_i^\varepsilon v_j^\varepsilon \varphi dx + \int_Q \left(1 - \eta\left(\frac{r}{\delta}\right)\right) v_i^\varepsilon v_j^\varepsilon \varphi \, dx\,.$$

By Lemma 2.4

$$I_1^{\varepsilon,\delta} \to \int_Q \eta\left(\frac{r}{\delta}\right) v_i v_j \varphi dx \text{ as } \varepsilon \to 0\,. \tag{40}$$

Hence the statement of theorem will be proved, if we show that for any $\kappa > 0$ there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ one can find $\varepsilon_0(\delta) > 0$ that

$$|I_2^{\varepsilon,\delta}| \le \kappa \quad \forall \varepsilon \in (0, \varepsilon_0)\,. \tag{41}$$

Indeed, in case either i or j equals 1 or 2, by Lemma 2.1 we have

$$\begin{aligned}
\left| \int_Q \left(1 - \eta\left(\frac{r}{\delta}\right)\right) v_i^\varepsilon v_j^\varepsilon \varphi \, dx \right| &\le 2\pi \int_0^T \int_{-\infty}^{+\infty} \int_0^\delta r |v_r^\varepsilon v_j^\varepsilon \varphi| \, dr\, dx_3\, dt \\
&\le C\delta \int_0^T \int_{-\infty}^{+\infty} \int_0^\delta |\varphi v_r^\varepsilon v_j^\varepsilon| \, dr\, dx_3\, dt \\
&\le C\delta \left(\int_0^T \int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{1}{1 + x_3^2} \frac{|v_r^\varepsilon|^2}{r} \, dr\, dx_3\, dt \right)^{\frac{1}{2}} \|v_0^\varepsilon\|_{V^0} \\
&\le C\delta \left(\|v_0\|_{V^0}^2 + \left\| \frac{\omega_0}{r} \right\|_{L^1(\mathbb{R}^3)} + 1 \right)^{\frac{1}{2}} (\|v_0\|_{V^0} + 1)\,.
\end{aligned} \tag{42}$$

Hence taking parameter $\varepsilon_0 = 1$ and parameter δ_0 sufficiently small, we obtain (41).

Let us consider the case $i = j = 3$. Then

$$|I_2^{\varepsilon,\delta}| \le \hat{C} \int_Q \left| 1 - \eta\left(\frac{r}{\delta}\right) \right| (v_3^\varepsilon)^2 \, dx\, dt. \tag{43}$$

Set $\rho(r) = 1 - \eta(r)$. Let $\delta_1 > 0$ be such that

$$\left| \int_Q \rho\left(\frac{r}{\delta}\right) ((v_r)^2 - (v_3)^2) \, dx\, dt \right| \le \frac{\kappa}{4\hat{C}} \quad \forall \delta \in (0, \delta_1). \tag{44}$$

In the above we also proved that for each $\kappa > 0$ there exists $\delta_2 > 0$ such that

$$\int_Q (v_r^\varepsilon)^2 \rho\left(\frac{r}{\delta}\right) \, dx\, dt \le \frac{\kappa}{4\hat{C}} \; \forall \delta \in (0, \delta_2),\; \varepsilon \in (0,1). \tag{45}$$

Let $\delta_0 = \min\{\delta_1, \delta_2\}$.

Note that by Lemma 2.2 for $\delta \in (0, \delta_0)$

$$\int_Q [(v_r^\varepsilon)^2 - (v_3^\varepsilon)^2]\rho\left(\frac{r}{\delta}\right)\, dx\, dt \to \int_Q [(v_r)^2 - (v_3)^2]\rho\left(\frac{r}{\delta}\right)\, dx\, dt\,,$$

as $\varepsilon \to +0$.

Thus for every $\kappa > 0$ and $\delta \in (0, \delta_0)$ one can find $\varepsilon_0(\delta)$ that

$$\left|\int_Q [(v_r^\varepsilon)^2 - (v_3^\varepsilon)^2 - (v_r)^2 + (v_3)^2]\rho\left(\frac{r}{\delta}\right)\, dx\, dt\right| \le \frac{\kappa}{4\hat{C}} \quad \forall\, \varepsilon \in (0, \varepsilon_0(\delta)). \tag{46}$$

Inequalities (43)-(46) imply (41). Since now (36) is proved for an arbitrary $i, j \in \{1, 2, 3\}$, the proof of the existence part of Theorem 2.1 is complete. The inequalities for the energy and the vorticity follow immediately by the energy conservation for velocity and the conservation of $\frac{\omega_\theta}{r}$ for the smooth approximate solutions, and taking limit for suitable subsequence. This completes the proof of the theorem. ■

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