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Existence and multiplicity of solutions to a p-Laplacian equation with nonlinear boundary condition *

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Abstract

We study the nonlinear elliptic boundary value problem

$$\begin{aligned} Au &= f(x,u) \quad \text{in } \Omega\,, \\ Bu &= g(x,u) \quad \text{on } \partial\Omega\,, \end{aligned}$$

where A is an operator of $p-$Laplacian type, Ω is an unbounded domain in $\mathbb{R}^N$ with non-compact boundary, and f and g are subcritical nonlinearities. We show existence of a nontrivial nonnegative weak solution when both f and g are superlinear. Also we show existence of at least two nonnegative solutions when one of the two functions f, g is sublinear and the other one superlinear. The proofs are based on variational methods applied to weighted function spaces.

1 Introduction

The objective of this paper is to study the nonlinear elliptic boundary value problem

$$-\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) = f(x,u) \quad \text{in} \quad \Omega \subset \mathbb{R}^N, \tag{1}$$

$$\mathsf{n}\cdot a(x)|\nabla u|^{p-2}\nabla u + b(x)|u|^{p-2}u = g(x,u) \quad \text{on} \quad \Gamma = \partial\Omega, \tag{2}$$

where Ω is an unbounded domain with noncompact, smooth boundary Γ (for example a cylindrical domain), and n is the unit outward normal vector on Γ. We assume throughout that $1 < p < N$, $0 < a_0 \le a \in L^\infty(\Omega)$ and b is a positive and continuous function defined on $\mathbb{R}^N$. The $p-$Laplace operator in (1) is a special case of the divergence–form operator $-\operatorname{div}(a(x,\nabla u))$ which appears in many

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nonlinear diffusion problems, in particular in the mathematical modeling of non–Newtonian fluids. For a discussion of some physical background see [5]. The boundary condition (2) describes a flux through the boundary which depends in a nonlinear manner on the solution itself. For some physical motivation of such boundary conditions see for example [10].

The energy functional corresponding to (1), (2) is defined as

$$J(u) = \frac{1}{p}\int_\Omega a(x)|\nabla u|^p\,dx + \frac{1}{p}\int_\Gamma b(x)|u|^p\,d\Gamma - \int_\Omega F(x,u)\,dx - \int_\Gamma G(x,u)\,d\Gamma\,,$$

where F and G denote the primitive functions of f and g with respect to the second variable, i. e. $F(x,u) = \int_0^u f(x,s)\,ds,\ G(x,u) = \int_0^u g(x,s)\,ds$. Then the weak solutions of (1), (2) are the critical points of J. We remark that, according to the regularity theorem of [14], every weak solution of (1), (2) belongs to $C^{1,\beta}_{\text{loc}}(\Omega)$. In addition, in [8] regularity up to the boundary was proved, but only under rather restrictive conditions on g.

In this paper we consider problem (1), (2) under several conditions on f and g. If both functions are subcritical and superlinear with respect to u, then we prove existence of a nontrivial nonnegative solution (Theorem 2). In the case, where f is sublinear and g superlinear, we show that there exist at least two nonnegative solutions, one with positive energy, the other one with negative energy (Theorem 3). The same result holds in the case where f is superlinear and g sublinear (Theorem 4).

Such kind of problems with combined concave and convex nonlinearities were studied recently by several authors, with the right hand side of (1) of the form $f+g$ and the boundary condition is $u=0$ on Γ. For a bounded domain Ω and $p=2$ see [1], for $1<p<N$ see [2] and [3] (which also includes the critical case). For the $p-$Laplacian in an exterior domain see [16]. Our proofs are based on weighted-norm estimates in Sobolev spaces, which imply some compactness properties of the functional J. For some related results on the existence of nontrivial solutions to equation (1) in $\mathbb{R}^N$ see for example [4], [6], [7], [9]. We remark that the results in this paper are new even in the semilinear elliptic case $p=2$.

This paper is organized as follows: In the next section we prove some preliminary results concerning equivalent norms and traces in weighted Sobolev spaces. Section 3 is devoted to the superlinear case (Theorem 2), and Section 4 contains the results on the mixed case (Theorems 3 and 4).

2 Preliminaries: Weighted Sobolev Spaces

Let $C^\infty_\delta(\Omega)$ be the space of $C^\infty_0(\mathbb{R}^N)$-functions restricted on Ω. We define the weighted Sobolev-space E as the completion of $C^\infty_\delta(\Omega)$ in the norm

$$\|u\|_E = \left(\int_\Omega |\nabla u(x)|^p + \frac{1}{(1+|x|)^p}|u(x)|^p\,dx\right)^{1/p}.$$

First we prove the following weighted Hardy-type inequality.

Lemma 1 *Let $1 < p < N$. Then there exist positive constants C_1 and C_2, such that for every $u \in E$*

$$\int_\Omega \frac{1}{(1+|x|)^p}|u|^p\,dx \le C_1 \int_\Omega |\nabla u|^p\,dx + C_2 \int_\Gamma \frac{|\mathsf{n}\cdot x|}{(1+|x|)^p}|u|^p\,d\Gamma\,. \tag{3}$$

Proof. Using the divergence theorem we obtain for $u \in C_\delta^\infty(\Omega)$

$$\int_\Omega x\cdot\nabla\left(\frac{1}{(1+|x|)^p}|u|^p\right)dx = \int_\Gamma (\mathsf{n}\cdot x)\frac{1}{(1+|x|)^p}|u|^p\,d\Gamma - N\int_\Omega \frac{1}{(1+|x|)^p}|u|^p\,dx\,.$$

This implies

$$\begin{aligned} N\int_\Omega \frac{1}{(1+|x|)^p}|u|^p\,dx \quad &\le \quad \int_\Gamma \frac{|\mathsf{n}\cdot x|}{(1+|x|)^p}|u|^p\,d\Gamma + p\int_\Omega \frac{1}{(1+|x|)^p}|u|^p\,dx \\ &\quad +p\int_\Omega \frac{1}{(1+|x|)^{p-1}}|u|^{p-1}|\nabla u|\,dx\,. \end{aligned}$$

Using Hölder's and Young's inequality, the last term can be estimated by

$$\begin{aligned} &p\left(\int_\Omega \frac{1}{(1+|x|)^p}|u|^p\,dx\right)^{(p-1)/p}\left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p} \\ &\le \quad \varepsilon(p-1)\int_\Omega \frac{1}{(1+|x|)^p}|u|^p\,dx + \varepsilon^{1-p}\int_\Omega |\nabla u|^p\,dx\,, \end{aligned}$$

where $\varepsilon > 0$ is an arbitrary real number. It follows that

$$(N-\varepsilon(p-1)-p)\int_\Omega \frac{1}{(1+|x|)^p}|u|^p\,dx \le \varepsilon^{1-p}\int_\Omega |\nabla u|^p\,dx + \int_\Gamma \frac{|\mathsf{n}\cdot x|}{(1+|x|)^p}|u|^p\,d\Gamma\,,$$

and for ε small enough, the desired inequality follows by standard density arguments. $\square$

Now denote by $L^r(\Omega; w_1)$ and $L^q(\Gamma; w_2)$ the weighted Lebesgue spaces with weight functions

$$w_i(x) = (1+|x|)^{\alpha_i}\,, \quad i = 1,2, \quad \alpha_i \in \mathbb{R} \tag{4}$$

and norm defined by

$$\|u\|_{r,w_1}^r = \int_\Omega w_1|u(x)|^r\,dx\,, \qquad \text{and} \qquad \|u\|_{q,w_2}^q = \int_\Gamma w_2|u(x)|^q\,dx\,.$$

Then we have the following embedding and trace theorem.

Theorem 1 *If*

$$p \le r \le \frac{pN}{N-p} \quad \text{and} \quad -N < \alpha_1 \le r\,\frac{N-p}{p} - N\,, \tag{5}$$

then the embedding $E \hookrightarrow L^r(\Omega; w_1)$ *is continuous. If the upper bounds for* r *in (5) are strict, then the embedding is compact. If*

$$p \le q \le \frac{p(N-1)}{N-p} \quad \text{and} \quad -N < \alpha_2 \le q\,\frac{N-p}{p} - N + 1\,, \tag{6}$$

then the trace operator $E \to L^q(\Gamma; w_2)$ *is continuous. If the upper bounds for* q *in (6) are strict, then the trace is compact.*

This theorem is a consequence of Theorem 2 and Corollary 6 of [11].

As a corollary of Lemma 1 and Theorem 1 we obtain

Lemma 2 *Let* b *satisfy* $c/(1+|x|)^{p-1} \le b(x) \le C/(1+|x|)^{p-1}$ *for some constants* $0 < c \le C$*. Then*

$$\|u\|_b^p = \int_\Omega a(x)|\nabla u|^p\,dx + \int_\Gamma b(x)|u|^p\,d\Gamma$$

defines an equivalent norm on E*.*

Proof. The inequality $\|u\|_E \le C_1\|u\|_b$ follows directly from Lemma 1, while from Theorem 1 (setting $p = q$ and $\alpha_2 = -(p-1)$) we obtain

$$\begin{aligned} \|u\|_b^p &\le \|a\|_{L^\infty}\int_\Omega |\nabla u|^p dx + C\int_\Gamma |u|^p(1+|x|)^{-(p-1)}d\Gamma \\ &\le \|a\|_{L^\infty}\int_\Omega |\nabla u|^p dx + C_2\|u\|_E^p, \end{aligned}$$

which shows the desired equivalence. □

Remark. In special geometries the lower bound for b required in Lemma 2 can be improved. In view of Lemma 1 it is sufficient to assume $b(x) \ge |\mathsf{n}\cdot x|/(1+|x|)^p$, where $\mathsf{n}\cdot x = |\mathsf{n}||x|\cos\gamma$ and γ is the angle between x and n. For a cylindrical domain $\Omega = B \times \mathbb{R}$, where $B \subset \mathbb{R}^{N-1}$ is bounded, we obtain $|\cos\gamma| \le C_B/|x|$, with a constant C_B depending only on the diameter of B. This shows that in cylindrical domains, Lemma 2 holds under the weaker assumption

$$\frac{c}{(1+|x|)^p} \le b(x) \le \frac{C}{(1+|x|)^{p-1}}\,.$$

We shall assume throughout the paper that b satisfies the assumption of Lemma 2 so that we can use $\|\cdot\|_b$ as an equivalent norm in E.

3 The superlinear case

We make the following assumptions

A1 f and g are Carathéodory functions on $\Omega \times \mathbb{R}$ and $\Gamma \times \mathbb{R}$, respectively, $f(\cdot, 0) = g(\cdot, 0) = 0$ and

$$|f(x,s)| \leq f_0(x) + f_1(x)|s|^{r-1} \quad , \quad p \leq r < pN/(N-p),$$
$$|g(x,s)| \leq g_0(x) + g_1(x)|s|^{q-1} \quad , \quad p \leq q < p(N-1)/(N-p),$$

where f_i, g_i are nonnegative, measurable functions which satisfy the following hypotheses: There exist α_1, α_2, $-N < \alpha_1 < r\frac{N-p}{p} - N$, $-N < \alpha_2 < q\frac{N-p}{p} - N + 1$, such that, with w_i defined as in (4), we have

$$0 \leq f_i(x) \leq C_f w_1 \text{ a. e. }, \qquad f_0 \in L^{r/(r-1)}(\Omega; w_1^{1/(1-r)}),$$
$$0 \leq g_i(x) \leq C_g w_2 \text{ a. e. }, \qquad g_0 \in L^{q/(q-1)}(\Gamma; w_2^{1/(1-q)})\,.$$

A2 $\lim_{s\to 0} f(x,s)/|s|^{p-1} = \lim_{s\to 0} g(x,s)/|s|^{p-1} = 0$ uniformly in x.

A3 There exists $\mu > p$ such that $\mu F(x,s) \leq f(x,s)s$, $\mu G(x,s) \leq g(x,s)s$ for a. e. $x \in \Omega$, resp. $x \in \Gamma$ and every $s \in \mathbb{R}$.

A4 One of the following conditions holds:

a) There is a nonempty open set $O \subset \Omega$ with $F(x,s) > 0$ for $(x,s) \in O \times (0,\infty)$

b) There is a nonempty open set $U \subset \Gamma$ with $G(x,s) > 0$ for $(x,s) \in U \times (0,\infty)$ and G satisfies $\bar{\mu} G(x,s) \leq g(x,s)s$ with some $\bar{\mu} > r$.

c) $G(x,s) > 0$ for $(x,s) \in U \times (0,\infty)$ and and there exist an open, nonempty subset $V \subset \Omega$, $\overline{V} \cap U \neq \emptyset$ and a constant C_F, such that $F(x,u) \geq -C_F$ on $V \times (0,\infty)$.

We denote by N_f, N_F, N_g, N_G the corresponding Nemytskii operators. Under the assumptions above we have the following result.

Lemma 3 *The operators*

$$N_f : L^r(\Omega; w_1) \to L^{r/(r-1)}(\Omega; w_1^{1/(1-r)})\,, \qquad N_F : L^r(\Omega; w_1) \to L^1(\Omega)\,,$$
$$N_g : L^q(\Gamma; w_2) \to L^{q/(q-1)}(\Gamma; w_2^{1/(1-q)})\,, \qquad N_G : L^q(\Gamma; w_2) \to L^1(\Gamma)$$

are bounded and continuous.

Proof. We only prove the statements for N_g and N_G, since the arguments for N_f and N_F are similar. Let $q' = q/(q-1)$ and $u \in L^q(\Gamma; w_2)$. Then, by Assumption A1,

$$\begin{aligned}\int_\Gamma |N_g(u)|^{q'} w_2^{1/(1-q)} d\Gamma &\le 2^{q'-1}\left(\int_\Gamma g_0^{q'} w_2^{1/(1-q)} d\Gamma + \int_\Gamma g_1^{q'} |u|^q w_2^{1/(1-q)} d\Gamma\right)\\ &\le 2^{q'-1}\left(C + C_g \int_\Gamma |u|^q w_2 d\Gamma\right),\end{aligned}$$

which shows that N_g is bounded. In a similar way we obtain

$$\begin{aligned}\int_\Gamma |N_G(u)| d\Gamma &\le \int_\Gamma g_0 |u| d\Gamma + \int_\Gamma g_1 |u|^q d\Gamma\\ &\le \left(\int_\Gamma g_0^{q'} w_2^{1/(1-q)} d\Gamma\right)^{\frac{1}{q'}} \left(\int_\Gamma |u|^q w_2\, d\Gamma\right)^{\frac{1}{q}} + C_g \int_\Gamma |u|^q w_2\, d\Gamma\end{aligned}$$

and again we claim that N_G is bounded. The continuity of these operators now follows from the usual properties of Nemytskii operators (cf. [15]). □

Lemma 4 *Under Assumptions A1–A4, J is Fréchet–differentiable on E and satisfies the Palais–Smale condition.*

Proof. We use the notation $I(u) = \frac{1}{p}\|u\|_b^p$, $K_F(u) = \int_\Omega F(x,u)\, dx$, $K_G(u) = \int_\Gamma G(x,u)\, d\Gamma$. Then the directional derivative of J in direction $h \in E$ is

$$\langle J'u, h\rangle = \langle I'u, h\rangle - \langle K_F'u, h\rangle - \langle K_G'u, h\rangle\,,$$

where

$$\begin{aligned}\langle I'(u), h\rangle &= \textstyle\int_\Omega a(x)|\nabla u|^{p-2}\nabla u \nabla h\, dx + \int_\Gamma b(x)|u|^{p-2}uh\, d\Gamma\,,\\ \langle K_F'(u), h\rangle &= \textstyle\int_\Omega f(x,u)h\, dx\,, \quad \langle K_G'(u), h\rangle \;=\; \int_\Gamma g(x,u)h\, d\Gamma\,.\end{aligned}$$

Clearly, $I' : E \to E'$ is continuous. The operator K_G' is a composition of operators

$$K_G' : E \to L^q(\Gamma; w_2) \xrightarrow{N_g} L^{q/(q-1)}(\Gamma; w_2^{1/(1-q)}) \xrightarrow{\ell} E',$$

where $\langle \ell(v), h\rangle = \int_\Gamma vh\, d\Gamma$. Since

$$\int_\Gamma |vh|\, d\Gamma \le \left(\int_\Gamma |v|^{q'} w_2^{1/(1-q)} d\Gamma\right)^{1/q'} \left(\int_\Gamma |h|^q w_2\, d\Gamma\right)^{1/q},$$

ℓ is continuous by Theorem 1. As a composition of continuous operators, K_G' is continuous, too. Moreover, by our assumptions on w_2 (see A1), the trace operator $E \to L^q(\Gamma; w_2)$ is compact and therefore, K_G' is also compact. In a similar way we obtain that K'_F is compact and the Fréchet-differentiability of J follows.

Now let $u_k \in E$ be a Palais–Smale sequence, i. e. $|J(u_k)| \leq C$ for all k and $J'(u_k) \to 0$ as $k \to \infty$. For k large enough we have $|\langle J'(u_k), u_k\rangle| \leq \|u_k\|_b$ and by Assumption A3

$$\begin{aligned} C + \|u_k\|_b &\geq J(u_k) - \frac{1}{\mu}\langle J'(u_k), u_k\rangle \\ &\geq \left(\frac{1}{p} - \frac{1}{\mu}\right) \|u\|_b^p . \end{aligned}$$

This shows that u_k is bounded in E. To show that u_k contains a Cauchy sequence we use the following inequalities for $\xi, \zeta \in \mathbb{R}^N$ (see [5], Lemma 4.10):

$$|\xi - \zeta|^p \leq C(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta)\,, \quad \text{for } p \geq 2, \tag{7}$$

$$|\xi - \zeta|^2 \leq C(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta)(\xi - \zeta)(|\xi| + |\zeta|)^{2-p}\,, \quad \text{for } 1 < p < 2\,. \tag{8}$$

Then we obtain in the case $p \geq 2$:

$$\begin{aligned} \|u_n - u_k\|_b^p &= \int_\Omega a(x)|\nabla u_n - \nabla u_k|^p dx + \int_\Gamma b(x)|u_n - u_p|^p d\Gamma \\ &\leq C\Big(\langle I'(u_n), u_n - u_k\rangle - \langle I'(u_k), u_n - u_k\rangle\Big) \\ &= C\Big(\langle J'(u_n), u_n - u_k\rangle - \langle J'(u_k), u_n - u_k\rangle + \langle K'_F(u_n) \\ &\quad + K'_G(u_n), u_n - u_k\rangle - \langle K'_F(u_n) + K'_G(u_k), u_n - u_k\rangle\Big) \\ &\leq C\Big(\|J'(u_n)\|_{E'} + \|J'(u_k)\|_{E'} + \|K'_F(u_n) - K'_F(u_k)\|_{E'} \\ &\quad + \|K'_G(u_n) - K'_G(u_k)\|_{E'}\Big)\|u_n - u_k\|_b\,. \end{aligned}$$

Since $J'(u_k) \to 0$ and K'_F, K'_G are compact, there exists a subsequence of u_k which converges in E.

If $1 < p < 2$, then we use (8) and Hölder's inequality to obtain the estimate

$$\|u_n - u_k\|_b^2 \leq C\Big|\langle I'(u_n), u_n - u_k\rangle - \langle I'(u_k), u_n - u_k\rangle\Big|\Big(\|u_n\|_b^{2-p} + \|u_k\|_b^{2-p}\Big).$$

Since $\|u_n\|_b$ is bounded, the same arguments as above lead to a convergent subsequence. □

Theorem 2 *There exists a nontrivial nonnegative solution of (1), (2) in E.*

Proof. We shall use the Mountain–Pass lemma [13] to obtain a solution. First we observe that, from Assumption A1 and A2, for every $\varepsilon > 0$ there is a C_ε

such that $|F(x,u)| \le \varepsilon f_0(x)|u|^p + C_\varepsilon f_1(x)|u|^r$, and $|G(x,u)| \le \varepsilon g_0(x)|u|^p + C_\varepsilon g_1(x)|u|^q$. Consequently

$$\begin{aligned} J(u) &\ge \frac{1}{p}\|u\|_b^p - \int_\Omega (\varepsilon f_0(x)|u|^p + C_\varepsilon f_1(x)|u|^r)\,dx \\ &\quad - \int_\Gamma (\varepsilon g_0(x)|u|^p + C_\varepsilon g_1(x)|u|^q)\,d\Gamma \\ &\ge \|u\|_b^p - \varepsilon C_1\|u\|_b^p - C_\varepsilon C_2(\|u\|_b^r + \|u\|_b^q) \end{aligned}$$

and for ε and $\|u\|_b = \rho$ sufficiently small, the right hand side is strictly greater than 0. It remains to show that there exists $u_0 \in E$, $\|u_0\|_b > \rho$ such that $J(u_0) \le 0$.

In the case A4 a), we choose a nontrivial nonnegative function $\varphi \in C_0^\infty(O)$. From A3 we see that $F(x,s) \ge C_1 s^\mu - C_2$ on $O \times (0,\infty)$. Then, for $t \ge 0$,

$$J(t\varphi) \le \frac{1}{p}\,t^p\|\varphi\|_b^p - C_1 t^\mu \int_O \varphi^\mu dx + C_2|O|\,.$$

Since $\mu > p$, the right hand side tends to $-\infty$ as $t \to \infty$ and for sufficiently large t_0, $u_0 = t_0\varphi$ has the desired properties.

In the case A4 b), we choose a nonnegative $\varphi \in C_\delta^\infty(\Omega)$ such that $\mathrm{supp}\varphi \cap \Gamma \subset U$ is not empty. Again from $G(x,s) \ge C_3 s^{\bar\mu} - C_4$ on $U \times (0,\infty)$ and Assumption A1 we claim

$$J(t\varphi) \le \frac{1}{p}\,t^p\|\varphi\|_b^p + C_5 \int_\Omega t\varphi + t^r\varphi^r dx - C_3 t^{\bar\mu} \int_U \varphi^{\bar\mu} d\Gamma + C_4|U|\,.$$

Since $\bar\mu > r \ge p$, we obtain $J(t\varphi) \to -\infty$ as $t \to \infty$.

In the case A4 c), we take $\varphi \in C_\delta^\infty(\Omega)$ with $\mathrm{supp}\varphi \cap \overline{\Omega} \subset \overline{V}$ and $\mathrm{supp}\varphi \cap U \ne \emptyset$. Then

$$J(t\varphi) \le \frac{1}{p}\,t^p\|\varphi\|_b^p + C_F|V| - C_3 t^\mu \int_U \varphi^\mu d\Gamma + C_4|U|$$

and again we claim $J(t\varphi) \to -\infty$ as $t \to \infty$.

Since J satisfies the Palais–Smale condition and $J(0) = 0$, the Mountain–Pass Lemma shows that there is a nontrivial critical point of J in E with critical value

$$c = \inf_{\gamma\in P} \max_{t\in[0,1]} J(\gamma(t)) > 0\,,$$

where $P = \{\gamma \in C([0,1],E) \mid \gamma(0) = 0,\ \gamma(1) = u_0\}$.

To obtain a nonnegative solution by this procedure, we introduce the truncated functions $\bar f$ and $\bar g$ such that $\bar f(x,s) = \bar g(x,s) = 0$ for all $s \le 0$. Then the arguments above remain true and we obtain a critical point u of the truncated functional $\bar J$, i. e. $\langle \bar J'(u), h\rangle = 0$ for all $h \in E$. In particular, setting $u_-(x) = \max\{-u(x), 0\}$ and $h = u_-$, we claim that $u \ge 0$. Since any nonnegative solution of the truncated problem is also a solution of the original equation, we have found a nonnegative solution of (1), (2). $\square$

4 Combined Sub- and Superlinear Nonlinearities

In this part we introduce an additional parameter into equation (1), i. e. we study

$$-\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) = \lambda f(x,u) \quad \text{in} \quad \Omega \qquad (1)_\lambda$$

with the same boundary condition (2) as before. Here, we assume the following

B1 Let g satisfy Assumptions A1–A3 with $g_0 \equiv 0$ and $|f(x,s)| \le f_1(x)|s|^{r-1}$, $1 \le r < p$, where f_1 is nonnegative, measurable and there exists α_1, $-N < \alpha_1 < r\frac{N-p}{p} - N$, such that for $w_1(x) = (1+|x|)^{\alpha_1}$, we have $f_1 \in L^{p/(p-r)}(\Omega; w_1^{r/(r-p)})$.

B2 $|f(x,s)| \ge f_2(x)|s|^{\bar r - 1}$, $1 \le \bar r \le r$, with $f_2 > 0$ in some nonempty open set $O \subset \Omega$.

B3 There is a nonempty open set $U \subset \Gamma$ with $G(x,s) > 0$ for $(x,s) \in U \times (0,\infty)$.

The Nemytskii operators N_g and N_G have the same properties as in Lemma 3, while for N_f and N_F we obtain

Lemma 5 *The operators $N_f : L^p(\Omega; w_1) \to L^{p/(p-1)}(\Omega; w_1^{1/(1-p)})$, and $N_F : L^p(\Omega; w_1) \to L^1(\Omega)$ are bounded and continuous.*

Proof. Since the first statement is trivial if $r = 1$, we may assume that $r > 1$. From B1 we obtain with Hölder's inequality (setting $p' = p/(p-1)$)

$$\begin{aligned}
\int_\Omega |f(x,u)|^{p'} w_1^{1/(1-p)} dx &\le \int_\Omega |f_1|^{p'} w_1^{r/(1-p)} |u|^{p'(r-1)} w_1^{(r-1)/(p-1)} dx \\
&\le \left(\int_\Omega |f_1|^{p/(p-r)} w_1^{r/(r-p)} \right)^{\frac{p-r}{p-1}} \left(\int_\Omega |u|^p w_1 \right)^{\frac{r-1}{p-1}} \\
&\le C \, \|u\|_{p,w_1}^{p(r-1)/(p-1)} .
\end{aligned}$$

For N_F we obtain

$$\begin{aligned}
\int_\Omega |F(x,u)| dx &\le \int_\Omega |f_1| w_1^{-r/p} |u|^r w_1^{r/p} dx \\
&\le \left(\int_\Omega |f_1|^{p/(p-r)} w_1^{r/(r-p)} dx \right)^{(p-r)/p} \left(\int_\Omega |u|^p w_1 dx \right)^{r/p} \\
&\le C \, \|u\|_{p,w_1}^{r} .
\end{aligned}$$

□

The differentiability for J now follows as above.

To obtain the Palais–Smale condition for J, let $u_k \in E$ be a sequence such that $|J(u_k)| \leq C$ and $J'(u_k) \to 0$ as $k \to \infty$. With Assumptions A3, B1 and Hölder's inequality we get

$$\begin{aligned}
&J(u_k) - \frac{1}{\mu}\langle J'(u_k), u_k\rangle \\
&= \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_k\|_b^p + \int_\Omega \frac{1}{\mu} f(x,u)u - F(x,u)\,dx + \int_\Gamma \frac{1}{\mu} g(x,u)u - G(x,u)\,d\Gamma \\
&\geq \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_k\|_b^p - \left(1 + \frac{1}{\mu}\right)\int_\Omega f_1(x)|u_k|^r dx \\
&\geq \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_k\|_b^p - \left(\int_\Omega f_1^{p/(p-r)} w_1^{r/(r-p)} dx\right)^{(p-r)/p} \left(\int_\Omega |u_k|^p dx\right)^{r/p} \\
&\geq \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_k\|_b^p - C_1\|f_1\|_*\|u_k\|_b^r\,,
\end{aligned}$$

where $\|f_1\|_*$ is the weighted norm of f_1 in $L^{p/(p-r)}(\Omega; w_1^{r/(r-p)})$. Since $r < p$ and $C + \|u_k\|_b \geq J(u_k) - \frac{1}{\mu}\langle J'(u_k), u_k\rangle$, we claim that u_k is bounded in E. The convergence of a subsequence of u_k then follows as above from the compactness properties of K'_F and K'_G.

Theorem 3 *Under Assumptions B1–B3 there exists $\lambda^* > 0$, such that for every $0 < \lambda < \lambda^*$, there are at least two nontrivial nonnegative solutions of (1)$_\lambda$, (2).*

Proof. First we show that for $\lambda \in (0, \lambda^*)$, we can find $\rho > 0$ such that $J(u) \geq c > 0$ if $\|u\|_b = \rho$. We denote by C_Ω, C_Γ the embedding and trace constants for the operators $E \hookrightarrow L^p(\Omega; w_1)$ and $E \to L^q(\Gamma; w_2)$, respectively. We obtain

$$\begin{aligned}
J_\lambda(u) &\geq \frac{1}{p}\|u\|_b^p - \frac{\lambda}{r}\int_\Omega f_1(x)|u|^r\,dx - \frac{1}{q}\int_\Gamma g_1(x)|u|^q\,d\Gamma \\
&\geq \frac{1}{p}\|u\|_b^p - \frac{\lambda}{r}\left(\int_\Omega f_1(x)^{p/(p-r)} w_1(x)^{r/(r-p)} dx\right)^{(p-r)/p}\left(\int_\Omega |u|^p w_1\,dx\right)^{r/p} \\
&\quad -\frac{1}{q}\int_\Gamma g_1(x)|u|^q\,d\Gamma \\
&\geq \frac{1}{p}\|u\|_b^p - \frac{\lambda}{r}C_\Omega\|f_1\|_*\|u\|_b^r - \frac{1}{q}C_\Gamma C_g\|u\|_b^q\,.
\end{aligned}$$

If $\|u\|_b = \rho$, we obtain

$$J_\lambda(u) \geq \frac{1}{p}\rho^p\left(1 - \frac{p\lambda}{r}C_\Omega\|f_1\|_*\rho^{r-p} - \frac{p}{q}\,C_\Gamma C_g\rho^{q-p}\right) \tag{9}$$

Elementary calculations show that the right hand side is maximal for

$$\rho_m = \left(\frac{q(p-r)\lambda\, C_\Omega\|f_1\|_*}{r(q-p)\,C_g C_\Gamma}\right)^{1/(q-r)}\,.$$

Inserting this into equation (9), we find that the right hand side is zero for

$$\lambda = \lambda^* := \left[\frac{p}{r}\,\|f_1\|_* C_\Omega C_0^{\frac{r-p}{q-r}} + \frac{p}{q}\,C_g C_\Gamma C_0^{\frac{q-p}{q-r}}\right]^{\frac{r-q}{q-p}} ,$$

where

$$C_0 = \left(\frac{\|f_1\|_* C_\Omega (p-r)q}{C_g C_\Gamma (q-p)r}\right) ,$$

and strictly greater than 0 for $\lambda < \lambda^*$. This shows that for every $\lambda < \lambda^*$, we find $\rho_\lambda > 0$ such that $J_\lambda \geq c_\lambda > 0$ for $\|u\|_b = \rho_\lambda$. The existence of a function $u_0 \in E$, $\|u_0\|_b > \rho_\lambda$ and $J_\lambda(u_0) \leq 0$ now follows as in the proof of Theorem 2 (case A4 b). Then the Mountain-Pass Lemma again implies the existence of a nontrivial solution u_1 with $J_\lambda(u_1) \geq c_\lambda$.

On the other hand, for $\varphi \in C_0^\infty(O)$ and $t > 0$ we obtain

$$J_\lambda(t\varphi) \leq \frac{t^p}{p}\|\varphi\|_b^p - \frac{t^{\bar r}}{\bar r}\int_O f_2(x)|\varphi|^{\bar r} dx\,.$$

This shows that $J_\lambda(t\varphi) < 0$ for sufficiently small t and consequently J_λ attains its minimum in the ball $B_{\rho_\lambda} \subset E$. We claim that there is a second solution $u_2 \in B_{\rho_\lambda}$ with $J_\lambda(u_2) < 0$.

In addition, with the same truncation procedure as in the proof of Theorem 2, we claim that there are two nonnegative solutions. □

Now we can prove the corresponding result for equation (1) with boundary condition

$$\mathsf{n} \cdot a(x)|\nabla u|^{p-2}\nabla u + b(x)|u|^{p-2}u = \lambda g(x,u) \quad \text{on} \quad \Gamma \qquad (2)_\lambda$$

if we interchange the roles of g and f in Assumptions B1–B3. That is, we assume now that f satisfies Assumptions A1–A4 a) (with $f_0 \equiv 0$) and g satisfies

B4 $|g(x,s)| \leq g_1(x)|s|^{q-1}$, $1 \leq q < p$, $g_1 \in L^{p/(p-q)}(\Gamma; w_2^{q/(q-p)})$, $|g(x,s)| \geq g_2(x)|s|^{\bar q-1}$, $1 \leq \bar q \leq q$ and $g_2 > 0$ in some nonempty open set $U \subset \Gamma$.

Theorem 4 *Let f satisfy Assumptions A1–A4 a) (with $f_0 \equiv 0$) and g satisfy B4. Then for every $0 < \lambda < \lambda^*$, there are at least two nontrivial nonnegative solutions of (1), $(2)_\lambda$.*

Proof. First we claim as in Lemma 5 that

$$N_g : L^p(\Gamma; w_2) \to L^{p/(p-1)}(\Gamma; w_2^{1/(1-p)}) \,, \qquad N_G : L^p(\Gamma; w_2) \to L^1(\Gamma)$$

are bounded and continuous. The estimate for J_λ now reads

$$J_\lambda(u) \geq \frac{1}{p}\|u\|_b^p - \frac{1}{r}C_\Omega C_f\|u\|_b^r - \frac{\lambda}{q}C_\Gamma\|g_1\|_*\|u\|_b^q,$$

where $\|g_1\|_*$ is the norm of g_1 in $L^{p/(p-q)}(\Gamma; w_2^{q/(q-p)})$. Now λ^* can be calculated as

$$\lambda^* := \left[\frac{p}{q}\,\|g_1\|_* C_\Gamma \bar{C}_0^{\frac{q-p}{r-q}} + \frac{p}{r}\,C_f C_\Omega \bar{C}_0^{\frac{r-p}{r-q}}\right]^{\frac{q-r}{r-p}}, \quad \bar{C}_0 = \left(\frac{\|g_1\|_* C_\Gamma (p-q) r}{C_f C_\Omega (r-p) q}\right).$$

The existence of u_0 with $\|u_0\|_b > \rho_\lambda$ and $J(u_0) < 0$ follows in the same way as in the proof of Theorem 2, case A4 a). Finally, for a nonnegative $\varphi \in C_\delta^\infty(\Omega)$ with $\operatorname{supp}\varphi \cap \Gamma \subset U$ not empty, we find

$$J_\lambda(t\varphi) \le \frac{t^p}{p}\|\varphi\|_b^p + C\frac{t^r}{r}\|\varphi\|_b^r - \frac{t^{\bar q}}{\bar q}\int_U g_2(x)|\varphi|^{\bar q}dx\,.$$

Since $\bar q < p \le r$, $J_\lambda(t\varphi) < 0$ for t sufficiently small and we claim that J_λ attains its minimum in $B_{\rho_\lambda} \subset E$. $\square$

We remark that, if Ω is of class $C^{1,\alpha}$ ($\alpha \le 1$) and, in addition to B4, g satisfies

$$|g(x,s) - g(y,t)| \le C\Big(|x-y|^\alpha + |s-t|^\alpha\Big), \qquad |g(x,s)| \le C$$

for all $x, y \in \Gamma$, $s, t \in \mathbb{R}$, then the regularity result of [8], Thm. 2, shows that the solution u belongs to $C^{1,\beta}(\overline{\Omega})$ for some $\beta > 0$.

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