A FAST SOLVER FOR DISCRETE SYSTEMS ARISING FROM THE EXPANDED MIXED FINITE ELEMENT METHOD

by

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ABSTRACT

We present a fast solver for expanded mixed finite element method with a discontinuous coefficient. Our result is based on the auxiliary space preconditioning method and the Crouzeix-Raviart Finite Element Method.

I. INTRODUCTION

The vast majority of the laws of physics for space and time dependent problems are usually expressed as partial differential equations. Partial differential equations are differential equations that contain unknown multi-variable functions, that can be used to describe sound, heat, electrostatics, electrodynamics, fluid mechanics, elasticity, and quantum mechanics. For certain geometries and problems, partial differential equations cannot be solved with analytical methods, instead an approximation of the equations can be constructed via discretization of the domain, Ω. The discretization approximates the partial differential equation with numerical models, which can be solved with numerical methods. One of these numerical methods is the finite element method . Benefits of the finite element method include freedom in choosing discretization in both the elements used to discretize the space and the basis functions. Common discretizations are using triangles or rectangles. Another benefit is that the theory is well developed because of the relation between the numerical formulation and the weak form of the partial differential equation, where the weak form is obtained from multiplying the strong form of a partial differential equation by a dummy variable, called a weight function, then using integration by parts. The finite element method in irreducible form is to solve for a single variable, say u, which is approximated by the discrete variable u_h where,

$$u_h = \sum_{i=1}^{n} u_i \varphi_i$$

where u_i represents the value of u_h at node i of the element and φ_i represents the basis function.

The mixed finite element method was developed around the 1970's. This method, unlike the irreducible finite element method, involves solving two variables explicitly, such as in Darcy's Law using either the Stokes equation or Poisson equation where, in a simple example, the goal is to find $(\mathbf{u}, p)$ such that

$$\begin{aligned} \nabla \cdot \mathbf{u} &= f && \text{in } \Omega \\ \mathbf{u} &= -K\nabla p && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $\partial\Omega$ represents the boundary of the domain, Ω. In applications to porous media, $\mathbf{u}$ is the Darcy velocity, or flux, p is the pressure, and K is the conductivity tensor. The benefit of the mixed finite element method is this ability to simultaneously solve two variables in relation to each other. Theory for the mixed finite element method was developed heavily by Brezzi, Raviert, Thomas, and Crouzeix [6, 7, 8]. Their developed methods known as Brezzi-Douglas-Marini [6], Raviart-Thomas [7], and Crouzeix-Raviart [8] use different methods of defining the basis functions on each node. In this paper we will make use the the Crouzeix-Ravairt finite element space. This method involves explicitly declaring continuity between elements strictly only on the midpoint of the element edge in two dimensions, or face if in three dimensions, more explanation will be presented later in the paper. The fact that this space is a subspace of the Sobolev space, $H^1 = \{v \in L^2(\Omega) | \nabla v \in L^2(\Omega)\}$ becomes useful for the preconditioner we propose later.

In this paper we present a fast solver for the expanded mixed finite element method. The expanded mixed method expands the usual mixed method by explicitly defining three unknowns, the scalar unknown, the negative of its gradient, and its flux. The resulting linear system is a saddle point problem where the tensor coefficient does not need to be inverted and has been shown to give better than optimal convergence for non-linear problems and optimal convergence for linear problems while the standard mixed method only gives suboptimal error estimates for nonlinear problems [1]. As a result, it works for differential equations with small diffusion or low permeability terms. In the case of the lowest order Raviart-Thomas elements using certain quadrature rules (i.e. trapezoidal and midpoint rules) results in a system that has a cell-centered finite difference scheme.

The expanded mixed finite element method has been studied extensively by Abogast [2], and Chen [1,4]. Chen had proposed the abstract framework of the method along with proving optimal convergence for the non-linear problem and showing the tensor coefficient does not need to be inverted in the expanded method. Abogast [2] had studied methods to solve the expanded mixed finite element method by using quadrature rules on certain integrals of the discrete form of the expanded method in order to change the problem from finite elements to a cell-centered finite differences In this case, solving the cell-centered problem results optimal error estimates in the interior, but they face issues of non consistent error along the boundary.

In order to overcome the issues faced by Abogast's [2] method of solving the expanded mixed finite element method we present a fast solver based on Xu's [3] auxiliary space preconditioning method. The goal of a preconditioner is to condition a given problem into a form that is more suitable for numerical solving methods. In this method we precondition the original problem by mapping to an auxiliary space. The auxiliary space does not need to be a subspace of the original solution space, but needs to be in a sense, "simpler". In the case that the auxiliary space is a subspace of the original space, we simply use the L^2 projection.

This paper is structured as follows. In Chapter 2 we present the expanded mixed finite element method and convergence. Then we follow with the consideration of using the quadrature rules presented by Abogast [2]. In Chapter 3 we show that the expanded mixed finite element method system of equations in operator form can be reduced into the Schur complement operator which is just the usual Lapacian. Following that we lay out the theoretical framework of the preconditioner and show our interpolation operator fit the framework. Lastly, in Chapter 4 we will discuss future works following this paper.

List of Notations

In this section we present a list of mathematical notations the reader may not be familiar with but will need to know in order to follow this paper. One of the first things we need to discuss is the general notation for Sobolev spaces, $W^{k,p}$. Named after the Russian mathematician, Sergei Sobolev, a Sobolev space is a vector space of functions with sufficiently many derivatives for the domain and equipped with a norm to measure both the size and regularity of the function. In the multidimensional case

$$W^{k,p} = \{v \in L^p(\Omega) \mid D^\alpha v \in L^p(\Omega) \ \forall |\alpha| \leq k\},$$

where D^α denotes all the partial derivatives of the function. In this paper we will mainly focus on the special case when $p = 2$, $W^{k,2}$.
In the case of $W^{0,2} = L^2$ the notation to address is the L^2-norm,

$$\|v\|_{L^2(\Omega)} = \left(\int_\Omega v^2 ds\right)^{1/2}.$$

Another norm we need to define that is closely related to the L^2-norm is the $W^{1,2} = H^1$ norm,

$$\|v\|_{H^1(\Omega)} = \left(\|v\|_{L^2(\Omega)} + \|\nabla v\|_{L^2(\Omega)}\right)^{1/2}$$

where ∇ represents the gradient of the function v.

II. GOVERNING EQUATIONS

In this section we present the governing equations with mixed boundary conditions presented by Abogast in [2] and present the stability and error estimates they derived. We consider the mixed finite element approximations of second order elliptic problems with Dirichlet, Neumann, and Robin boundary conditions. In this form the problem is to find $(\mathbf{u}, p)$ such that

$$\nabla \cdot \mathbf{u} = f, \qquad \text{in } \Omega$$

$$\mathbf{u} = -K\nabla p, \qquad \text{in } \Omega$$

where $p = g^D$ on Γ^D, $\mathbf{u} \cdot n = g^N$ on Γ^N and $\mathbf{u} \cdot n - g_1^R p = g_2^R$ on Γ^R and Ω is a bounded domain in $\mathbb{R}^d$ (d=2 or 3) with boundary $\partial\Omega = \overline{\Gamma}^D \cup \overline{\Gamma}^N \cup \overline{\Gamma}^R$ ($\Gamma^D \cap \Gamma^N = \Gamma^D \cap \Gamma^R = \Gamma^N \cap \Gamma^R = \emptyset$); $K(\mathbf{x})$ is a symmetric, positive definite second order tensor with components in $L^\infty(\Omega)$; n is the outward unit normal vector on $\partial\Omega$; and $g_1^R(\mathbf{x}) \geq 0$. In application to flow in porous media, p is pressure, $\mathbf{u}$ is the Darcy velocity, and K is the conductivity tensor. Now to expanded the usual mixed method we let $\widetilde{\mathbf{u}}$ = -∇p in Ω, thus $\mathbf{u} = K\widetilde{\mathbf{u}}$ in Ω.
Let V^0 and V^N be the sub-spaces of

$$V^H = H(div; \Omega) = \{v \in L^2(\Omega) | \nabla \cdot v \in L^2(\Omega)\}$$

consisting of the functions with normal trace on Γ^N equal to zero and g^N respectively; let $\widetilde{V} = (L^2(\Omega))^d$; let W $= L^2(\Omega)$; and let

$$\Lambda = H^{1/2}(\Omega) = \{v \in L^2(\partial\Omega) | \text{ there exists a } \widetilde{v} \text{ such that } v = tr(\widetilde{v}).$$

Let $(\cdot, \cdot)_S$ denote the $L^2(S)$-inner-product of the duality pairing, where we omit

S if S = Ω. We now have the following weak variation of the expanded system: where the problem is to find $\mathbf{u} \in V^N$, $\widetilde{\mathbf{u}} \in \widetilde{V}$, p ∈ W and $\lambda \in \Lambda$ such that

$$\begin{aligned}
(\nabla \cdot \mathbf{u}, w) &= (f, w), && w \in W, \\
(\widetilde{\mathbf{u}}, \mathbf{v}) = (p, \nabla \cdot \mathbf{v}) - (g^D, \mathbf{v} \cdot v)_{\Gamma^D} - (\lambda, \mathbf{v} \cdot v)_{\Gamma^D}&, && \mathbf{v} \in V^0 \\
(\mathbf{u}, \widetilde{\mathbf{v}}) &= (K\widetilde{\mathbf{u}}, \widetilde{\mathbf{v}}) && \widetilde{\mathbf{v}} \in \widetilde{V} \\
(\mathbf{u} \cdot v, \mu)_{\Gamma^R} &= (g_2^R + g_1^R \lambda, \mu)_{\Gamma^R} && \mu \in \Lambda
\end{aligned}$$

Now since Γ^R and Γ^N both affect the flux or flow, let Γ^F denote the interior of $\overline{\Gamma}^N \cup \overline{\Gamma}^R$ and define

$$g_1^F = \begin{cases} 0, & \text{on } \Gamma^N. \\ g_1^R, & \text{on } \Gamma^R. \end{cases}$$

$$g_2^F = \begin{cases} g^N, & \text{on } \Gamma^N. \\ g_1^R, & \text{on } \Gamma^R. \end{cases}$$

An equivalent formulation is to find $\mathbf{u}$ ∈ V, $\widetilde{\mathbf{u}} \in \widetilde{V}$, p ∈ W, $\lambda \in \Lambda$ such that

$$\begin{aligned}
(\nabla \cdot \mathbf{u}, w) &= (f, w), && w \in W \\
(\widetilde{\mathbf{u}}, \mathbf{v}) = (p, \nabla \cdot \mathbf{v}) - (g^D, \mathbf{v} \cdot n)_{\Gamma^D} - (\lambda, \mathbf{v} \cdot n)_{\Gamma^F}&, && \mathbf{v} \in V, \\
(\mathbf{u}, \widetilde{\mathbf{v}}) &= (K\widetilde{\mathbf{u}}, \widetilde{\mathbf{v}}), && \widetilde{\mathbf{v}} \in \widetilde{V}, \\
(\mathbf{u} \cdot n, \mu)_{\Gamma^F} &= (g_2^F + g_1^F \lambda, \mu)_{\Gamma^F}, && \mu \in \Lambda.
\end{aligned}$$

The Discrete Formulation

Let $(\mathcal{T}_h)_{h\geq 0}$ be a quasi-uniform family of finite element partitions of Ω such that no elements crosses the boundaries of Γ^D, Γ^N, or Γ^R, and where h is the maximal element diameter. Let $V_h \times W_h$ be any of the usual mixed finite element approximating subspaces of H(Ω; div)$\times$W, such as Brezzi-Douglas-Marini or Raviart-Thomas. Let $\Lambda_h \subset L^2(\partial\Omega)$ be the corresponding hybrid space of Lagrange multipliers for the pressure restricted to $\partial\Omega$. Define $V_h^0 = V_h \cap V^0$,

$$V_h^N = \{\mathbf{v} \in V_h : (\mathbf{v}\cdot n - g^N, \mu)_{\Gamma^N} = 0\}$$

for all $\mu \in \Lambda_h$, and $\Lambda_h^F = \Lambda_h|_{\Gamma^F}$.

Let $\widetilde{V}_h$ be a finite element subspace of $\widetilde{V}$ satisfying $V_h^N \subseteq \widetilde{V}_h$. In the mixed finite element approximation, we seek $\mathbf{u}_h \in V_h$, $\widetilde{\mathbf{u}} \in \widetilde{V}_h$, $p_h \in W_h$, and $\lambda_h \in \Lambda_h^F$ such that

$$\begin{aligned}
(\nabla\cdot\mathbf{u}_h, w) &= (f, w), && w \in W_h\\
(\widetilde{\mathbf{u}}_h, \mathbf{v}) &= (p_h, \nabla\cdot\mathbf{v}) - (g^D, \mathbf{v}\cdot n)_{\Gamma^D} - (\lambda_h, \mathbf{v}\cdot n)_{\Gamma^F}, && \mathbf{v} \in V_h,\\
(\mathbf{u}_h, \widetilde{\mathbf{v}}) &= (K\widetilde{\mathbf{u}}_h, \widetilde{\mathbf{v}}), && \widetilde{\mathbf{v}} \in \widetilde{V}_h,\\
(\mathbf{u}_h\cdot n, \mu)_{\Gamma^F} &= (g_2^F + g_1^F\lambda_h, \mu)_{\Gamma^F}, && \mu \in \Lambda_h^F.
\end{aligned}$$

Convergence of the Expanded Mixed Method

For a domain S let, $\|\cdot\|_{j,q,S}$ denote the norm of $W^{s,j}$(S), the Sobolev space of j-differentiable functions in L^q(S), and let H^j(S) $= W^{j,2}$(S), $\|\cdot\|_{j,S}$ is its norm and $\|\cdot\|_{-j,S}$ denote the norm of its dual space H^{-j}(S) $= (H^j(\mathrm{S}))'$. We may omit S if S $= \Omega$. The error will be measured in the norms of L^2 and H^{-j}. Let C denote a generic positive constant that is independent of the discretization parameter h. We make the following five hypotheses:

(H1) Problem (1.1) is 2-regular; i.e. given f, g^D, and g_f^2, there exists a unique solution $p \in H^2(\Omega)$ such that

$$\|p\|_2 \le C(\|f\|_0 + \|g^D\|_{3/2,\Gamma^D} + \|g_2^F\|_{1/2,\Gamma^F}),$$

where C depends only on Ω, K, and g_1^F;

(H2) $\nabla \cdot V_h = W_h$;

(H3) $V_h \cdot \mathrm{n}|_{\partial\Omega} \subset \Lambda_h$;

(H4) $V_h^N \subset V_h$

(H5) K is uniformly positive definite in Ω and $g_1^R \ge 0$.

We need four projection operators and their approximation properties. Let P_h denote L^2-projection of W onto W_h: for $\varphi \in$ W, $P_h\varphi \in W_h$ is defined by

$$(P_h\varphi - \varphi, w) = 0, \quad w \in W_h.$$

For $\varphi \in$ W,

$$\|P_h\varphi - \varphi\|_{-s} \le C\|\varphi\|_j h^{h+j}, \quad 0 \le s \le l, 0 \le j \le l,$$

where l is associated with the degree of the polynomials in W_h. Similarly , let $\widetilde{\Pi}$ denote the L^2-projection of $\widetilde{V}$ onto $\widetilde{V_h}$ and Q_h denote $L^2(\partial\Omega)$-projection onto Λ_h. For $\mathbf{q} \in H^j(\Omega)$ and $\psi \in H^j(\partial\Omega)$,

$$\|\mathbf{q} - \widetilde{\Pi}\mathbf{q}\|_{-s} \le C\|\mathbf{q}\|_j h^{j+s}, \quad 0 \le s \le k, 0 \le j \le k,$$

and

$$\|Q_h\psi - \psi\|_{-s,\partial\Omega} \le C\|\psi\|_{j,\partial\Omega} h^{j+s}, \quad 0 \le s \le m, 0 \le j \le m,$$

where k and m are associated with the degree of the polynomials in V_h and Λ_h, respectively, and where in (2.8) we can restrict to Γ^F.

Each of the mixed spaces we consider has a projection Π: $(H^1(\Omega))^d \to V_h$ with the three properties

$$
\begin{aligned}
&(\nabla \cdot \Pi \mathbf{q}, w) = (\nabla \cdot \mathbf{q}, w), \quad w \in W_h \\
&\|\Pi \mathbf{q} - \mathbf{q}\|_0 \le C \|\mathbf{q}\|_j h^j, \quad 1 \le j \le k, \\
&(\Pi \mathbf{q} \cdot n, \mu)_e = (\mathbf{q} \cdot n, \mu)_e, \quad \mu \in \Gamma_h
\end{aligned}
$$

where e is any element edge or face. The divergence and normal fluxes are well approximated by (2.6) and (2.8).

We now present the proof that the solution exists and is both unique and stable from Abogast [2].

Proposition *Assume (H1)-(H5). If ($\boldsymbol{u}_h$, $\widetilde{\boldsymbol{u}}_h$, p_h, λ_h) is a solution to the finite element approximation, then*

$$\|\nabla \cdot \boldsymbol{u}_h\|_0 \le \|f\|_0,$$

$$
\begin{aligned}
\|\boldsymbol{u}_h\|_0 + \|\widetilde{\boldsymbol{u}}_h\|_0 + \|\boldsymbol{u}_h \cdot n\|_{0,\Gamma^F} + \|p_h\|_0 + \left\| \sqrt{g_2^F} \lambda_h \right\|_{0,\Gamma^F} + \|\lambda_h\|_{-1/2,\Gamma^F} \\
\le C(\|f\|_0 + \|g^D\|_{1/2,\Gamma^D} + \|g_2^F\|_{1/2,\Gamma^F}),
\end{aligned}
$$

where C depends on Ω, $\|K\|_{1,\infty}$, and $\|g_1^F\|_{0,\infty,\Gamma^F}$.

Corollary *Assume (H1)-(H5). There exists a unique solution to the finite element approximation*

The next theorem expresses the error in approximating the weak form of the original problem by the finite element approximation.

Proposition *Assume (H1)-(H5). There exists a constant C, independent of h*

and dependent on Ω*,* p*,* $\boldsymbol{u}$*,* $\|K\|_{0,\infty}$*, and* $\|g_1^F\|_{0,\infty,\Gamma^F}$ *such that*

$$\|\mathbf{u}-\mathbf{u}_h\|_0 + \|\widetilde{\mathbf{u}}-\widetilde{\mathbf{u}}_h\|_0 + \|\sqrt{g_1^F}(\lambda-\lambda_h\|_{0,\Gamma^F}$$

$$\leq C(\|\mathbf{u}-\Pi\mathbf{u}\|_0 + \|\widetilde{\mathbf{u}}-\widetilde{\Pi}\widetilde{\mathbf{u}}_h\|_0 + \|\lambda-Q_h\lambda\|_{0,\Gamma^F})$$

$$\leq Ch^j,\ 0 \leq j \leq min(k,m),$$

$$\|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_{-s} = \|\nabla\cdot(\mathbf{u}-\Pi\mathbf{u}_h)\|_{-s}$$

$$\leq Ch^{j+s}, \quad \text{for } 0 \leq s \leq l, 0 \leq j \leq l,$$

$$\|(\mathbf{u}-\mathbf{u}_h)\cdot n\|_{0,\Gamma^F} \leq C(\|\sqrt{g_1^F}(\lambda-\lambda_h)\|_{0,\Gamma^F} + \|(\mathbf{u}-\Pi\mathbf{u})\cdot n\|_{0,\Gamma^F})$$

$$\leq Ch^j, \quad \text{for } 0 \leq j \leq min(k,m)$$

Moreover, if $0 \leq s \leq min(k,l,m)-1$*,* Ω *is (s+2) regular, and* C *depends also on* $\|K\|_{s+1,\infty}$ *and* $\|g_1^F\|_{s+1,\infty,\Gamma^F}$*, then for any* $0 \leq j \leq min(k,l,m)$*,*

$$\|P_hp-p_h\|_{-s} + \|Q_h\lambda-\lambda_h\|_{-s-1/2,\Gamma^F}$$

$$\leq C(\|\widetilde{\mathbf{u}}-\widetilde{\mathbf{u}}_h\|_0 + \|\mathbf{u}-\mathbf{u}_h\|_0 + \|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_0 + \|(\mathbf{u}-\mathbf{u}_h)\cdot n\|_{0,\Gamma^F}$$

$$+\|\sqrt{g_1^F}(\lambda-\lambda_h)\|_{0,\Gamma^F} + \|Q_h\lambda-\lambda_h\|_{0,\Gamma^F})h^{s+1} \leq Ch^{j+s+1},$$

$$\|p-p_h\|_{-s} \leq Ch^{j+s}$$

$$\|\lambda-\lambda_h\|_{-s-1/2,\Gamma^F} \leq Ch^{j+s+1/2},$$

$$
\begin{aligned}
&\|\mathbf{u}-\mathbf{u}_h\|_{-s}+\|\widetilde{\mathbf{u}}-\widetilde{\mathbf{u}}_h\|_{-s}\\
&\qquad\le C([\|\widetilde{\mathbf{u}}-\widetilde{\mathbf{u}}_h\|_0]h^s\\
&\qquad+[\|P_hp-p_h\|_0+\|Q_h\lambda-\lambda_h\|_{0,\Gamma^F}]h^{s-1}\\
&\qquad+\|P_hp-p_h\|_{-s+1}+\|Q_h\lambda-\lambda_h\|_{-s+1/2,\Gamma^F}\le Ch^{j+s}
\end{aligned}
$$

Cell-Centered Finite Difference Method

In this section we derive a finite difference stencil for the pressure in the case of the lowest-order Raviart-Thomas spaces on rectangles. Recall that on that element $E \in \mathcal{T}_h$,

$$V_h(E)=\{(\alpha_1x_1+\beta_1,\alpha_2x_2+\beta_2,\alpha_3x_3+\beta_3)^T:\alpha_i,\beta_i\in\mathbb{R}\}$$

$$W_h(E)=\{\alpha:\alpha\in\mathbb{R}\}$$

and on an edge or face e,

$$\Lambda_h(e)=\{\alpha;\alpha\in\mathbb{R}\},$$

where the last component of V_h should be deleted if $d=2$. We use the standard nodal basis, where for V_h and Λ_h the nodes are at the midpoints of the edges or faces of the elements and for W_h the nodes are at the midpoints of the elements (cell centers). We choose $\widetilde{V}_h = V_h$ in the finite element approximation.

In this paper $(\cdot,\cdot)_{\mathbf{M}}$ and $(\cdot,\cdot)_{\mathbf{T}}$ mean an application of the midpoint and trapezoidal rule, respectively (in each coordinate direction), and for $\mathbf{v}$, $\mathbf{q} \in R^d$,

$$(\mathbf{v},\mathbf{q})_{\mathbf{TM}}=\begin{cases}(v_1,q_1)_{\mathbf{T}X\mathbf{M}}+(v_2,q_2)_{\mathbf{M}X\mathbf{T}} \text{ if d} = 2\\ (v_1,q_1)_{\mathbf{T}X\mathbf{M}X\mathbf{M}}+(v_2,q_2)_{\mathbf{M}X\mathbf{T}X\mathbf{M}}+(v_3,q_3)_{\mathbf{M}X\mathbf{M}X\mathbf{T}} \text{ if d} = 3\end{cases}$$

Our goal is to express approximately $\mathbf{u}_h$ and $\widetilde{\mathbf{u}}_h$ in terms of p_h and λ_h from in the last three equations of the finite element approximation; then the first equation of the finite element approximation will give us an equation for the pressures. To do this, we use numerical quadrature rules for evaluating some of the integrals of the finite element approximation. The approximate problem is to solve for $\mathbf{u}_h \in V_h$, $\widetilde{\mathbf{u}}_h \in V_h$, $p_h \in W_h$, and $\lambda_h \in \Lambda_h^F$ such that

$$
\begin{aligned}
(\nabla \cdot \mathbf{u}_h, w) &= (f, w), && w \in W_h \\
(\widetilde{\mathbf{u}}_h, \mathbf{v})_{\mathbf{TM}} &= (p_h, \nabla \cdot \mathbf{v}) - (g^D, \mathbf{v} \cdot n)_{\Gamma^D} - (\lambda_h, \mathbf{v} \cdot n)_{\Gamma^F}, && \mathbf{v} \in V_h, \\
(\mathbf{u}_h, \widetilde{\mathbf{v}})_{\mathbf{TM}} &= (K\widetilde{\mathbf{u}}_h, \widetilde{\mathbf{v}})_{\mathbf{T}}, && \widetilde{\mathbf{v}} \in \tilde{V}_h, \\
(\mathbf{u}_h \cdot n, \mu)_{\Gamma^F} &= (g_2^F + g_1^F \lambda_h, \mu)_{\Gamma^F}, && \mu \in \Lambda_h^F.
\end{aligned}
$$

In other words, for computing an integral on the i^{th} component of the vectors, $i = 1, 2(, 3)$, we apply the trapezoid rule in the i^{th} direction and the midpoint rule in the other directions. The choice of quadrature rules is compatible with the nodal basis functions for V_h; it gives diagonal coefficient matrices for $\widetilde{\mathbf{u}}_h$ in the second equation and also for $\mathbf{u}_h$ in third equation. This technique is sometimes referred to as a lumped mass approximation. It happens that for $\mathbf{v}$,$\mathbf{q} \in V_h$ we have,

$$(\mathbf{v}, \mathbf{q})_{\mathbf{TM}} = (\mathbf{v}, \mathbf{q})_{\mathbf{T}}.$$

Also the matrix given by $(\mathbf{v}_i,\, \mathbf{v}_j)_{\mathbf{TM}}$, where i and j run over a nodal basis of V_h, is diagonal, independently of whether K is diagonal or not. This explains why the expanded mixed method was used. The second equation expresses the normal component of $\widetilde{\mathbf{u}}_h$ at any nodal point as a difference of the pressure at the midpoints of the two adjacent elements, or, near the boundary as a difference of a pressure and either a Lagrange multiplier pressure or Dirichlet pressure. This corresponds to a finite difference approximation of the equation $\widetilde{\mathbf{u}} = -\nabla p$. The third equation expresses the normal component of $\mathbf{u}_h$ at ay node by the normal components of $\widetilde{\mathbf{u}}_h$ at the nodes of the adjacent elements. Note that $\mathbf{u}_h$ does not depend on the components of $\widetilde{\mathbf{u}}_h$ on the far left and right edges. Thus we get a relatively compact finite difference approximation of the equation $\mathbf{u} = K\widetilde{\mathbf{u}}$.

Finally substituting the last three equations in the first equation, we obtain a finite difference stencil for the pressure, and approximation of the elliptic problem $-\nabla \cdot K\nabla p = f$. We get a 9-point stencil in two dimensions and a 19-point stencil in three dimensions. If K is a diagonal tensor, the stencil is reduced to five, or seven points. If a uniform mesh and a constant K are used, we obtain a standard finite difference procedure. In the strict interior

$$f_{ij}h^2 = 2(K_{11} + K_{22})p_{h,ij} - K_{11}(p_{h,i-1,j} + p_{h,i+1,j}) - K_{22}(p_{h,i,j-1} + p_{h,i,j+1})$$

$$+\frac{1}{2}K_{12}(p_{h,i+1,j-1} + p_{h,i-1,j+1} - p_{h,i-1,j-1} - p_{h,i+1,j+1}).$$

The local truncation error is $O(h^2)$, except near the boundary. Many other $O(h^2)$ finite difference schemes can be constructed that vary mainly in how K is treated and the second-order derivatives are approximated. The next section shows that our scheme has global convergence properties. Moreover, it is symmetric and locally conservative, and it has a compact 9-, or 19-point stencil and connections to mixed finite elements methods. Moreover, it can be extended easily to non rectangular grids.

III. A FAST SOLVER

In this section, we shall design a preconditioner for expanded mixed finite element method. We first cast the equation into an operator form and show it is equivalent to the Schur complement operator:

Theorem *The expanded mixed finite element method can be written as follows:*

$$\begin{pmatrix} A_h & B_h \\ B_h^T & 0 \end{pmatrix} \begin{pmatrix} p \\ \hat{u} \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

where A_h and B_h are given as follows:

$$A_h = \begin{pmatrix} 0 & I \\ I & -K_h \end{pmatrix} \text{ and } B_h = (-\nabla_h \cdot, 0)^T.$$

and

$$\hat{u} = \begin{pmatrix} \widetilde{u} \\ u \end{pmatrix}$$

The system is therefore, a symmetric system and it is spectrally equivalent to

$$\begin{pmatrix} A_h & 0 \\ 0 & -\nabla_h \cdot K_h \nabla_h \end{pmatrix} \begin{pmatrix} p \\ \hat{u} \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

Proof. We will use the following Woodbury Matrix Identity [11]

$$\begin{pmatrix} A & U \\ V & C \end{pmatrix} = \begin{pmatrix} I & 0 \\ VA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & C - VA^{-1}U \end{pmatrix} \begin{pmatrix} I & A^{-1}U \\ 0 & I \end{pmatrix}$$

So using the Woodbury Matrix Identity on the left hand side of the original system we get that

$$\begin{pmatrix} A_h & B_h \\ B_h^T & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ B_h^T A_n^{-1} & I \end{pmatrix} \begin{pmatrix} A_h & 0 \\ 0 & -B_h^T A_h^{-1} B_h \end{pmatrix} \begin{pmatrix} I & A_h^{-1} B_h \\ 0 & I \end{pmatrix}$$

where

$$\begin{aligned} B_h^T A_h^{-1} B_h &= \begin{pmatrix} -\nabla\cdot & 0 \end{pmatrix} \begin{pmatrix} 0 & I \\ I & -K_h \end{pmatrix}^{-1} \begin{pmatrix} \nabla \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} -\nabla\cdot & 0 \end{pmatrix} \begin{pmatrix} -K_h & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \nabla \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \nabla \cdot K_h & 0 \end{pmatrix} \begin{pmatrix} \nabla \\ 0 \end{pmatrix} = \nabla \cdot K_h \nabla \end{aligned}$$

Thus,

$$\begin{pmatrix} A_h & B_h \\ B_h^T & 0 \end{pmatrix} \approx \begin{pmatrix} A_h & 0 \\ 0 & -B_h^T A_h^{-1} B_h \end{pmatrix}$$

and this completes the proof □

Abstract Framework of the Auxiliary Space Preconditioner

In this section we will present the auxiliary space preconditioner and we will follow Xu's [3] framework. Assume that a linear inner product space $\mathcal{V}$ is given together with a linear operator A: $\mathcal{V} \to \mathcal{V}$ that is symmetric positive definite (SPD) with respect to an inner product $(\cdot, \cdot)$. Consider the following linear equation

$$Au = f$$

The main ingredient of the auxiliary space preconditioning method is another auxiliary linear inner product space $\mathcal{V}_0$ together with operator $A_0 : \mathcal{V}_0 \to \mathcal{V}_0$ that is symmetric positive definite with respect to an inner product $[\cdot, \cdot]$ on $\mathcal{V}_0$. This space may be viewed as a certain approximation for $\mathcal{V}$. The space $\mathcal{V}_0$ needs not be a subspace of $\mathcal{V}$ in general, but it should be, in a sense, simpler than $\mathcal{V}$. A_0 is assumed to be preconditioned by another symmetric positive definite operator $B_0 : \mathcal{V}_0 \to \mathcal{V}_0$.

The auxiliary space $\mathcal{V}_0$ is linked with the original space $\mathcal{V}$ by an operator $\Pi : \mathcal{V}_0 \to \mathcal{V}$. The restriction operator is given by its adjoint $\Pi^t : \mathcal{V} \to \mathcal{V}_0$ defined by

$$[\Pi^t v, w] = (v, \Pi w) \qquad v \in \mathcal{V}, w \in \mathcal{V}_0$$

Another ingredient is an symmetric positive definite operator R: $\mathcal{V} \to \mathcal{V}$. The role of R is to resolve what can not be resolved by the aforementioned space $\mathcal{V}_0$ and the operators defined on $\mathcal{V}_0$.
With th ingredients described above, the proposed preconditioner is as follows.

$$B = R + \Pi B_0 \Pi$$

By definition, for any $u, v \in \mathcal{V}_h$,

$$(BAu, v)_A = (RAu, v)_A + (B_0 A_0 \Pi^* u, \Pi^* v)_{A_0},$$

where $(\cdot, \cdot)_A = (\mathrm{A}\cdot, \cdot)$, $(\cdot, \cdot)_{A_0} = (A_0 \cdot, \cdot)$ and $\Pi^* = A_0^{-1} \Pi^t A$ satisfying

$$(\Pi^* v, w)_{A_0} = (v, \Pi w)_A \qquad\qquad v \in \mathcal{V}, w \in \mathcal{V}_0.$$

Denote $\rho_A = \rho(A)$, the spectral radius of A. Also denote $\| \cdot \|$ to be the norm induced by $(\cdot, \cdot)$.

Thus we need to present the following theorem proved by Xu [3] that places a bound on the condition number of the preconditioner.

Theorem *Assume that there are some non-negative constants* α_0, α_1, λ_0, λ_1 *and* β_1 *such that, for all* $v \in \mathcal{V}$ *and* $w \in \mathcal{V}_0$

$$\alpha_0 \rho_A^{-1}(v,v) \leq (Rv,v) \leq \alpha_1 \rho_A^{-1}(v,v),$$

$$\lambda_0 [w,w]_{A_0} \leq [B_0 A_0 w, w]_{A_0} \leq \lambda_1 [w,w]_{A_0},$$

$$\|\Pi w\|_A^2 \leq \beta_1 \|w\|_{A_0}^2.$$

and furthermore, assume that there exists a linear operator P*:* $\mathcal{V} \to \mathcal{V}_0$ *and positive constants* β_0 *and* γ_0 *such that,*

$$\|Pv\|_{A_0}^2 \leq \beta_0^{-1} \|v\|_A^2$$

$$\|v - \Pi P v\|^2 \leq \gamma_0^{-1} \rho_A^{-1} \|v\|_A^2.$$

Then the preconditioner satisfies

$$\kappa(BA) \leq (\alpha_1 + \beta_1 \lambda_1)((\alpha_0 \lambda_0)^{-1} + (\beta_0 \lambda_0)^{-1})$$

In particular, if P *is a right inverse of* Π*, namely* $\Pi P v = v$*, for* $v \in \mathcal{V}$ *then,*

$$\kappa((\Pi \beta_0 \Pi^t) A) \leq \frac{\beta_1}{\beta_0} \frac{\lambda_1}{\lambda_0}.$$

A Nonconforming P1 Interpolant

In this section we present a non-conforming P1 inetrpolant that will be used in the auxiliary space preconditioner following Brenner [5]. The goal of interpolation is to create a new set of points with better properties. Interpolation creates this new set of points within the range of the known discrete set of points. Let $S(\mathcal{T},\Omega)$ be the set of interior open edges ($d=2$) or open faces ($d=3$). Also denote the set of boundary edges of faces by $S(\mathcal{T},\partial\Omega)$ and the minimum angle of the triangles or tetrahedra in $\mathcal{T}$ by $\theta_{\mathcal{T}}$.

The nonconforming P1 finite element space associated with the triangulation $\mathcal{T}$ is $V_0 = \{v \in L_2(\Omega) : v_T = v|_T \in P_1(\mathcal{T})$ for any $\tau \in \mathcal{T}$ and v is continuous at the center of the common side of any two neighboring triangles$\}$, which is the Crouzeix-Raviert finite element space. A function in V_0 is completely determined by its nodal values at the centers of the sides of the triangles.

The interpolation operator $P_0 : H^1(\Omega,\mathcal{T}) \to V_0$ is defined by

$$(P_0\xi)(c_\sigma) = \frac{1}{|\sigma|}\int_\sigma \{\xi\} ds \qquad \forall \sigma \in S(\mathcal{T},\Omega) \cup S(\mathcal{T},\partial\Omega),$$

where c_σ is the center of the side σ and $\{\xi\}$ is the average of the traces from the two sides of σ. For $\sigma \subset \partial\Omega$, we take $\{\xi\}$ to be ξ.

Let $\Pi_T : H^1(\mathcal{T}) \to P_1(\mathcal{T})$ be the local interpolation operator defined by

$$(\Pi_T\xi)(c_\sigma) = \frac{1}{|\sigma|}\int_\sigma \xi dz \qquad \forall \sigma \subset \partial\Omega.$$

From (3.11) and (3.12) we see the difference of the two interpolants on $\tau \in \mathcal{T}$ is given by

$$(P_0\xi - \Pi_T\xi)(c_\sigma) = \begin{cases} \frac{1}{2|\sigma|}\int_\sigma [\![\xi]\!] ds & \text{if } \sigma \subset \partial\mathcal{T} \setminus \partial\Omega \\ 0 & \text{if } \sigma \subset \partial\mathcal{T} \cap \partial\Omega \end{cases}$$

where the jump $[\![\xi]\!]$ is measured by subtracting the interior trace from the exterior trace.

Using the difference between the above estimates and standard finite element estimates (cf. [9,10]), we find

$$\begin{aligned}
|P_0\xi - \Pi_T\xi|^2_{H^1(T)} &\leq |\tau|^{1-(2/d)} \sum_{\sigma\subset\partial\mathcal{T}} [(P_0\xi - \Pi_T\xi)(c_\sigma]^2 \\
&\leq |\tau|^{1-(2/d)} \sum_{\sigma\subset\partial\mathcal{T}\setminus\partial\Omega} \frac{1}{|\sigma|^2}\left(\int_\sigma [\![\xi]\!] ds\right)^2 \\
&\leq \sum_{\sigma\subset\partial\mathcal{T}\setminus\partial\Omega} |\sigma|^{d/(1-d)}\left(\int_\sigma [\![\xi]\!] ds\right)^2
\end{aligned}$$

$$\begin{aligned}
\|P_0\xi - \Pi_T\xi\|^2_{L^2(T)} &\leq |\tau| \sum_{\sigma\subset\partial\mathcal{T}} [(P_0\xi - \Pi_T\xi)(c_\sigma]^2 \\
&\leq \sum_{\sigma\subset\partial\mathcal{T}\setminus\partial\Omega} |\sigma|^{(2-d)/(1-d)}\left(\int_\sigma [\![\xi]\!] ds\right)^2
\end{aligned}$$

where $|\tau|$ is the d-dimensional volume of τ. Note that

$$|\tau| \approx |\sigma|^{d/(d-1)} \qquad \text{for } \sigma \subset \partial\tau.$$

On the other hand, we also have the following well known estimates for the local interpolation operator (cf. [8]) :

$$\|\xi - \Pi_T\xi\|^2_{L_2(\mathcal{T})} + |\tau|^{2/d}|\Pi_T\xi|^2_{H^1(\mathcal{T})} \leq |\tau|^{2/d}|\xi|^2_{H^1(\mathcal{T})}$$

Combing the estimates and summing over all $\tau \in \mathcal{T}$ we find

$$\begin{aligned}
|P_0\xi|^2_{H^1(\Omega,\mathcal{T})} \leq |\xi|^2_{H^1(\Omega,\mathcal{T})} &+ \sum_{\sigma\in S(\mathcal{T},\Omega)} |\sigma|^{d/(1-d)}\left(\int_\sigma [\![\xi]\!] ds\right)^2 \\
\|\xi - P_0\xi\|^2_{L^2(\mathcal{T})} &\leq \sum_{\tau\in\mathcal{T}} |\tau|^{2/d}|\xi|^2_{H^1(T)} \\
&+ \sum_{\sigma\in S(T,\Omega} |\sigma|^{(2-d)/(d-1)}\left(\int_\sigma [\![\xi]\!] ds\right)^2 .
\end{aligned}$$

Preconditioner

In this section we lay out the preconditioner following Xu's framework for the auxiliary space method. We use Crouziex-Raviart finite element space as the auxiliary space to precondition $H^1(\Omega)$ functions. We know define the following spaces:

$\mathcal{V} = H^1(\Omega)$

$\mathcal{V}_0 = \{v \in L^2(\Omega), v \in P_1(\tau), \forall \tau \in \mathcal{T}_h, v$ is continuous at the midpoints of the interior edges (faces)$\}$

Since we can see that $\mathcal{V}_0 \subset \mathcal{V}$, we can use B_0 from the above auxiliary framework and use Q_0 in place of Π^t, since Π is the identity operator from the inclusion of the spaces, and where $Q_0 : \mathcal{V} \to \mathcal{V}_0$ is the $L^2(\Omega)$ projection operator. Thus we have the preconditioner

$$B = R + B_0 Q_0,$$

where R : $\mathcal{V} \to \mathcal{V}$ is a smoother. We also define the operator, P, from Xu's Theorem as P = $P_0 : \mathcal{V} \to \mathcal{V}_0$, where P_0 is the above $P1$ non-conforming interpolation operator. We can then see the estimates, for v $\in \mathcal{V}$ needed for the auxiliary space preconditioner are shown in the theory for the $P1$ non-conforming interpolation.

IV. FUTURE WORK

Future work will include finishing numerical results for this preconditioner and taking this preconditioning method one step further to $P1$ conforming finite element space because it is easier to solve, and again get numerical results to compare with the one proposed in this paper and the original work on by Abogast [2]. We also wish to try solving the original by preconditioning with Abogast's [2] quadrature rules form that gives the cell-centered finite difference method. Then we want to expand the problem to a non-linear case which can be applied further to work done by engineers and physicists.

REFERENCES

[1] Chen, Zhangxin. "Expanded mixed finite element methods for linear second-order elliptic problems, I." ESAIM: Mathematical Modelling and Numerical Analysis 32.4 (1998): 479-499.

[2] Abogast, Todd, Mary F. Wheeler, and Ivan Yotov. "Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences." SIAM Journal on Numerical Analysis 34.2 (1997): 828-852.

[3] Xu, Jinchao. "The auxiliary space method and optimal multigrid preconditioning techniques for unstructured grids." Computing 56.3 (1996): 215-235.

[4]Chen, Zhangxin. "Expanded mixed finite element methods for quasilinear second order elliptic problems, II" ESAIM: M2AN 32 (4) (1998) 501-520

[5]Brenner, Susanne C. "Poincar–Friedrichs Inequalities for Piecewise H 1 Functions." SIAM Journal on Numerical Analysis 41.1 (2003): 306-324.

[6] F. Brezzi; J. Douglas, and L. D Marini, Two families of mixed finite elements for second order elliptic problems, Numer. Math., vol 47. no. 2., 1985. 217235. 1, 2.2

[7] P.-A. Raviart and J. M. Thomas. A mixed finite element method for 2nd order elliptic problems. In Mathematical aspects of finite element methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975), pages 292315. Lecture Notes in Math., Vol. 606. Springer, Berlin, 1977.

[8] Crouzeix, Michel, and P-A. Raviart. "Conforming and nonconforming finite element methods for solving the stationary Stokes equations I." Revue franaise d'automatique informatique recherche oprationnelle. Mathmatique 7.R3 (1973): 33-75.

[9] S. C. Brenner and L. R. Scott,The Mathematical Theory of Finite Element Methods,2nd ed., Springer-Verlag, New York, Berlin, Heidelberg, 2002.

[10] P. Ciarlet,The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam,1978

[11] Woodbury, Max A. Inverting Modified Matrices. Princeton, NJ: Department of Statistics, Princeton University , 1950.