Electronic Journal of Differential Equations, Vol. 1998(1998), No. 35, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
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EIGENVALUE COMPARISONS FOR DIFFERENTIAL EQUATIONS ON A MEASURE CHAIN

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Abstract. The theory of $\mathbf{u_0}$-positive operators with respect to a cone in a Banach space is applied to eigenvalue problems associated with the second order Δ-differential equation (often referred to as a differential equation on a measure chain) given by

$$y^{\Delta\Delta}(t) + \lambda p(t)y(\sigma(t)) = 0, \qquad t \in [0,1],$$

satisfying the boundary conditions $y(0) = 0 = y(\sigma^2(1))$. The existence of a smallest positive eigenvalue is proven and then a theorem is established comparing the smallest positive eigenvalues for two problems of this type.

1. Background

In this paper, we are concerned with comparing the smallest positive eigenvalues for second order Δ-differential equations satisfying conjugate boundary conditions. Much recent attention has been given to differential equations on measure chains, and we refer the reader to [4, 8, 15] for some historical works as well as to the more recent papers [1, 9, 10] and the book [17] for excellent references on these types of equations. Before introducing the problems of interest for this paper, we present some definitions and notation which are common to the recent literature. Our sources for this background material are the two papers by Erbe and Peterson [9, 10].

Definition 1.1. Let T be a closed subset of $\mathbb{R}$, and let T have the subspace topology inherited from the Euclidean topology on $\mathbb{R}$. The set T is referred to as a *measure chain* or, in some places in the literature, a *time scale*. For $t < \sup T$ and $r > \inf T$, define the *forward jump operator*, σ, and the *backward jump operator*, ρ, respectively, by

$$\sigma(t) = \inf\{\tau \in T \mid \tau > t\} \in T,$$
$$\rho(r) = \sup\{\tau \in T \mid \tau < r\} \in T,$$

for all $t, r \in T$. If $\sigma(t) > t$, t is said to be *right scattered*, and if $\rho(r) < r$, r is said to be *left scattered*. If $\sigma(t) = t$, t is said to be *right dense*, and if $\rho(r) = r$, r is said to be *left dense*.

1991 *Mathematics Subject Classification.* 34B99, 39A99 .
Key words and phrases. Measure chain, eigenvalue problem .

Submitted November 23, 1998. Published December 19, 1998.

Definition 1.2. For $x : T \to \mathbb{R}$ and $t \in T$ (if $t = \sup T$, assume t is not left scattered), define the *delta derivative of* $x(t)$, denoted by $x^{\Delta}(t)$, to be the number (when it exists), with the property that, for any $\epsilon > 0$, there is a neighborhood, U, of t such that

$$\Big|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]\Big| \le \epsilon\Big|\sigma(t) - s\Big|,$$

for all $s \in U$. The *second delta derivative* of $x(t)$ is defined by

$$x^{\Delta\Delta}(t) = (x^{\Delta})^{\Delta}(t).$$

If $F^{\Delta}(t) = h(t)$, then define the *integral* by

$$\int_a^t h(s)\Delta s = F(t) - F(a).$$

Throughout, we will assume that T is a closed subset of $\mathbb{R}$ with $0, 1 \in T$.

Definition 1.3. Define the closed interval, $[0, 1] \subset T$ by

$$[0, 1] := \{t \in T \mid 0 \le t \le 1\}.$$

Other closed, open, and half-open intervals in T are similarly defined.

For convenience, we will use interval notation, $[0, 1]$ and inequalities such as $0 \le t \le 1$ interchangeably.

We are concerned with the comparison of the eigenvalues for the eigenvalue problems

$$y^{\Delta\Delta}(t) + \lambda_1\, p(t)y(\sigma(t)) = 0, \qquad t \in [0, 1], \tag{1.1}$$

$$y^{\Delta\Delta}(t) + \lambda_2\, q(t)y(\sigma(t)) = 0, \qquad t \in [0, 1], \tag{1.2}$$

satisfying the two-point conjugate boundary conditions

$$y(0) = 0 = y(\sigma^2(1)), \tag{1.3}$$

where we assume $0 < p(t) \le q(t)$ for $t \in [0, 1]$.

To be more precise, we will first establish the existence of smallest positive eigenvalues for (1.1), (1.3) and (1.2), (1.3), respectively, and then we will compare these smallest positive eigenvalues. Our techniques involve applications from the theory of $\mathbf{u_0}$-positive operators with respect to a cone in a Banach space as it is developed in Krasnosel'skii's book [20] or in the book by Krein and Rutman [21]. Also, we make use of the sign properties of an appropriate Green's function.

Results of this type are not without motivation. The cone theory techniques we apply here have been successfully applied by several authors in comparing eigenvalues for boundary value problems for ordinary differential equations including two-point, multipoint, focal, right focal, and Lidstone problems; for example, see [2, 3, 6, 7, 11, 12, 16, 18, 19, 22, 23, 24, 25]. In addition, a few smallest eigenvalue comparison results have been obtained for boundary value problems for finite difference equations. A representative set of references for these works would be [5, 13, 14].

In the development of this paper, we include in Section 2 preliminary definitions and fundamental results from the theory of $\mathbf{u_0}$-positive operators with respect to a cone in a Banach space. Then, in Section 3, we apply the results of Section 2 in comparing the smallest positive eigenvalues of (1.1), (1.3) and (1.2), (1.3).

2. Cones and $\mathbf{u_0}$-Positive Operators

In this section, we provide definitions and auxillary results from cone theory which we will apply in the next section to the eigenvalue problems (1.1), (1.3) and (1.2), (1.3). Most of the discussion of this section involving the theory of cones in a Banach space arises from results in Krasnosel'skii's book [20].

Definition 2.1. Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$. A nonempty, closed set $\mathcal{P} \subset \mathcal{B}$ is said to be a *cone* provided

(i) $\alpha\mathbf{u} + \beta\mathbf{v} \in \mathcal{P}$ for all $\mathbf{u}, \mathbf{v} \in \mathcal{P}$ and all $\alpha, \beta \geq 0$, and
(ii) $\mathbf{u}, -\mathbf{u} \in \mathcal{P}$ implies $\mathbf{u} = \mathbf{0}$.

A cone is said to be *reproducing* if $\mathcal{B} = \mathcal{P} - \mathcal{P}$. A cone is said to be *solid* if $\mathcal{P}^\circ \neq \emptyset$, where $\mathcal{P}^\circ$ denotes the interior of $\mathcal{P}$.

Remark. *Krasnosel'skii* [20] *proved that every solid cone is reproducing.*

Definition 2.2. A Banach space $\mathcal{B}$ is called a *partially ordered Banach space* if there exists a partial ordering $\preceq$ on $\mathcal{B}$ satisfying

(i) $\mathbf{u} \preceq \mathbf{v}$, for $\mathbf{u}, \mathbf{v} \in \mathcal{B}$ implies $t\mathbf{u} \preceq t\mathbf{v}$, for all $t \geq 0$, and
(ii) $\mathbf{u}_1 \preceq \mathbf{v}_1$ and $\mathbf{u}_2 \preceq \mathbf{v}_2$, for $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{B}$ imply $\mathbf{u}_1 + \mathbf{u}_2 \preceq \mathbf{v}_1 + \mathbf{v}_2$.

Let $\mathcal{P} \subset \mathcal{B}$ be a cone and define $\mathbf{u} \preceq \mathbf{v}$ if and only if $\mathbf{v} - \mathbf{u} \in \mathcal{P}$. Then $\preceq$ is a partial ordering on $\mathcal{B}$ and we will say that $\preceq$ is the partial ordering induced by $\mathcal{P}$. Moreover, $\mathcal{B}$ is a partially ordered Banach space with respect to $\preceq$.

Definition 2.3. If $L_1, L_2 : \mathcal{B} \to \mathcal{B}$ are bounded, linear operators, then we say that $L_1 \preceq L_2$ *with respect to* $\mathcal{P}$ provided $L_1\mathbf{u} \preceq L_2\mathbf{u}$ for every $\mathbf{u} \in P$. A bounded, linear operator $L_1 : \mathcal{B} \to \mathcal{B}$ is $\mathbf{u_0}$*-positive with respect to* $\mathcal{P}$ if there exists $\mathbf{u}_0 \in \mathcal{P}$, $\mathbf{u}_0 \neq \mathbf{0}$, such that for each nonzero $\mathbf{u} \in \mathcal{P}$, there exist $k_1(\mathbf{u}), k_2(\mathbf{u})$ such that $k_1\mathbf{u}_0 \preceq L_1\mathbf{u} \preceq k_2\mathbf{u}_0$.

Of the next three results, the first two can be found in Krasnosel'skii's book [20] and the third result is proved by Keener and Travis [18] as an extension of results from [20].

Theorem 2.1. *Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$ and let $\mathcal{P} \subset \mathcal{B}$ be a solid cone. If $L_1 : \mathcal{B} \to \mathcal{B}$ is a linear operator such that $L_1 : \mathcal{P} \setminus \{\mathbf{0}\} \to \mathcal{P}^\circ$, then L_1 is $\mathbf{u_0}$-positive with respect to $\mathcal{P}$.*

Theorem 2.2. *Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$ and let $\mathcal{P} \subset \mathcal{B}$ be a reproducing cone. Let $L_1 : \mathcal{B} \to \mathcal{B}$ be a compact, linear operator which is $\mathbf{u_0}$-positive with respect to $\mathcal{P}$. Then L_1 has an essentially unique eigenvector in $\mathcal{P}$, and the corresponding eigenvalue is simple, positive, and larger than the absolute value of any other eigenvalue.*

Theorem 2.3. *Let $\mathcal{B}$ be a Banach space over $\mathbb{R}$ and let $\mathcal{P} \subset \mathcal{B}$ be a cone. Let $L_1, L_2 : \mathcal{B} \to \mathcal{B}$ be bounded, linear operators, and assume that at least one of the operators is $\mathbf{u_0}$-positive with respect to $\mathcal{P}$. If $L_1 \preceq L_2$ with respect to $\mathcal{P}$, and if there exist nonzero $\mathbf{u_1}, \mathbf{u_2} \in \mathcal{P}$ and positive real numbers λ_1 and λ_2 such that $\lambda_1\mathbf{u_1} \preceq L_1\mathbf{u_1}$ and $L_2\mathbf{u_2} \preceq \lambda_2\mathbf{u_2}$, then $\lambda_1 \leq \lambda_2$. Moreover, if $\lambda_1 = \lambda_2$, then $\mathbf{u_1}$ is a scalar multiple of $\mathbf{u_2}$.*

3. Eigenvalue Comparisons for the Boundary Value Problems

In order to apply the results of Section 2 concerning the theory of $\mathbf{u_0}$-positive operators, we now introduce a suitable Banach space, $\mathcal{B}$, and a cone, $\mathcal{P}$, in the Banach space. Define $\mathcal{B}$ by

$$\mathcal{B} := \{x : [0, \sigma^2(1)] \to \mathbb{R} \mid x^\Delta \text{ exists and is bounded on } [0, \sigma(1)], \quad \text{and } x \text{ satisfies the boundary conditions (1.3)}\}$$

and let the norm $||\cdot||$ on $\mathcal{B}$ be defined by

$$||x|| := \max\Big\{ \sup_{t\in[0,\sigma^2(1)]} |x(t)|, \sup_{t\in[0,\sigma(1)]} |x^\Delta(t)| \Big\}.$$

Notice that if $||x|| = 0$, then $x(t) \equiv 0$. Define the cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} := \{x \in \mathcal{B} \mid x(t) \geq 0 \text{ for } t \in [0, \sigma^2(1)]\}.$$

Lemma 3.1. *The cone $\mathcal{P}$ has nonempty interior and*

$$Q := \{x \in \mathcal{P} \mid x(t) > 0 \ \text{on}\ (0, \sigma^2(1)),\ x^\Delta(0) > 0,\ x^\Delta(\sigma(1)) < 0\} \subset \mathcal{P}^\circ.$$

Proof. Choose $x(t) \in Q$. Our only concern is the positivity of $x(t)$ in a right deleted neighborhood of $t = 0$ and in a left deleted neighborhood of $t = \sigma^2(1)$. If $t = 0$ is right dense, then by the definition of Q we have $x'(0) > 0$. If $t = 0$ is right scattered, then $x(\sigma(0)) > 0$. In either case, $x(t) > 0$ on any right deleted neighborhood of $t = 0$. Now consider the right endpoint. If $t = \sigma^2(1)$ is left dense, then $x^\Delta(\sigma(1)) = x'(\sigma^2(1)) < 0$. If $t = \sigma^2(1)$ is left scattered, then $x(\sigma(1)) > 0$. Again, in either case, $x(t) > 0$ on any left deleted neighborhood of $t = \sigma^2(1)$. □

Corollary 3.1. *The cone $\mathcal{P}$ is solid and hence reproducing.*

Next we define the linear operators $L_1, L_2 : \mathcal{B} \to \mathcal{B}$ by

$$L_1x(t) = \int_0^{\sigma(1)} G(t,s)p(s)x(\sigma(s))\Delta s, \tag{3.1}$$

$$L_2x(t) = \int_0^{\sigma(1)} G(t,s)q(s)x(\sigma(s))\Delta s, \tag{3.2}$$

respectively, where $G(t,s)$ is the Green's function for

$$-x^{\Delta\Delta}(t) = 0$$

satisfying (1.3). That is,

$$G(t,s) = \begin{cases} \frac{t\left(\sigma^2(1)-\sigma(s)\right)}{\sigma^2(1)}, & 0 \leq t \leq s \leq \sigma(1), \\ \frac{\sigma(s)\left(\sigma^2(1)-t\right)}{\sigma^2(1)}, & 0 \leq \sigma(s) \leq t \leq \sigma^2(1), \end{cases}$$

on $[0, \sigma^2(1)] \times [0, \sigma(1)]$; see Erbe and Peterson [9, 10]. Note that

$$G(t,s) > 0 \quad \text{on } (0, \sigma^2(1)) \times (0, \sigma(1)).$$

Lemma 3.2. *Let λ_1 be an eigenvalue of* (1.1), (1.3) *and $u(t)$ be the corresponding eigenvector. Then*

$$u(t) = \lambda_1 \int_0^{\sigma(1)} G(t,s)p(s)u(\sigma(s))\Delta s.$$

That is, $\frac{1}{\lambda_1}u = L_1u$. *Hence, the eigenvalues of* (1.1), (1.3) *are reciprocals of the eigenvalues of* (3.1) *and conversely.*

Lemma 3.3. *The linear operators* L_1 *and* L_2 *are* $\mathbf{u_0}$*-positive with respect to* $\mathcal{P}$.

Proof. We prove the statement is true for the operator L_1. By Theorem 2.1, we only need to show that $L_1 : \mathcal{P} \setminus \{\mathbf{0}\} \to \mathcal{P}^\circ$. To this end, choose $v \in \mathcal{P} \setminus \{\mathbf{0}\}$. Then, for $t \in (0, \sigma^2(1))$,

$$L_1v(t) = \int_0^{\sigma(1)} G(t,s)p(s)v(\sigma(s))\Delta s > 0.$$

A direct computation yields

$$G^\Delta(0,s) = \frac{\sigma^2(1) - \sigma(s)}{\sigma^2(1)} > 0, \qquad 0 \le s < 1, \tag{3.3}$$

$$G^\Delta(\sigma(1),s) = -\frac{\sigma(s)}{\sigma^2(1)} < 0, \qquad 0 < s \le \sigma(1). \tag{3.4}$$

By (3.3), we obtain

$$\begin{aligned}(L_1v)^\Delta(0) &= \int_0^{\sigma(1)} G^\Delta(0,s)p(s)v(\sigma(s))\Delta s \\ &= \int_0^{\sigma(1)} \frac{\sigma^2(1) - \sigma(s)}{\sigma^2(1)} p(s)v(\sigma(s))\Delta s \\ &> 0.\end{aligned}$$

Similarly, $(L_1v)^\Delta(\sigma(1)) < 0$ by using (3.4). Hence $L_1v \in Q \subset \mathcal{P}^\circ$. □

By the way the operators were defined, $L_1, L_2 : \mathcal{P} \to \mathcal{P}$ and therefore L_1 and L_2 are bounded. It follows from standard arguments involving the Arzela-Ascoli Theorem that L_1 and L_2 are in fact compact operators. We may now apply Theorems 2.2 and 2.3 to obtain the eigenvalue comparison we seek.

Theorem 3.1. *Suppose* $0 < p(t) \le q(t)$ *for* $0 \le t \le 1$. *Then the operator* L_1 *has an essentially unique eigenvector* $u \in \mathcal{P}^\circ \setminus \{\mathbf{0}\}$, *and the corresponding eigenvalue* Λ *is simple, positive, and larger than the absolute value of any other eigenvalue.*

Proof. The existence of such an eigenvalue Λ with eigenvector $u \in \mathcal{P}$ follows from Theorem 2.2. Since $u \not\equiv \mathbf{0}$, the proof of Lemma 3.3 shows $L_1u \in \mathcal{P}^\circ$. Since $L_1u = \Lambda u$, it follows that $u \in \mathcal{P}^\circ$. □

Theorem 3.2. *Suppose* $0 < p(t) \le q(t)$ *for* $0 \le t \le 1$. *Let* Λ_1 *and* Λ_2 *be the largest positive eigenvalues of* L_1 *and* L_2, *respectively. Then* $\Lambda_1 \le \Lambda_2$. *Furthermore,* $\Lambda_1 = \Lambda_2$ *if and only if* $p(t) \equiv q(t)$ *for* $0 \le t \le 1$.

Proof. Let Λ_1 and Λ_2 be as in the statement of the theorem. Since by assumption $p(t) \le q(t)$, we have, for $u \in \mathcal{P}$,

$$\begin{aligned}L_1u(t) &= \int_0^{\sigma(1)} G(t,s)p(s)u(\sigma(s))\Delta s \\ &\le \int_0^{\sigma(1)} G(t,s)q(s)u(\sigma(s))\Delta s \\ &= L_2u(t)\end{aligned}$$

and hence $L_1 \preceq L_2$ with respect to $\mathcal{P}$. If $u_1, u_2 \in \mathcal{P}^\circ$ are the essentially unique eigenvectors given by Theorem 3.1 that correspond to Λ_1 and Λ_2, respectively. Theorem 2.3 then yields $\Lambda_1 \leq \Lambda_2$.

For the final statement of the theorem, suppose that $p(t_0) < q(t_0)$ for some $t_0 \in (0,1)$. The proof of Theorem 3.1 shows $L_1 u_1(t_0) > 0$. It can be argued just as in Lemma 3.3 that $(L_2 - L_1)u_1 \in \mathcal{P}^\circ$. But $u_1 \in \mathcal{P}^\circ$ so for sufficiently small $\varepsilon > 0$, it must be that $(L_2 - L_1)u_1 \geq \varepsilon u_1$. Therefore

$$L_2 u_1 \geq L_1 u_1 + \varepsilon u_1 = (\Lambda_1 + \varepsilon) u_1.$$

Since $L_2 u_2 = \Lambda_2 u_2$, if we apply Theorem 2.2 to the operator L_2 we have $\Lambda_1 + \varepsilon \leq \Lambda_2$ or equivalently $\Lambda_1 < \Lambda_2$. Conversely, $\Lambda_1 = \Lambda_2$ implies $p(t) = q(t)$ for all $t \in (0,1)$. □

In view that the eigenvalues of L_1 are reciprocals of the eigenvalues of (1.1), (1.3), and conversely, and in view of Theorems 3.1 and 3.2, we see that

$$\lambda_1 = \frac{1}{\Lambda_1} \geq \frac{1}{\Lambda_2} = \lambda_2.$$

Moreover, if $p(t) \leq q(t)$ and $p(t) \not\equiv q(t)$, then

$$\frac{1}{\Lambda_1} > \frac{1}{\Lambda_2}.$$

We are now able to state the following comparison theorem for smallest positive eigenvalues, λ_1 and λ_2, of (1.1), (1.3) and (1.2), (1.3).

Theorem 3.3. *Assume the hypotheses of Theorem 3.2. Then there exist smallest positive eigenvalues λ_1 and λ_2 of* (1.1), (1.3) *and* (1.2), (1.3), *respectively, each of which is simple and less than the absolute value of any other eigenvalue for the corresponding problem, and the eigenvectors corresponding to λ_1 and λ_2 may be chosen to belong to $\mathcal{P}^\circ$. Finally, $\lambda_1 \geq \lambda_2$ with $\lambda_1 = \lambda_2$ if and only if $p(t) \equiv q(t)$ on* $0 \leq t \leq 1$.

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