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RELATIONSHIP BETWEEN DIFFERENT TYPES OF STABILITY FOR LINEAR ALMOST PERIODIC SYSTEMS IN BANACH SPACES

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Abstract. For the linear equation $x' = A(t)x$ with recurrent (almost periodic) coefficients in an arbitrary Banach space, we prove that the asymptotic stability of the null solution and of all limit equations implies the uniform stability of the null solution.

Introduction

In 1962, W. Hahn [13] posed the problem of whether asymptotic stability implies uniform stability for linear equation

$$x' = A(t)x \qquad (x \in \mathbb{R}^n) \tag{0.1}$$

with almost periodic coefficients. In 1965, C. C. Conley and R. K. Miller [12] gave a negative answer to this question, by constructing a scalar equation $x' = a(t)x$ with the property that every solution $\varphi(t, x, a) \to 0$ as $t \to +\infty$, but the null solution is not uniformly stable (see also [4]). From the results by R. J. Sacker and G. R. Sell [17] and I. U. Bronshteyn [2, p.141], the uniform stability of the null solution to (0.1) holds under the following conditions: The matrix $A(t)$ in (0.1) is recurrent (in particular, almost periodic), and the asymptotic stability holds for the null solution of (0.1) and for the null solutions of all systems

$$x' = B(t)x\,, \tag{0.2}$$

where $B \in H(A) = \overline{\{A_\tau : \tau \in \mathbb{R}\}}$, with A_τ denoting the translation of the matrix A by τ and the bar denoting the closure in the topology of the uniform convergence on compact subsets of $\mathbb{R}$. Also we want to point out that from the results of the author in [5], the result mentioned above is valid for (0.1) with compact matrix (i. e., when $H(A)$ is compact). The goal of the present paper is to study the relationship between the asymptotical stability and uniform stability of the null solution of system (0.1) in arbitrary Banach spaces.

Our main result is that for (0.1) with recurrent coefficients in an arbitrary Banach space the following statement holds: If the null solution of (0.1) and the null solutions of all equations (0.2) are asymptotically stable, then the null solution of (0.1) is uniformly stable.

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1. Linear non-autonomous dynamical systems

Assume that X and Y are complete metric spaces, $\mathbb{R}$ is the set of real numbers, $\mathbb{Z}$ is the set integer numbers, $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, $\mathbb{T}_+ = \{t \in \mathbb{T} : t \geq 0\}$, and $\mathbb{T}_- = \{t \in \mathbb{T} : t \leq 0\}$. Denote by $(X, \mathbb{T}_+, \pi)$ a semigroup dynamical system on X, and by $(Y, \mathbb{T}, \sigma)$ a group on Y. A triple $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$, where h is a homomorphism of $(X, \mathbb{T}_+, \pi)$ onto $(Y, \mathbb{T}, \sigma)$, is called a non-autonomous dynamical system.

The system $(X, \mathbb{T}_+, \pi)$ is called: [6-7]

point dissipative, if there is $K \subseteq X$ such that for all $x \in X$

$$\lim_{t\to+\infty} \rho(xt, K) = 0, \tag{1.1}$$

where $xt = \pi^t x = \pi(t, x)$;

compactly dissipative, if (1.1) holds uniformly with respect to x on compact subsets of X;

locally dissipative, if for a point $p \in X$ there is $\delta_p > 0$ such that (1.1) holds uniformly with respect to $x \in B(p, \delta_p)$;

A non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is said to be point (compact, local) dissipative, if the autonomous dynamical system $(X, \mathbb{T}_+, \pi)$ is so.

Let (X, h, Y) be a locally trivial Banach fibre bundle over Y [3]. A non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is said to be linear if the mapping $\pi^t : X_y \to X_{yt}$ is linear for every $t \in \mathbb{T}_+$ and $y \in Y$, where $X_y = \{x \in X | h(x) = y\}$ and $yt = \sigma(t, y)$. Let $|\cdot|$ be a norm on (X, h, Y), i. e., $|\cdot|$ is co-ordinated with the metric ρ (that is $\rho(x_1, x_2) = |x_1 - x_2|$ for any $x_1, x_2 \in X$ such that $h(x_1) = h(x_2)$). In [8], the author obtained a point (compact, local) dissipativity criterion for linear systems.

Theorem 1.1 [8]. *Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be a linear non-autonomous dynamical system and Y be compact, then the following assertions hold*

1. $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is point dissipative if and only if $\lim_{t\to+\infty} |xt| = 0$ for all $x \in X$;

2. A non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is compactly dissipative if and only if $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is point dissipative and there exists a positive number M such that for all $x \in X$ and $t \in \mathbb{T}_+$,

$$|xt| \leq M|x|\,; \tag{1.2}$$

3. A non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is locally dissipative if and only if there exist positive numbers N and ν such that $|xt| \leq Ne^{-\nu t}|x|$ for all $x \in X$ and $t \in \mathbb{T}_+$.

¿From the Banach-Steinhauss theorem it follows that point dissipativity and compact dissipativity are equivalent for autonomous linear systems. For an example of a linear autonomous dynamical system which is compactly dissipative but not locally dissipative, see [8].

Theorem 1.2. *Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be a linear non-autonomous dynamical system and the following conditions hold*

1. Y is compact and minimal (i. e., $Y = H(y) = \overline{\{yt : t \in \mathbb{T}\}}$ for all $y \in Y$);

2. for each $x \in X$ there exists $C_x \geq 0$ such that for all $t \in \mathbb{T}_+$,

$$|xt| \leq C_x\,; \tag{1.3}$$

3. the mapping $y \mapsto \|\pi^t_y\|$ is continuous for every $t \in \mathbb{T}_+$, where $\|\pi^t_y\|$ is the norm of the linear operator $\pi^t_y = \pi^t|_{X_y}$.

Then there exists $M \geq 0$ such that (1.2) holds for all $t \in \mathbb{T}_+$ and $x \in X$.

Proof. From Condition 2. and the Banach-Steinhauss theorem, it follows the uniform boundedness of the family of linear operators $\{\pi^t_y : t \in \mathbb{T}_+\}$ for every $y \in Y$, i. e., for each $y \in Y$ there exists $M_y \geq 0$ such that $\|\pi^t_y\| \leq M_y$ for all $t \in \mathbb{T}_+$. We put

$$d(y) = \sup_{t\geq 0} \|\pi^t_y\| \tag{1.4}$$

and claim that $d : Y \to \mathbb{R}_+$ is lower semi-continuous, i. e., $\liminf_{y_n\to y} d(y_n) \geq d(y)$ for all $y \in Y$ and $\{y_n\} \to y$. Suppose that this is not true, then there exist $y \in Y, \{y_n\}$ and $\varepsilon > 0$ such that

$$\liminf_{y_n\to y} d(y_n) = d(y) - \varepsilon \tag{1.5}$$

From (1.4) it follows that $d(y) = \lim_{n\to+\infty} \|\pi^{t_n}_y\|$ for some sequence $\{t_n\} \subseteq \mathbb{T}_+$ and, consequently, there exists k such that

$$|\|\pi^{t_n}_y\| - d(y)| < \frac{\varepsilon}{4} \tag{1.6}$$

for all $n \geq k$. By the the continuity of mapping $y \mapsto \|\pi^t_y\|$ there exists $n(k)$ such that

$$|\|\pi^{t_k}_{y_n}\| - \|\pi^{t_k}_y\|| < \frac{\varepsilon}{4} \tag{1.7}$$

for all $n \geq n(k)$. From (1.6) and (1.7),

$$|d(y) - \|\pi^{t_k}_{y_n}\|| < \frac{\varepsilon}{2} \tag{1.8}$$

for all $n \geq n(k)$. From (1.8),

$$|d(y) - d(y_n)| \leq \frac{\varepsilon}{2} \tag{1.9}$$

for all $n \geq n(k)$. Notice that (1.9) contradicts (1.5), and this contradiction proves that $d : Y \to \mathbb{R}_+$ is lower semicontinuous. Hence, this function has a set of points of continuity $D \subset Y$ of the type G_δ. Let $p \in D$, then there exist positive numbers δ_p and M_p such that $d(y) \leq M_p$ for all $y \in S[p, \delta_p] = \{y \in Y | \rho(y, p) \leq \delta_p\} \subset Y$.

Since Y is minimal, there are negative numbers $t_1, t_2, \dots, t_m$ such that $Y = \bigcup_{i=1}^m \sigma(S[p, \delta_p], t_i)$ (see [16, p.134]). We put $L = \max\{t_i | i = 1, 2, \dots, m\}$. Assume that $m \in Y$, $y \in S[p, \delta_p]$ and t_i are such that $m = yt_i$. Then

$$|xt| = |\pi^{t+t_i}_y(\pi^{-t_i}_{yt_i}(x))| \leq M_p C|x| \tag{1.10}$$

for all $x \in X$ with $h(x) = m$ and $t \geq L$, where

$$C = \max\{\max\{\|\pi^{-t_i}_y\| : y \in Y\}, i = 1, 2, \dots, m\}\,.$$

We claim that the family of operators $\{\pi^t : t \in [0, L]\}$ is uniformly continuous, that is, for any $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that $|x| \leq \delta$ implies $|xt| \leq \varepsilon$ for all

$t \in [0, L]$. On the contrary, assume that there are $\varepsilon_0 > 0$, $\delta_n > 0$ with $\delta_n \to 0$, $|x_n| < \delta_n$ and $t_n \in [0, L]$ such that

$$|x_n t_n| \geq \varepsilon_0. \tag{1.11}$$

Since (X, h, Y) is a locally trivial Banach fibre bundle and Y is compact, then the zero section $\Theta = \{\theta_y : y \in Y\}$ of (X, h, Y) is compact and, consequently, we can assume that the sequences $\{x_n\}$ and $\{t_n\}$ are convergent. Put $x_0 = \lim_{n\to+\infty} x_n$ and $t_0 = \lim_{n\to+\infty} t_n$, then $x_0 = \theta_{y_0}$ $(y_0 = h(x_0))$. Passing to the limit in (1.11) as $n \to +\infty$, we obtain $0 = |x_0 t_0| \geq \varepsilon_0$. This last inequality contradicts the choice of ε_0, and hence proves the above assertion. If $\gamma > 0$ is such that $|\pi^t x| \leq 1$ for all $|x| \leq \gamma$ and $t \in [0, L]$, then

$$|xt| \leq \frac{1}{\gamma}|x| \tag{1.12}$$

for all $t \in [0, L]$ and $x \in X$. We put $M = \max\{\gamma^{-1}, M_p C\}$, then from (1.10) and (1.12) it follows (1.2) for all $t \geq 0$ and $x \in X$. The theorem is proved.

Remark 1.3. *a.) If the fibre bundle (X, h, Y) is finite-dimensional, then condition 3 in Theorem 1.2 holds.*

b.) Let $X = E \times Y$, where E is a Banach space and $\pi = (\varphi, \sigma)$, i. e., $\pi^t x = (\varphi(t, u, y), \sigma^t y)$ for all $t \in \mathbb{T}_+$ and $x = (u, y) \in X = E \times Y$. Then condition 3 in Theorem 1.2 holds, if for every $t \in \mathbb{T}_+$ the mapping $U(t, \cdot) : Y \to [E]$ is continuous, where $U(t, y)u = \varphi(t, u, y)$ with $(t, u, y) \in \mathbb{T}_+ \times E \times Y$ and $[E]$ the Banach space of continuous operators acting on E equipped with the operator norm.

Theorem 1.4. *Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be a linear non-autonomous dynamical system, Y be a compact minimal set and the mapping $y \mapsto \|\pi^t_y\|$ be continuous for each $t \in \mathbb{T}_+$. Then the point dissipativity of $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ implies its compact dissipativity.*

Proof. Assume that the conditions of Theorem 1.4 are fulfilled and the non-autonomous dynamical system $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ is point dissipative, then according to Theorem 1.1 for every $x \in X$ there exists a constant $C_x \geq 0$ such that (1.3) holds for all $x \in X$ and $t \in \mathbb{T}_+$. Then by referring to Theorem 1.1, the present proof is complete.

Theorem 1.5 [6]. *Let $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ be a linear non-autonomous dynamical system, Y be compact, then the following assertions take place.*

1. If $(X, \mathbb{T}_+, \pi)$ is completely continuous (i. e., for all bounded subset $A \subset X$ there exists a positive number $l = l(A)$ such that $\pi^l A$ is precompact), then from the point dissipativity of $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ follows its local dissipativity;

2. If $(X, \mathbb{T}_+, \pi)$ is asymptotically compact (i. e., for all bounded sequence $\{x_n\} \subset X$ and $\{t_n\} \to +\infty$ the sequence $\{x_n t_n\}$ is precompact), then from the compact dissipativity of $\langle (X, \mathbb{T}_+, \pi), (Y, \mathbb{T}, \sigma), h\rangle$ results its local dissipativity.

2. Some classes of linear non-autonomous differential equations

Let Λ be a complete metric space of linear operators that act on Banach space E and $C(\mathbb{R}, \Lambda)$ be a space of all continuous operator-functions $A : \mathbb{R} \to \Lambda$ equipped with the open-compact topology and $(C(\mathbb{R}, \Lambda), \mathbb{R}, \sigma)$ be the dynamical system of shifts on $C(\mathbb{R}, \Lambda)$.

Ordinary linear differential equations. Let $\Lambda = [E]$ and consider the linear differential equation

$$u' = \mathcal{A}(t)u\,, \tag{2.1}$$

where $\mathcal{A} \in C(\mathbb{R}, \Lambda)$. Along with equation (2.1), we shall also consider its $H-$class, that is, the family of equations

$$v' = \mathcal{B}(t)v\,, \tag{2.2}$$

where $\mathcal{B} \in H(\mathcal{A}) = \overline{\{\mathcal{A}_\tau : \tau \in \mathbb{R}\}}, \mathcal{A}_\tau(t) = \mathcal{A}(t+\tau) \quad (t \in \mathbb{R})$ and the bar denotes closure in $C(\mathbb{R}, \Lambda)$. Let $\varphi(t, u, \mathcal{B})$ be the solution of equation (2.2) that satisfies the condition $\varphi(0, v, \mathcal{B}) = v$. We put $Y = H(\mathcal{A})$ and denote the dynamical system of shifts on $H(\mathcal{A})$ by $(Y, \mathbb{R}, \sigma)$, then the triple $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ is a linear non-autonomous dynamical system, where $X = E \times Y, \pi = (\varphi, \sigma) \quad$ (i. e., $\pi((v, \mathcal{B}), \tau) = (\varphi(\tau, v, \mathcal{B}), \mathcal{B}_\tau)$ and $h = pr_2 : X \to Y$. Applying Theorem 1.4 to this system, we obtain the following assertion.

Theorem 2.1. *Let $\mathcal{A} \in C(\mathbb{R}, \Lambda)$ be recurrent (i. e., $H(\mathcal{A})$ is compact minimal set of $(C(\mathbb{R}, \Lambda), \mathbb{R}, \sigma)$) and the zero solutions of equation (2.1) and all equations (2.2) are asymptotically stable, i. e., $\lim\limits_{t\to+\infty} |\varphi(t, v, \mathcal{B})| = 0$ for all $v \in E$ and $\mathcal{B} \in H(\mathcal{A})$. Then the zero solution of equation (2.1) is uniformly stable, i. e., there exists $M \geq 0$ such that $|\varphi(t, v, \mathcal{B})| \leq M|v|$ for all $t \geq 0, v \in E$ and $\mathcal{B} \in H(\mathcal{A})$.*

Proof. By Lemma 2 in [9], the mapping $\mathcal{B} \mapsto \varphi(t, \cdot, \mathcal{B})$ from $H(\mathcal{A})$ into $[E]$ is continuous for all $t \in \mathbb{R}$. Then applying Theorem 1.3 this proof is complete.

Partial linear differential equations. Let Λ be some complete metric space of linear closed operators acting on a Banach space $E \quad$ (for example $\Lambda = \{A_0 + B : B \in [E]\}$, where A_0 is a closed operator that acts on E). We assume that the following conditions are fulfilled for equation (2.1) and its $H-$ class (2.2).

a.) For every $v \in E$ and $\mathcal{B} \in H(\mathcal{A})$ equation (2.2) has exactly one solution that is defined on $\mathbb{R}_+$ and satisfies the condition $\varphi(0, v, \mathcal{B}) = v$.

b.) The mapping $\varphi : (t, v, \mathcal{B}) \to \varphi(t, v, \mathcal{B})$ is continuous in the topology of $\mathbb{R}_+ \times E \times C(\mathbb{R}; \Lambda)$.

c.) For every $t \in \mathbb{R}_+$ the mapping $U(t, \cdot) : H(\mathcal{A}) \to [E]$ is continuous, where $U(t, \cdot)$ is the Cauchy operator of equation (2.2), i. e., $U(t, \mathcal{B})v = \varphi(t, v, \mathcal{B}) \quad (t \in \mathbb{R}_+, v \in E$ and $\in H(\mathcal{A})$).

Under the above assumptions, (2.1) generates a linear non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$, where $X = E \times Y$, $\pi = (\varphi, \sigma)$ and $h = pr_2 : X \to Y$. Applying Theorem 1.4 to this system, we will obtain the analogue to Theorem 2.1 for different classes of partial differential equations.

We will consider an example of a partial differential equation which satisfies conditions a)-c) above. Let $\mathcal{H}$ be a Hilbert space with a scalar product $\langle \cdot, \cdot \rangle = |\cdot|^2$, $\mathcal{D}(\mathbb{R}_+, \mathcal{H})$ be the set of all infinite differentiable and bounded functions on $\mathbb{R}_+$ with values in $\mathcal{H}$.

Denote by $(C(\mathbb{R}, [\mathcal{H}]), \mathbb{R}, \sigma)$ the dynamical system of shifts on $C(\mathbb{R}, [\mathcal{H}])$. Consider the equation

$$\int\limits_{\mathbb{R}_+} \langle u(t), \varphi'(t)\rangle + \langle \mathcal{A}(t)u(t), \varphi(t)\rangle \, dt = 0\,, \tag{2.3}$$

along with the family of equations

$$\int_{\mathbb{R}_+} \langle u(t), \varphi'(t)\rangle + \langle \mathcal{B}(t)u(t), \varphi(t)\rangle \, dt = 0\,, \tag{2.4}$$

where $\mathcal{B} \in H(\mathcal{A}) = \overline{\{\mathcal{A}_\tau : \tau \in \mathbb{R}\}}$, $\mathcal{A}_\tau(t) = (t+\tau)$ and the bar denotes closure in $C(\mathbb{R}, [\mathcal{H}])$.

A function $u \in C(\mathbb{R}_+, \mathcal{H})$ is called a solution of (2.3), if the equality in (2.3) is satisfied for all $\varphi \in \mathcal{D}(\mathbb{R}_+, \mathcal{H})$.

Assume that the operator $\mathcal{A}(t)$ is self-adjoint. Let $(H(\mathcal{A}), \mathbb{R}, \sigma)$ be the dynamical system of shifts on $H(\mathcal{A})$, $\varphi(t, v, \mathcal{B})$ be a solution of (2.4) with the condition $\varphi(0, v, \mathcal{B}) = v$, $\overline{X} = \mathcal{H} \times H(\mathcal{A})$, X be a set of all the points $\langle u, \mathcal{B}\rangle \in \overline{X}$ such that through point $u \in \mathcal{H}$ passes a solution $\varphi(t, u, \mathcal{A})$ of (2.3) defined on $\mathbb{R}_+$. According to Lemma 2.21 in [10] the set X is closed in $\overline{X}$. By Lemma 2.22 in [10] the triple $(X, \mathbb{R}_+, \pi)$ is a dynamical system on X (where $\pi = (\varphi, \sigma)$) and $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ is a linear non-autonomous dynamical system, with $h = pr_2 : X \to Y = H(\mathcal{A})$. Applying the results from [1] it is possible to show that for every t the mapping $\mathcal{B} \mapsto U(t, \mathcal{B})$ (where $U(t, \mathcal{B})v = \varphi(t, v, \mathcal{B})$) from $H(\mathcal{A})$ into $[\mathcal{H}]$ is continuous and, consequently, for this system we can apply Theorem 1.3. Thus the following assertion takes place.

Theorem 2.2. *Let $\mathcal{A} \in C(\mathbb{R}, [\mathcal{H}])$ be recurrent and the zero solution of (2.1) and the zero solutions of (2.2) be asymptotically stable, i. e., $\lim_{t\to+\infty} |\varphi(t, v, \mathcal{B})| = 0$ for all $v \in E$ and $\mathcal{B} \in H(\mathcal{A})$. Then the zero solution of (2.1) is uniformly stable, i. e., there exists $M \geq 0$ such that $|\varphi(t, v, \mathcal{B})| \leq M|v|$ for all $t \geq 0, v \in \mathcal{H}$ and $\mathcal{B} \in H(\mathcal{A})$.*

We will give an example of a boundary-value problem reduced to an equation of type (2.3). Let Ω be a bounded domain in $\mathbb{R}^n$, Γ be boundary of Ω, $Q = \mathbb{R}_+ \times \Omega$ and $S = \mathbb{R}_+ \times \Gamma$. In Q consider the initial boundary-value problem

$$\frac{\partial u}{\partial t} = L(t)u \qquad (u|_{t=0} = \varphi, u|_S = 0)\,, \tag{2.5}$$

where

$$L(t)u = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(t,x)\frac{\partial u}{\partial x_j}) - a(t,x)u\,.$$

By the Riesz representation theorem,

$$\langle \mathcal{A}(t)u, \varphi\rangle = -\int_{\Omega} [\sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i} + a(t,x)u\varphi]dx.$$

If $a_{ij}(t,x) = a_{ji}(t,x)$ and the functions $a_{ij}(t,x)$ and $a(t,x)$ are recurrent (almost periodic) with respect to $t \in \mathbb{R}$ uniformly with respect to $x \in \Omega$, then we can apply Theorem 2.2 to equation (2.5), if $\mathcal{H} = \dot{W}_2^1(\Omega)$.

Linear functional-differential equations. Let $r > 0$, $C([a,b], \mathbb{R}^n)$ be the Banach space consisting of continuous functions from $[a,b]$ to $\mathbb{R}^n$ with the supremum norm. Then we put $C = C([-r, 0], \mathbb{R}^n)$. Let $\sigma \in \mathbb{R}$, $A \geq 0$ and $u \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$. For $t \in [\sigma, \sigma + A]$ we define $u_t \in C$ by $u_t(\theta) = u(t+\theta)$,

$-r \leq \theta \leq 0$. Denote by $\mathfrak{A} = \mathfrak{A}(C, \mathbb{R}^n)$ the Banach space consisting of linear continuous operators from C into $\mathbb{R}^n$, equipped with the operator norm. Consider the equation

$$u' = \mathcal{A}(t)u_t\,, \tag{2.6}$$

where $\mathcal{A} \in C(\mathbb{R}, \mathfrak{A})$. We put $H(\mathcal{A}) = \overline{\{\mathcal{A}_\tau : \tau \in \mathbb{R}\}}$, $\mathcal{A}_\tau(t) = \mathcal{A}(t+\tau)$ and the bar denotes closure in the topology of uniform convergence on compact subsets of $\mathbb{R}$.

Along with equation (2.6) we also consider the family of equations

$$u' = \mathcal{B}(t)u_t\,, \tag{2.7}$$

where $\mathcal{B} \in H(\mathcal{A})$. Let $\varphi(t, v, \mathcal{B})$ be a solution of (2.7) with $\varphi_0(v, \mathcal{B}) = v$ on $\mathbb{R}_+$. We put $Y = H(\mathcal{A})$ and denote by $(Y, \mathbb{R}, \sigma)$ the dynamical system of shifts on $H(\mathcal{A})$. Let $X = C \times Y$ and $\pi = (\varphi, \sigma)$ be the dynamical system on X, defined by $\pi(t, (v, \mathcal{B})) = (\varphi_\tau(v, \mathcal{B}), \mathcal{B}_\tau)$. The non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ ($h = pr_2 : X \to Y$) is linear, and the following assertion takes place.

Lemma 2.3. *Let $H(\mathcal{A})$ be compact in $C(\mathbb{R}, \mathfrak{A})$, then the non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ generated by (2.6) is completely continuous.*

Proof. Let B be a bounded subset of $C[-r, 0]$, and $t \geq r$. By the continuity of the mapping $\varphi : \mathbb{R}_+ \times C \times H(A) \mapsto C$ and the compactness of $H(\mathcal{A})$ there exists a positive number M such that $|\varphi_\tau(v, \mathcal{B})| \leq M$ and $|\mathcal{B}(\tau)\varphi_\tau(v, \mathcal{B})| \leq M$ for all $\tau \in [0, t]$, $\mathcal{B} \in H(\mathcal{A})$ and $v \in B$. Consequently, $|\dot{\varphi}(\tau, v, \mathcal{B})| \leq M$ for all $\tau \in [0, t]$, $\mathcal{B} \in H(\mathcal{A})$, and $v \in B$, i. e., the family of functions $\{\varphi_t(v, B) : \mathcal{B} \in H(\mathcal{A}), v \in B\}$ (for $t \geq r$) is uniformly continuous on $[-r, 0]$. Therefore, this family of functions is precompact, and the present proof is complete.

Theorem 2.4. *Let $H(\mathcal{A})$ be compact. Then the following assertion are equivalent.*

1. For any $\mathcal{B} \in H(\mathcal{A})$ the zero solution of (2.7) is asymptotically stable, i. e., $\lim_{t\to+\infty} |\varphi_t(v, \mathcal{B})| = 0$ for all $v \in C$ and $\mathcal{B} \in H(\mathcal{A})$.

2. The zero solution of (2.6) is uniformly asymptotically stable, i. e., there are the positive numbers N and ν such that $|\varphi_t(v, \mathcal{B})| \leq Ne^{-\nu t}|v|$ for all $t \geq 0$, $v \in C$ and $\mathcal{B} \in H(\mathcal{A})$.

Proof. Let $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ be the linear non-autonomous dynamical system, generated (2.6). By Lemma 2.3 this system is completely continuous. Then using Theorems 1.1 and 1.5, we conclude the present proof.

Consider the neutral functional differential equation

$$\frac{d}{dt}Dx_t = \mathcal{A}(t)x_t\,, \tag{2.8}$$

where $\mathcal{A} \in C(\mathbb{R}, \mathfrak{A})$ and $D \in \mathfrak{A}$ is non-atomic at zero [14, p. 67]. As in the case of (2.6), the equation (2.8) generates a linear dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$, where $X = C \times Y$, $Y = H(\mathcal{A})$ and $\pi = (\varphi, \sigma)$.

Theorem 2.5 [4, 11]. *Let $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ be a non-autonomous dynamical system, and the mapping $\pi^t = \pi(\cdot, t) : X \to X(t\mathbb{R}_+)$ be representable as a sum $\pi(x, t) = \varphi(x, t) + \psi(x, t)$ for all $t\mathbb{R}_+$ and $x \in X$, and the following conditions be are fulfilled.*

1. $|\varphi(x,t)| \leq m(t,r)$ *for all* $t\mathbb{R}_+$, $r > 0$ *and* $|x| \leq r$, *where* $m : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *and* $m(t,r) \to 0$ *for* $t \to +\infty$;

2. The mappings $\psi(\cdot,t) : X \to X (t > 0)$ *are conditionally completely continuous, i. e.,* $\psi(A,t)$ *is relatively compact for any* $t > 0$ *and any bounded positively invariant set* $A \subseteq X$.

Then the dynamical system $(X, \mathbb{R}_+, \pi)$ *is asymptotically compact.*

Lemma 2.6. *Let* $H(\mathcal{A})$ *be compact and the operator* D *be stable, i. e., the zero solution of the homogeneous difference equation* $Dy_t = 0$ *is uniformly asymptotically stable. Then a linear non-autonomous dynamical system* $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$, *generated by (2.8), is asymptotically compact.*

Proof. According to [15, p.119, formula (5.18)] the mapping $\varphi_t(\cdot, \mathcal{B}) : C \to C$ can be written as

$$\varphi_t(\cdot, \mathcal{B}) = S_t(\cdot) + U_t(\cdot, \mathcal{B})$$

for all $\mathcal{B} \in H(\mathcal{A})$, where $U_t(\cdot, \mathcal{B})$ is conditionally completely continuous for $t \geq r$. Also there exist positive constants N, ν such that $\|S_t\| \leq Ne^{-\nu t}(t \geq 0)$. Then this proof is complete by referring to Theorem 2.5.

Theorem 2.7. *Let* $\mathcal{A} \in C(\mathbb{R}, \mathfrak{A})$ *be recurrent (i. e.,* $H(\mathcal{A})$ *is compact minimal in the dynamical system of shifts* $(C(\mathbb{R}, \mathfrak{A}), \mathbb{R}, \sigma)$ *) and let* D *be stable. Then the following assertions are equivalent.*

1. The zero solution of (2.6) and the zero solutions of all equations

$$\frac{d}{dt} Dx_t = \mathcal{B}(t)x_t\,, \tag{2.9}$$

where $\mathcal{B} \in H(\mathcal{A})$, *are asymptotically stable, i. e.,* $\lim_{t\to+\infty} |\varphi(t, v, \mathcal{B})| = 0$ *for all* $v \in C$ *and* $\mathcal{B} \in H(\mathcal{A})$ *(*$\varphi(t, v, \mathcal{B})$ *is a solution of (2.9) with* $\varphi(0, v, \mathcal{B}) = v$*).*

2. The zero solution of (2.8) is uniformly exponentially stable, i. e., there are a positive numbers N *and* ν *such that* $|\varphi(t, v, \mathcal{B})| \leq Ne^{-\nu t}|v|$ *for all* $t \geq 0$, $v \in C$ *and* $\mathcal{B} \in H(\mathcal{A})$.

Proof. Let $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h\rangle$ be a linear non-autonomous dynamical system, generated by (2.8). By Lemma 2.6 this system is asymptotically compact. To complete this proof it is sufficient to refer to Theorems 1.1 and 1.5.

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