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On some nonlinear potential problems *

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Dedicated to Prof. Dr. K. Kalik
on the occasion of his 70th birthday 1998

Abstract

The degree theory of mappings is applied to a two–dimensional semilinear elliptic problem with the Laplacian as principal part subject to a nonlinear boundary condition of Robin type. Under some growth conditions we obtain existence. The analysis is based on an equivalent coupled system of domain–boundary variational equations whose principal parts are the Dirichlet bilinear form in the domain and the single layer potential bilinear form on the boundary, respectively. This system consists of a monotone and a compact part. Additional monotonicity implies convergence of an appropriate Richardson iteration.

The degree theory also provides the instrument for showing convergence of a subsequence of a nonlinear finite element — boundary element Galerkin scheme with decreasing mesh width. Stronger assumptions provide strong monotonicity, uniqueness and convergence of the discrete Richardson iterations. Numerical experiments show that the Richardson parameter as well as the number of iterations (for given accuracy) are independent of the mesh width.

1 Introduction

In a bounded domain $\Omega \in \mathbb{R}^n$ we consider the nonlinear boundary value problem,

$$\begin{aligned} -\Delta u &= \Psi(x,u,\nabla u) \quad \text{in } \Omega, \\ -\frac{\partial u}{\partial n} &= \Phi(x,u) \quad \text{on } \partial\Omega = \Gamma. \end{aligned}$$

For the homogeneous differential equation with $\Psi = 0$ and strictly monotone Φ, this problem can be reduced to a strongly monotone boundary integral equation of Hammerstein type. These equations and their finite element – boundary

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element approximations with Galerkin as well as collocation methods have been investigated in [2], [7], [10], [11], [15], [16], [21], [22], [23]. Spectral methods for these equations have been considered in [5], [12]. In this paper, we consider more general Φ and Ψ. With the general theory of the degree of mappings in connection with a–priori estimates, we obtain existence and regularity results. We also consider a finite element – boundary element Galerkin scheme which approximates these equations in two and three dimensions. Additional restrictions for the nonlinear terms provide uniqueness of the solution u and allow at the same time a convergence and error analysis of Galerkin schemes. The solution of the nonlinear problem is constructed by an appropriate relaxation method in combination with successive approximation — a constructive method which also works for the discretizations.

Our paper is organized as follows: In Section 2 we present a brief introduction to the degree theory of mappings in Banach spaces for a class of operators which is adequate for our nonlinear boundary value problems. Here we follow the presentations in [9], [14], [26]. If a–priori estimates are available then by using homotopy we obtain the existence of solutions.

In Section 3 we apply this technique to our nonlinear boundary value problem. For this purpose we decompose the problem into a strongly monotone and a non–monotone compact mapping where the latter also satisfies appropriate growth conditions. These assumptions allow us to show the above–mentioned a–priori estimates yielding existence and regularity of solutions.

Section 4 is devoted to uniqueness results which are obtained for “small” perturbations of strongly monotone operators. Using potential methods we also obtain convergence of a certain relaxation method and a successive iteration scheme. Both iterations can also be performed for a coupled domain finite element and boundary element approximation which is considered in Section 5.

The relaxation method is an improvement of the iteration in [2]. Our approach gives a constructive solution procedure for the discrete systems of nonlinear approximate equations, too. We also show asymptotic energy norm error estimates of optimal order in terms of the corresponding mesh width.

In Section 6 we present numerical experiments for this iteration procedure with finite and boundary elements.

2 Mapping degree theory

The degree theory of mappings in Banach spaces became one of the most important branches of global nonlinear analysis with applications in mathematical physics. In particular, it can be applied to nonlinear elliptic boundary value problems which can be reformulated as problems of infinite–dimensional geometry in Banach spaces. Then the solutions of the nonlinear boundary value problems can be considered as fixed points of nonlinear mappings or as the preimage of a point under a nonlinear mapping or as the intersection of finitely many submanifolds. Here we use the concept of preimages in connection with degree theory of mappings in Banach spaces which extends the classical finite–

dimensional theory. For the finite–dimensional case, let G be a domain in the n–dimensional space X^n and consider a continuous mapping $A : \overline{G} \to X_n$. Then, for arbitrary $y \in X_n$ with distance $\operatorname{dist}(A(\partial G), y) \geq \delta > 0$, we can define an integer $d(A, G, y)$, the **degree** of the mapping A with respect to G and y in the following way: To A choose a smooth mapping $\widetilde{A} : \overline{G} \to X_n$ with

1) $\sup_{x \in G} \|\widetilde{A}(x) - A(x)\| < \delta/2$,

2) $\widetilde{A}^{-1}(y) = \{x^{(1)}, \dots, x^{(N)}\}$ consists of a finite number of points such that

$$\det \frac{\partial}{\partial x} \widetilde{A}(x^{(j)}) \neq 0, \quad \text{for} \quad j = 1, \dots, N.$$

Now define the degree $\deg(\widetilde{A}, G, y)$ of $\widetilde{A}$ with respect to G and y in the usual manner by

$$\deg(\widetilde{A}, G, y) := \sum_{j=1}^{N} \operatorname{sgn} \det d\widetilde{A}(x^{(j)}).$$

It is well known that d does not depend on the special choice of $\widetilde{A}$ for all $\widetilde{A}$ satisfying the above properties 1) and 2) which justifies the definition of the degree d of A by

$$\deg(A, G, y) := \deg(\widetilde{A}, G, y).$$

The basic properties of the degree $\deg(A, G, y)$ are the following (see [24]):

P1. If $\deg(A, G, y) \neq 0$ then the equation $Ax = y$ has at least one solution $x \in G$.

P2. Let $\mathcal{A} : \overline{G} \times [0, 1] \to X_n$ be a continuous mapping satisfying $\operatorname{dist}(A_\lambda(\partial G), y) \geq \delta > 0$ with $A_\lambda := \mathcal{A}(\cdot, \lambda)$ for all $\lambda \in [0, 1]$. Then the degree is constant: $\deg(A_\lambda, G, y) = \text{const}$.

P3. For the identity mapping I there holds $\deg(I, G, y) \neq 0$ if $y \in \overset{\circ}{G}$.

It is well known that these properties of the degree $\deg(A, G, y)$ provide a method for proving existence of solutions of a nonlinear equation $Ax = y$. In fact, if the mapping $A_1 := A$ admits a homotopy A_t to a simple operator A_0 such that the conditions on ∂G are fulfilled, then $\deg(A_0, G, y) \neq 0$ yields the existence of a solution $x \in G$ of the original equation according to the properties P1, P2. However, this theory does not permit a simple analogous procedure in the infinite-dimensional case. This can be seen from the following two most important paradoxical examples:

1. According to Kuiper's theorem (see [17]), the group of invertible linear continuous operators in a Hilbert space is connected and, hence, there is no concept of orientation in the Hilbert space. Analogous results are also true for most Banach spaces (see [19]).

2. There exists a smooth diffeomorphic mapping of the unit ball B_1 in a Hilbert space onto $B_1\backslash\{0\}$ under which the boundary ∂B_1 remains motionless. Nevertheless, this mapping admits a homotopy to the identity (see [3]), in the contrary to the properties P1-P3.

This shows that the mappings which are admissible in degree theory are singled out among the general continuous ones by special additional geometrical properties allowing the definition of the degree and other topological invariants. In fact, there are various degree theories of nonlinear mappings which generalize the classical Leray-Schauder degree theory [4], [9], [14],[24], [26].

In our paper we will use the degree theory of mappings A which admit a decomposition $A = B + T$ with a strongly monotone operator B and a compact operator T. For this class of mappings it is possible to define the degree, such that the properties P1-P3 remain valid (see [9], [14], [26]). Following [9], we give a brief description of the case when X is a Hilbert space.

Let G be a bounded domain in the Hilbert space X and $A : \overline{G} \to X$ be a continuous mapping of the form $A = B + T$ as above. Then, for all $y \notin A(\partial G)$ we define the degree $\deg(A, G, y)$ in the following way:
Let $\mathcal{T}(X)$ denote the set of all finite–dimensional subspaces of X. Then $\mathcal{T}(X)$ is partially ordered by inclusion. For $T \in \mathcal{T}(X)$, the orthogonal projection onto T will be denoted by P_T. One can show that there exists a space $T_0 \in \mathcal{T}(X)$ with $y \in T_0$ such that for any $T \in \mathcal{T}(X)$ with $T_0 \subset T$ the set $P_T(\partial G \cap T)$ does not contain y. Then the Brower degree $\deg_B(P_T A, G \cap T, y)$ is well defined and independent of the special choice of T. This mapping degree then has the following additional property:

P4. If B is a strongly monotone operator, then $\deg(B, G, y) \neq 0$ if $y \in \overset{\circ}{G}$.

In many applications, the bounded domain $G \subset X$ can be defined from an a–priori estimate for all solutions u_λ of the set of equations

$$A_\lambda u_\lambda = 0$$

associated with the homotopy A_λ. If such a uniform estimate

$$\|u_\lambda\|_H \leq C$$

is available then we can choose $\overline{G} := \{u \in H | \, \|u\|_H \leq C + 1\}$.

3 The potential problem

Let $\Omega \subset \mathbb{R}^n, (n = 2 \text{ or } 3)$, be a bounded domain with smooth boundary Γ satisfying $\operatorname{diam}(\Omega) < 1$ for $n = 2$ which is just a scaling assumption. We now specialize the nonlinear boundary value problem to

$$-\Delta u = f(x, u, \nabla u) + d \quad \text{in } \Omega\,, \tag{3.1}$$

$$\tfrac{\partial u}{\partial n} + b_0(x, u) = b_1(x, u) + g \quad \text{on } \Gamma\,. \tag{3.2}$$

As in [2], [22] we suppose b_0 to be a Carathéodory function i.e. $b_0(\cdot, u)$ is measurable for all $u \in \mathbb{R}$ and $b_0(x, \cdot)$ is continuous for almost all $x \in \Gamma$. Further we assume that $\frac{\partial}{\partial u} b_0(x,u)$ is Borel measurable and satisfies

$$0 < c \leq \frac{\partial}{\partial u} b_0(x,u) \leq C < \infty \quad \text{ for almost all } x \in \Gamma \text{ and all } u \in \mathbb{R}.$$

These conditions imply that the Nemitzky operator $B_0 : L^2(\Gamma) \to L^2(\Gamma)$ defined by

$$[B_0 u](x) := b_0(x, u(x)) \quad \text{for a.e. } \ x \in \Gamma \tag{3.3}$$

is Lipschitz continuous and strongly monotone; i.e. there are positive constants $l, L > 0$ such that

$$\begin{gathered} \|B_0 u - B_0 v\|_{L^2(\Gamma)} \leq L \|u - v\|_{L^2(\Gamma)} \\ \text{and} \\ (B_0 u - B_0 v, u - v)_{L^2(\Gamma)} \geq l \|u - v\|^2_{L^2(\Gamma)} \quad \text{for all } u, v \in L^2(\Gamma). \end{gathered} \tag{3.4}$$

Here and in the sequel we denote by $(z, w)_\Omega = \int\limits_\Omega z\overline{w} dx$ and $(u, v)_\Gamma = \int\limits_\Gamma u\overline{v} ds_\Gamma$ the corresponding L^2–dualities, by $H^s(\Omega)$ and $H^s(\Gamma)$ the Sobolev spaces of order s in Ω and on Γ, respectively. In particular, $H^{-s}(\Omega) = (\widetilde{H}^s(\Omega))'$ where $\widetilde{H}^s$ denotes the completion of $C_0^\infty(\Omega)$ in $H^s(\mathbb{R}^n)$.

Furthermore, for every $u \in H^s(\Gamma)$ and $0 \leq s \leq 1$, we have $B_0 u \in H^s(\Gamma)$ and $B_0 : H^s(\Gamma) \to H^s(\Gamma)$ is bounded (see [22]). We suppose that b_1, f are also Carathéodory functions for which there exist positive constants c and $\alpha < 1$ and functions $\beta \in L^2(\Gamma)$, $\varphi \in L^2(\Omega)$ such that

$$\begin{aligned} &|b_1(x,u)| \leq \beta(x) + c(1 + |u|)^\alpha, \\ &|f(x,u,v)| \leq \varphi(x) + c(1 + |u| + |v|)^\alpha \end{aligned} \tag{3.5}$$

for almost all $x \in \Gamma$ and $u \in \mathbb{R}$, $v \in \mathbb{R}$. Then, the corresponding Nemitzky operators $B_1 : L^2(\Gamma) \to L^2(\Gamma)$ with $B_1 u(x) = b_1(x, u(x))$ and $F : H^1(\Omega) \to L^2(\Omega)$ with $Fu(x) = f(x, u(x), \nabla u(x))$ are Lipschitz continuous and satisfy the estimates

$$\begin{aligned} &\|B_1 u\|_{L^2(\Gamma)} \leq c(1 + \|u\|_{L^2(\Gamma)})^\alpha \quad \text{for all } u \in L^2(\Gamma) \ \text{ and} \\ &\|Fv\|_{L^2(\Omega)} \leq c(1 + \|v\|_{H^1(\Omega)})^\alpha \quad \text{ for all } \ v \in H^1(\Omega). \end{aligned} \tag{3.6}$$

These estimates result from the following modification of a classical result [27, pp. 561–562]:

Theorem 3.1 *Suppose that $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a Caratheodory function which satisfies the growth condition:*

$$|f(x,u)| \leq a(x) + b \sum_{i=1}^m |u_i|^{\frac{\alpha r_i}{q}} \qquad \textit{for all } \ (x,u) \in \Omega \times \mathbb{R}^m$$

with fixed positive numbers α, b, with $0 < \alpha < 1, a(\cdot) \in L^q(\Omega)$ and $1 \le q,\ r_i < \infty$ for $i = 1, \ldots, m$. Then, the corresponding Nemitzky operator

$$(Fu)(x) := f(x, u_1(x), \ldots, u_m(x)) \text{ with } F : \prod_{i=1}^{m} L^{r_i}(\Omega) \to L^q(\Omega)$$

is continuous and bounded satisfying

$$\|Fu\|_{L^q} \le c \left(\|a\|_{L^q} + \sum_{i=1}^{m} (\|u_i\|_{L^{r_i}})^{\frac{\alpha r_i}{q}} \right) \quad \text{for all } u \in \prod_{i=1}^{m} L^{r_i}(\Omega)\,.$$

Inserting (3.1) and (3.2) into Green's formula

$$\int_\Omega (\Delta u) v \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Gamma \frac{\partial u}{\partial n} \, v \, ds_\Gamma = 0\,, \tag{3.7}$$

we obtain the weak formulation of our problem:

Let $d \in \widetilde{H}^{-1}$. Find $u \in H^1(\Omega)$ such that for all $v \in H^1(\Omega)$,

$$\begin{aligned} (Au, v)_{H^1(\Omega)} \quad := \quad & (\nabla u, \nabla v)_\Omega + (B_0 u|_\Gamma, v|_\Gamma)_\Gamma - (B_1 u|_\Gamma, v|_\Gamma)_\Gamma \\ & -(g, v|_\Gamma)_\Gamma - (Fu, v)_\Omega - (d, v)_\Omega = 0 \end{aligned} \tag{3.8}$$

In order to apply mapping degree theory we consider the parameter–dependent problem:

Find $u_\lambda \in H^1(\Omega)$ such that for all $v \in H^1(\Omega)$,

$$\begin{aligned} (A_\lambda u_\lambda, v) \quad := \quad & (\nabla u_\lambda, \nabla v)_\Omega + (B_0 u_\lambda|_\Gamma, v|_\Gamma)_\Gamma - \lambda(B_1 u_\lambda|_\Gamma, v|_\Gamma)_\Gamma \\ & -(g, v|_\Gamma)_\Gamma - \lambda(F u_\lambda, v)_\Omega - (d, v)_\Omega = 0 \end{aligned} \tag{3.9}$$

Theorem 3.2 *There is a constant $R > 0$ not depending on $\lambda \in [0, 1]$ such that all solutions u_λ of (3.9) are uniformly bounded:*

$$\|u_\lambda\|_{H^1(\Omega)} \le R. \tag{3.10}$$

Proof. First, we get with (3.9)

$$\begin{aligned} 0 \quad = \quad & (A_\lambda u_\lambda, u_\lambda)_{H^1(\Omega)} \ge |u_\lambda|^2_{H^1(\Omega)} + l\|u_\lambda\|^2_{L^2(\Gamma)} - \|B_0 u_\lambda\|_{L^2(\Gamma)} \|u_\lambda\|_{L^2(\Gamma)} \\ & -\|B_1 u_\lambda\|_{L^2(\Gamma)} \|u_\lambda\|_{L^2(\Gamma)} - \|g\|_{H^{-\frac{1}{2}}(\Gamma)} \|u_\lambda\|_{H^{\frac{1}{2}}(\Gamma)} \\ & -\|F u_\lambda\|_{L^2(\Omega)} \|u_\lambda\|_{L^2(\Omega)} - \|d\|_{\widetilde{H}^{-1}(\Omega)} \|u_\lambda\|_{H^1(\Omega)}. \end{aligned} \tag{3.11}$$

We use the trace lemma $\|v\|_{L^2(\Gamma)} \le \|v\|_{H^{\frac{1}{2}}(\Gamma)} \le c\|v\|_{H^1(\Omega)}$ for all $v \in H^1(\Omega)$ and the Friedrichs inequality [20]

$$C\|v\|^2_{L^2(\Omega)} \le \|\nabla v\|^2_{L^2(\Omega)} + l\|v\|^2_{L^2(\Gamma)}. \tag{3.12}$$

Using (3.12) in (3.11) and dividing by $\|u_\lambda\|_{H^1(\Omega)}$ we get

$$C\|u_\lambda\|_{H^1(\Omega)} \le \|B_0 u_\lambda\|_{L^2(\Gamma)} + \|B_1 u_\lambda\|_{L^2(\Gamma)} + \|g\|_{H^{-\frac{1}{2}}(\Gamma)} + \|F u_\lambda\|_{L^2(\Omega)} + \|d\|_{\widetilde{H}^{-1}(\Omega)} .$$

Inserting (3.6) into the inequality right above we obtain

$$\|u_\lambda\|_{H^1(\Omega)} \le c(1 + \|u_\lambda\|_{H^1(\Omega)})^\alpha$$

with $\alpha \in (0,1)$ and, thus, the boundedness of $\|u_\lambda\|_{H^1(\Omega)}$. ♢

Theorem 3.3 *The operator $A_\lambda : H^1(\Omega) \to H^1(\Omega)$ is of the form $A_\lambda = A_0 + \lambda A_1$ with a Lipschitz continuous strongly monotone operator $A_0 : H^1(\Omega) \to H^1(\Omega)$ and a compact operator $A_1 : H^1(\Omega) \to H^1(\Omega)$.*

Proof. Define A_0 by

$$(A_0 u, v)_{H^1(\Omega)} := (\nabla u, \nabla v)_\Omega + (B_0 u|_\Gamma, v|_\Gamma)_\Gamma. \tag{3.13}$$

Because of the Lipschitz continuity of B_0 in $L^2(\Gamma)$, the operator $A_0 : H^1(\Omega) \to H^1(\Omega)$ is also Lipschitz continuous. Inequality (3.4) yields

$$(A_0 u - A_0 v, u - v)_{H^1(\Omega)} \ge \|\nabla(u-v)\|^2_{L^2(\Omega)} + l\|u - v\|^2_{L^2(\Gamma)} \ge C\|u - v\|^2_{H^1(\Omega)}.$$

The operator A_1 defined by

$$(A_1 u, v)_{H^1(\Omega)} := -(B_1 u|_\Gamma, v|_\Gamma)_\Gamma - (g, v|_\Gamma)_\Gamma - (Fu, v)_\Omega - (d, v)_\Omega$$

is compact since $F : H^1(\Omega) \to L^2(\Omega)$ is continuous and the imbedding $L^2(\Omega) \hookrightarrow \widetilde{H}^{-1}(\Omega)$ is compact. $B_1 : L^2(\Gamma) \to L^2(\Gamma)$ is continuous and the imbeddings $H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^2(\Gamma) \hookrightarrow H^{-\frac{1}{2}}(\Gamma)$ are also compact. ♢

Applying the mapping degree theory sketched in Section 2 we finally have shown the following theorem.

Theorem 3.4 *For any given $h \in \widetilde{H}^{-1}(\Omega)$ and $g \in H^{-\frac{1}{2}}(\Gamma)$, the problem (3.8) has at least one solution $u \in H^1(\Omega)$. Furthermore, the set of solutions is a compact subset of $H^1(\Omega)$. If $h \in L^2(\Omega)$ and $g \in L^2(\Gamma)$ then we obtain the regularity result $u \in H^{\frac{3}{2}}(\Omega)$. If, in addition, we assume b_1 to be Lipschitz continuous, i. e.*

$$|b_1(x,u) - b_1(x,v)| \le L|u - v| \text{ for all } u, v \in \mathbb{R} \text{ and } x \in \Gamma \tag{3.14}$$

and if $g \in H^{\frac{1}{2}}(\Gamma)$ then $u \in H^2(\Omega)$.

This regularity result follows from the well known regularity properties of linear elliptic equations [18] and the mapping properties of F, B_0, and B_1.

4 Perturbations of the strongly monotone case

If the operators F and B_1 are sufficiently small, then the solution of (3.8) is unique due to the contraction principle.

Here we present a potential method in order to apply these arguments. As is well known, any smooth function v satisfies the Green representation formula

$$v = K_{\Gamma,\Omega}\ v|_\Gamma + V_{\Gamma,\Omega}\ \left.\frac{\partial v}{\partial n}\right|_\Gamma + V_{\Omega,\Omega}\ \triangle v \quad \text{in } \Omega \tag{4.1}$$

with the potentials

$$\begin{aligned}
V_{\Omega,\Omega}\ \psi(x) &= -\frac{1}{2\pi}\int_\Omega \psi(y)\ \log|x-y|\ dy && \text{for } x\in\Omega,\\
V_{\Gamma,\Omega}\ \psi(x) &= -\frac{1}{2\pi}\int_\Gamma \psi(y)\ \log|x-y|\ ds_\Gamma(y) && \text{for } x\in\Omega,\\
K_{\Gamma,\Omega}\ \psi(x) &= \frac{1}{2\pi}\int_\Gamma \psi(y)\ \frac{\partial}{\partial n_y}\log|x-y|\ ds_\Gamma(y) && \text{for } x\in\Omega.
\end{aligned} \tag{4.2}$$

Inserting the differential equation (3.1) and the boundary condition (3.2) in (4.1) we obtain the equation

$$u = K_{\Gamma,\Omega}\ u|_\Gamma - V_{\Gamma,\Omega}\ (B_0u|_\Gamma - B_1u|_\Gamma - g) - V_{\Omega,\Omega}\ (Fu + d) \ \text{ in } x\in\Omega\,. \tag{4.3}$$

The continuity of the single layer potential and the jump relations for the double layer potential yield the following equation for the boundary values:

$$(I - K_{\Gamma,\Gamma})\ u|_\Gamma + V_{\Gamma,\Gamma}\ (B_0u|_\Gamma - B_1u|_\Gamma - g) - V_{\Omega,\Gamma}\ (Fu + d) = 0 \ \text{ on } \Gamma\,. \tag{4.4}$$

The equations (4.3) and (4.4) can be considered as a system of equations for u in Ω and the boundary values $u|_\Gamma$ on Γ.

Here, the operators that map into the boundary spaces are defined by

$$\begin{aligned}
V_{\Omega,\Gamma}\ \psi(x) &= -\frac{1}{\pi}\int_\Omega \psi(y)\ \log|x-y|\ dy && \text{for } x\in\Gamma,\\
V_{\Gamma,\Gamma}\ \psi(x) &= -\frac{1}{\pi}\int_\Gamma \psi(y)\ \log|x-y|\ ds_\Gamma(y) && \text{for } x\in\Gamma,\\
K_{\Gamma,\Gamma}\ \psi(x) &= \frac{1}{\pi}\int_\Gamma \psi(y)\ \frac{\partial}{\partial n_y}\log|x-y|\ ds_\Gamma(y) && \text{for } x\in\Gamma.
\end{aligned} \tag{4.5}$$

It is well known [6] that the potential operators are linear continuous operators in the spaces

$$\begin{aligned}
&V_{\Omega,\Omega} : \widetilde{H}^{-1}(\Omega) \to H^1(\Omega), && V_{\Omega,\Gamma} : \widetilde{H}^{-1}(\Omega) \to H^{\frac12}(\Gamma),\\
&V_{\Gamma,\Omega} : H^{-\frac12}(\Gamma) \to H^1(\Omega), && V_{\Gamma,\Gamma} : H^{-\frac12}(\Gamma) \to H^{\frac12}(\Gamma),\\
&K_{\Gamma,\Omega} : H^{\frac12}(\Gamma) \to H^1(\Omega), && K_{\Gamma,\Gamma} : H^{\frac12}(\Gamma) \to H^{\frac32}(\Gamma).
\end{aligned} \tag{4.6}$$

According to (4.4), let us introduce the operator

$$L_0 := I - K_{\Gamma,\Gamma} + V_{\Gamma,\Gamma}B_0 : H^{\frac12}(\Gamma) \to H^{\frac12}(\Gamma). \tag{4.7}$$

Theorem 4.1 *The operator L_0 is Lipschitz continuous, invertible and has a Lipschitz continuous inverse.*

Proof. First, the Lipschitz continuity is clear from (3.4) and the mapping properties (4.6). We show that L_0 is $V_{\Gamma,\Gamma}^{-1}$–monotone i.e. there is a constant γ such that

$$(L_0\varphi - L_0\psi, V_{\Gamma,\Gamma}^{-1}(\varphi-\psi))_\Gamma \geq \gamma\|\varphi-\psi\|^2_{H^{\frac{1}{2}}(\Gamma)}\,. \tag{4.8}$$

Using the symmetry of $V_{\Gamma,\Gamma}$, we know that for T, the Steklov-Poincaré operator on harmonic functions, which is defined by

$$T \;:=\; V_{\Gamma,\Gamma}^{-1}(I-K_{\Gamma,\Gamma}) \tag{4.9}$$

Gårding's inequality is valid:

$$(T\varphi,\varphi)_\Gamma \;\geq\; \tilde{\gamma}\|\varphi\|^2_{\frac{1}{2}} - c\|\varphi\|^2_0 \qquad \text{for all } \varphi \in H^{\frac{1}{2}}(\Gamma)\,. \tag{4.10}$$

Hence (4.8) follows from the semidefiniteness of T and inequality (3.4) (see [2] and [22]).

For the construction of the inverse to L_0 we consider the sequence φ_n defined by

$$\varphi_{n+1} := \varphi_n - \alpha(L_0\varphi_n - f) \quad \text{for } n = 0,1,\ldots \tag{4.11}$$

with an appropriate $\alpha \in \mathbb{R}$ for some starting value $\varphi_0 \in H^{\frac{1}{2}}(\Gamma)$ and a given right hand side $f \in H^{\frac{1}{2}}(\Gamma)$. We get the estimate

$$\begin{aligned}
&(\varphi_{n+1}-\varphi_n, V_{\Gamma,\Gamma}^{-1}(\varphi_{n+1}-\varphi_n))_\Gamma \\
&\quad= (\varphi_n-\varphi_{n-1}, V_{\Gamma,\Gamma}^{-1}(\varphi_n-\varphi_{n-1}))_\Gamma - 2\alpha(L_0\varphi_n - L_0\varphi_{n-1}, V_{\Gamma,\Gamma}^{-1}(\varphi_n-\varphi_{n-1}))_\Gamma \\
&\qquad +\alpha^2(L_0\varphi_n - L_0\varphi_{n-1}, L_0\varphi_n - L_0\varphi_{n-1})_\Gamma \\
&\quad\leq (\varphi_n-\varphi_{n-1}, V_{\Gamma,\Gamma}^{-1}(\varphi_n-\varphi_{n-1}))_\Gamma - 2\alpha\gamma\|\varphi_n-\varphi_{n-1}\|^2_{H^{\frac{1}{2}}(\Gamma)} \\
&\qquad +\alpha^2 C\|\varphi_n-\varphi_{n-1}\|^2_{H^{\frac{1}{2}}(\Gamma)} \\
&\quad\leq (1-c_1\alpha+c_2\alpha^2)(\varphi_n-\varphi_{n-1}, V_{\Gamma,\Gamma}^{-1}(\varphi_n-\varphi_{n-1}))_\Gamma\,.
\end{aligned}$$

Hence, for $\alpha < \dfrac{c_1}{c_2}$ the sequence φ_n is a Cauchy sequence in the norm $\|\cdot\|^2_{V^{-1}} := (\,\cdot\,, V_{\Gamma,\Gamma}^{-1}\,\cdot\,)_\Gamma$ which is equivalent to the $H^{\frac{1}{2}}(\Gamma)$–norm. The limit $\varphi_0 = \lim_{n\to\infty}\varphi_n$ is necessarily a solution of $L_0\varphi_0 = f$. Due to (4.8) this solution is unique. For f_1, $f_2 \in H^{\frac{1}{2}}(\Gamma)$ we get

$$\begin{aligned}
\|L_0^{-1}f_1 - L_0^{-1}f_2\|^2_{H^{\frac{1}{2}}(\Gamma)} &\leq \gamma^{-1}((f_1-f_2), V_{\Gamma,\Gamma}^{-1}(L_0^{-1}f_1 - L_0^{-1}f_2))_\Gamma \\
&\leq c\|L_0^{-1}f_1 - L_0^{-1}f_2\|_{H^{\frac{1}{2}}(\Gamma)}\,\|f_1-f_2\|_{H^{\frac{1}{2}}(\Gamma)}
\end{aligned}$$

and, hence, L_0^{-1} is Lipschitz continuous. ♢

For the solution of (4.3) and (4.4) now we consider the sequences u_n, v_n defined by $u_0 \in H^1(\Omega)$ and successive approximation

$$\begin{aligned} v_{n+1} &:= L_0^{-1}(V_{\Gamma,\Gamma} B_1 u_n|_\Gamma + V_{\Gamma,\Gamma} g - V_{\Omega,\Gamma} F u_n - V_{\Gamma,\Omega} d) \\ u_{n+1} &:= K_{\Gamma,\Omega} v_{n+1} - V_{\Gamma,\Omega}(B_0 v_{n+1} - B_1 v_{n+1} - g) - V_{\Omega,\Omega}(F u_n + d)\,. \end{aligned} \tag{4.12}$$

Since L_0^{-1}, B_1 and F are Lipschitz continuous we obtain

$$\begin{aligned} &\|v_{n+1} - v_n\|_{H^{\frac12}(\Gamma)} \\ &\le \|L_0^{-1}\| \left(\|V_{\Gamma,\Gamma}\ B_1\|\ \|u_n - u_{n-1}\|_{H^{\frac12}(\Gamma)} + \|V_{\Omega,\Gamma}\ F\|\ \|u_n - u_{n-1}\|_{H^1(\Omega)} \right) \end{aligned}$$

and

$$\begin{aligned} \|u_{n+1} - u_n\|_{H^1(\Omega)} \le\ & \|K_{\Gamma,\Omega} - V_{\Gamma,\Omega} B_0 + V_{\Gamma,\Omega} B_1\| \|v_{n+1} - v_n\|_{H^{\frac12}(\Gamma)} \\ & + \|V_{\Omega,\Omega} F\| \|u_n - u_{n-1}\|_{H^1(\Omega)}\,. \end{aligned}$$

By the trace theorem we obtain

$$\|u_{n+1} - u_n\|_{H^1(\Omega)} \le \left(c_1\|B_1\| + c_2\|B_1\|^2 + c_3\|F\|\right)\ \|u_n - u_{n-1}\|_{H^1(\Omega)} \tag{4.13}$$

with

$$\begin{aligned} c_1 &= (\|K_{\Gamma,\Omega}\| + \|V_{\Gamma,\Omega}\| \cdot \|B_0\|)\ \|L_0^{-1}\|\ \|V_{\Gamma,\Gamma}\|\ \|\mathrm{Trace}\|\,, \\ c_2 &= \|V_{\Gamma,\Omega}\|\ \|L_0^{-1}\|\ \|V_{\Gamma,\Gamma}\|\ \|\mathrm{Trace}\|\,, \\ c_3 &= (\|K_{\Gamma,\Omega}\| + \|V_{\Gamma,\Omega}\| \cdot \|B_0\|)\ \|L_0^{-1}\|\ \|V_{\Omega,\Gamma}\|\ +\ \|V_{\Omega,\Omega}\|\,. \end{aligned} \tag{4.14}$$

where $\mathrm{Trace}(u) = u|_\Gamma$ denotes the trace operator: $H^1(\Omega) \to H^{\frac12}(\Gamma)$.

Hence, if the Lipschitz constants $\|B_1\|$ and $\|F\|$ satisfy the additional condition

$$c_1\|B_1\| + c_2\|B_1\|^2 + c_3\|F\| < 1$$

we obtain u_n as a Cauchy sequence in $H^1(\Omega)$ and v_n as a Cauchy sequence in $H^{\frac12}(\Gamma)$.

For the limits u and v one gets from (4.13) on the boundary

$$L_0 u|_\Gamma = L_0 v$$

which implies $u|_\Gamma = v$. Hence, u is the solution of (4.3). Inserting two solutions u, v into (4.12) we find

$$\|u - v\|_{H^1(\Omega)} \le \left(c_1\|B_1\| + c_2\|B_1\|^2 + c_3\|F\|\right)\ \|u - v\|_{H^1(\Omega)}$$

by the same procedure; hence, uniqueness follows.

5 Finite element – boundary element approximations

In order to solve (4.3), (4.4) numerically by a boundary element scheme we introduce finite dimensional subspaces of $H^{\frac{1}{2}}(\Gamma)$ and of $H^1(\Omega)$, respectively. For this purpose let Δ_h^Γ be a sequence of quasi-uniform grids on Γ with meshsize $h \to 0$. Let $\mathcal{S}(\Delta_h^\Gamma)$ be the corresponding space of piecewise linear continuous splines with respect to a fixed parametric representation of Γ. By P_h^Γ we denote the orthogonal projection of $L^2(\Gamma)$ onto $\mathcal{S}(\Delta_h^\Gamma)$.

Theorem 5.1 *For any $0 \le t < \frac{3}{2}, t \le s \le 2, \frac{1}{2} \le s$ there exists a constant $c > 0$ such that the operator P_h^Γ satisfies the approximation property*

$$\left\| P_h^\Gamma\, v - v \right\|_t \le c\, h^{s-t}\, \|v\|_s \qquad \textit{for all } v \in H^s(\Gamma)\,. \tag{5.1}$$

For the proof of this proposition see e.g. [8, Theorem 6.1.2]. Let Δ_h^Ω be a sequence of triangulations of Ω with mesh size $h \to 0$. Again, $\mathcal{H}(\Delta_h^\Omega)$ denotes the corresponding space of piecewise linear continuous finite element functions (see [1]).

The $L^2(\Omega)$–projection onto this spline space is denoted by P_h^Ω. For $0 \le t < 2, 1 \le s \le 2, t \le s, P_h^\Omega$ satisfies the estimate [1]

$$\left\| P_h^\Omega\, u - u \right\|_t \le c\, h^{s-t}\, \|u\|_s\,. \tag{5.2}$$

First, we follow the approach of [22] by approximating L_0 defined in (4.7) by the discrete approximation

$$L_0^h := P_h^\Gamma\; L_0|_{\mathcal{S}(\Delta_h^\Gamma)} = I - K_h + V_h + P_h\; B_0|_{\mathcal{S}(\Delta_h^\Gamma)} \tag{5.3}$$

where K_h, V_h are defined by $K_h := P_h^\Gamma\; K_{\Gamma,\Gamma}|_{\mathcal{S}(\Delta_h^\Gamma)}$ and $V_h := P_h^\Gamma\; V_{\Gamma,\Gamma}|_{\mathcal{S}(\Delta_h^\Gamma)}$, respectively. It is well known that the operators V_h, V_h^{-1} are invertible and satisfy the stability estimates

$$\begin{aligned} \|V_h\varphi\|_{H^{\frac{1}{2}}(\Gamma)} &\le c\,\|\varphi\|_{H^{-\frac{1}{2}}(\Gamma)} &&\text{for all } \varphi \in H^{-\frac{1}{2}}(\Gamma)\,, \\ \left\|V_h^{-1}P_h^\Gamma\psi\right\|_{H^{-\frac{1}{2}}(\Gamma)} &\le c\,\|\psi\|_{H^{\frac{1}{2}}(\Gamma)} &&\text{for all } \psi \in H^{\frac{1}{2}}(\Gamma) \end{aligned} \tag{5.4}$$

where the constants c do not depend on h. Furthermore, we have

$$(V_h\varphi,\varphi)_\Gamma \ge \tilde\gamma\; \|\varphi\|^2_{H^{-\frac{1}{2}}(\Gamma)} \quad \text{for all } \varphi \in H^{-\frac{1}{2}}(\Gamma) \tag{5.5}$$

and

$$\left(V_h^{-1}\psi,\psi\right)_\Gamma \ge \gamma\; \|\psi\|^2_{H^{\frac{1}{2}}(\Gamma)} \quad \text{for all } \psi \in \mathcal{S}(\Delta_h^\Gamma)\,. \tag{5.6}$$

This implies that the forms on the left hand sides are equivalent to the inner products in the spaces appearing on the right hand sides. These results were proven in [13]. In order to analyze the convergence $L_0^h \to L_0$, we need

Lemma 5.2 *The operator L_0^h defined in* (5.3) *is uniformly Lipschitz continuous with respect to the $H^{\frac{1}{2}}(\Gamma)$–norm and V_h^{-1}–strongly monotone, i.e. there exist constants $l, \gamma > 0$, not depending on h, such that*

$$\begin{aligned}\left(L_0^h\varphi - L_0^h\psi, V_h^{-1}(L_0^h\varphi - L_0^h\psi)\right)_\Gamma &\leq l\cdot\left(\varphi-\psi, V_h^{-1}(\varphi-\psi)\right)_\Gamma\,, &(5.7)\\ (L_0^h\varphi - L_0^h\psi, V_h^{-1}(\varphi-\psi))_\Gamma &\geq \gamma\cdot\|\varphi-\psi\|^2_{\frac{1}{2}} &(5.8)\end{aligned}$$

for all $\varphi,\ \psi\in\mathcal{S}(\Delta_h^\Gamma)$.

Proof. Inequality (5.7) follows immediately from the uniform boundedness of V_h^{-1} in $H^{\frac{1}{2}}(\Gamma)\cap\varphi(\Delta_h^\Gamma)$ which can be found in [8] and with Theorem 4.1. By the symmetry of V_h, inequality (5.8) is equivalent to

$$(\tilde{L}_0^h\varphi - \tilde{L}_0^h\psi, \varphi-\psi)_\Gamma \geq \gamma\cdot\|\varphi-\psi\|^2_{\frac{1}{2}} \quad \text{for all } \varphi,\psi\in\mathcal{S}(\Delta_h^\Gamma) \tag{5.9}$$

with

$$\tilde{L}_0^h := T_h + P_h^\Gamma B_0 := V_h^{-1}(I-K_h) + P_h^\Gamma B_0\,. \tag{5.10}$$

In [2] inequality (5.9) was derived from Gårding's inequality for the Steklov-Poincaré operator $T := V^{-1}(I-K)$ and the compactness of the double layer potential operator K and [25]. ♢

Theorem 5.3 *The operator* L_0^h *defined in* (5.3) *is invertible. For a suitable choice of the constant* $\alpha>0$*, the sequence*

$$u_{n+1} \;:=\; u_n - \alpha(L_0^h u_n - \psi) \tag{5.11}$$

converges to u_h*, the solution of*

$$L_0^h u_h = \psi \tag{5.12}$$

for any function $\psi\in\mathcal{S}(\Delta_h^\Gamma)$ *and any starting value* $u_0\in\mathcal{S}(\Delta_h^\Gamma)$. L_0^h *satisfies the inverse stability estimate*

$$\|\varphi-\psi\|_{H^{\frac{1}{2}}(\Gamma)} \leq c\cdot\left\|L_0^h\varphi - L_0^h\psi\right\|_{H^{\frac{1}{2}}(\Gamma)} \quad \textit{for all } \varphi,\ \psi\in\mathcal{S}(\Delta_h^\Gamma) \tag{5.13}$$

where c *does not depend on* h.

Proof. The inequality (5.8) implies that L_0^h is one to one. For $\alpha>0$, equation (5.12) is equivalent to the fixed–point–equation

$$u_h = u_h - \alpha(L_0^h u_h - \psi) \tag{5.14}$$

and (5.11) defines the corresponding iteration scheme. By using the estimates (5.7) and (5.8) we obtain

$$\begin{aligned}&\left(V_h^{-1}(u_{n+1}-u_n),(u_{n+1}-u_n)\right)_\Gamma\\ &\quad= \left(V_h^{-1}(u_n-u_{n+1}),u_n-u_{n+1}\right)_\Gamma - 2\alpha\left(L_0^h u_n - L_0^h u_{n-1}, V_h^{-1}(u_n-u_{n-1})\right)_\Gamma\\ &\qquad+\alpha^2\left(L_0^h u_n - L_0^h u_{n-1}, V_h^{-1}\left(L_0^h u_n - L_0^h u_{n-1}\right)\right)_\Gamma \qquad (5.15)\\ &\quad\leq (1-2\gamma\alpha+l\alpha^2)\left(V_h^{-1}(u_{n+1}-u_n),(u_{n+1}-u_n)\right)_\Gamma\,.\end{aligned}$$

Choosing $0 < \alpha < \frac{2\gamma}{l}$ we find that (u_n) is a Cauchy sequence (in any norm in $\mathcal{S}(\Delta_h^\Gamma)$). Taking the limit in (5.11) shows that (u_n) converges to a solution of (5.12) and, hence, L_0^h is surjective.

The stability estimate (5.13) follows immediately from inequality (5.8) and the properties of V_h^{-1}. ♢

The following approximation result for the operator L_0^h can be derived from Theorem 5.3 and was proven in [2]. Related results for the boundary element collocation method can be found in [11].

Theorem 5.4 *For $f \in H^{s-1}(\Gamma)$ and $\frac{1}{2} < s \leq 2$ there holds the optimal asymptotic error estimate*

$$\|L_0^{-1}Vf - (L_0^h)^{-1}V_hP_hf\|_{H^{\frac{1}{2}}(\Gamma)} \leq c \cdot h^{s-\frac{1}{2}} \cdot \|f\|_{H^{s-1}(\Gamma)}. \tag{5.16}$$

Now we are able to approximate the solution $u \in H^1(\Omega)$ of (4.3) by the iterative scheme

$$\begin{aligned} v_{n+1} &:= (L_0^h)^{-1}\left(P_h^\Gamma V_{\Gamma,\Gamma}P_h^\Gamma(B_1v_n - g) - P_h^\Gamma V_{\Omega,\Gamma}P_h^\Omega(Fu_n + d)\right) , \\ u_{n+1} &:= P_h^\Omega K_{\Gamma,\Omega}v_{n+1} - P_h^\Omega V_{\Gamma,\Omega}P_h^\Gamma(B_0v_{n+1} - B_1v_{n+1} - g) \\ &\quad -P_h^\Omega V_{\Omega,\Omega}P_h^\Omega(Fu_n + d) \end{aligned} \tag{5.17}$$

with starting values $v_0 \in \mathcal{S}(\Delta_h^\Gamma)$, $u_0 \in \mathcal{H}(\Delta_h^\Omega)$. With similar arguments as in Section 4 we see that the scheme is convergent due to the contraction principle. The limits u_h, v_h satisfy the equations

$$\begin{aligned} v_h &:= (L_0^h)^{-1}\left(P_h^\Gamma V_{\Gamma,\Gamma}P_h^\Gamma(B_1v_h - g) - P_hV_{\Omega,\Gamma}P_h^\Omega(Fu_h + d)\right) , \\ u_h &:= P_h^\Omega K_{\Gamma,\Omega}v_h - P_h^\Omega V_{\Gamma,\Omega}P_h^\Gamma(B_0v_h - B_1v_h - g) - P_h^\Omega V_{\Omega,\Omega}P_h^\Omega(Fu_h + d) . \end{aligned} \tag{5.18}$$

Theorem 5.5 *Let $d \in L^2(\Omega)$. Then the solutions u_h, v_h of (5.18) satisfy the optimal asymptotic error estimate*

$$\|u|_\Gamma - v_h\|_{H^{\frac{1}{2}}(\Gamma)} + \|u - u_h\|_{H^1(\Omega)} \leq c\,\left(1 + \|u\|_{H^2(\Omega)}\right)\cdot h . \tag{5.19}$$

Proof. The inverse stability (5.16) of L_0^h and the approximation and boundedness properties of P_h^Ω and P_h^Γ yield the estimates

$$\begin{aligned} \|z - u_h\|_{H^1(\Omega)} &\leq c_1\,\left(1 + \|u\|_{H^2(\Omega)}\right)\cdot h , \\ \|w - v_h\|_{H^{\frac{1}{2}}(\Gamma)} &\leq c_2\,\left(1 + \|u\|_{H^2(\Omega)}\right)\cdot h \end{aligned}$$

with

$$\begin{aligned} w &:= L_0^{-1}\left(V_{\Gamma,\Gamma}(B_1v_h - g) - V_{\Gamma,\Omega}(Fu_h + d)\right) , \\ z &:= K_{\Gamma,\Omega}v_h - V_{\Gamma,\Omega}(B_0v_h - B_1v_h - g) - V_{\Omega,\Omega}(Fu_h + d) . \end{aligned} \tag{5.20}$$

Inserting v_h, u_h as starting values into the iteration scheme (4.12), the contraction principle yields the estimates

$$\begin{aligned} \|u - z\|_{H^1(\Omega)} &\leq c_1 \|\omega - u_h\|_{H^1(\Omega)} , \\ \|u|_\Gamma - w\|_{H^{\frac{1}{2}}(\Gamma)} &\leq c_2 \|w - v_h\|_{H^{\frac{1}{2}}(\Gamma)} \end{aligned}$$

where the constants depend on the contraction properties of (4.12). Hence, the desired estimate follows. $\diamondsuit$

6 Numerical results

The solution scheme (5.11) was implemented in the programming language C and – in the case of the homogeneous differential equation in Ω – the iteration (5.18), so that the program could be used either on a PC or on an arbitrary UNIX-system. By a version partially written in FORTRAN we were able to use the vector facility on an IBM 3090E which was rather efficient for analyzing the dependence on the parameters practically. In order to keep the programming effort low we used an interpolation $I_h\, B_0$ instead of the orthogonal projection $P_h\, B_0$ in the iteration schemes. That means that we used schemes which may perform less efficiently than the theoretical schemes analyzed here.

Example 6.1 *We choose Ω to be the circle of radius 0.25 centered at the origin. Here, the harmonic function $u(x,y) := x^2 - y^2$ satisfies the nonlinear boundary condition (see [22])*

$$-\frac{\partial u}{\partial n} = \left\{2u + \sin u - 4(x^2 - y^2) - \sin(x^2 - y^2)\right\} / (x^2 + y^2)^{\frac{1}{2}} \tag{6.1}$$

which is of the type (3.2) with $b_1(x,u) = 0$ and

$$b_0(x,u) = \{2u + \sin u\} / (x^2 + y^2)^{\frac{1}{2}} . \tag{6.2}$$

As a parametric representation of Γ we used

$$(x(t), y(t)) := 0.25(\cos 2\pi t, \sin 2\pi t) \quad \text{for } t \in [0,1].$$

The nodal points of Δ_h^Γ were defined by $t_i := \frac{i}{p}$, $i = 1, ..., p$. Table 1 shows the optimal choice of α in (5.11) and the resulting number of iterations N for the different mesh sizes. The corresponding optimal results for the scheme in [2] are listed in Table 2 and illustrate the advantage of our improved method. The values of the $H^{\frac{1}{2}}$–error performed better than predicted by Proposition 5.4 due to the high regularity of the solution; and they lie between $9 \cdot 10^{-3}$ and $3.5 \cdot 10^{-6}$.

p	—	20	40	80	160
α	—	0.15	0.15	0.15	0.15
N	—	13	13	13	13

Table 1: Number of iterations in (5.11)

p	10	20	40	80	160
α	0.06	0.038	0.021	0.011	0.0058
N	8	10	35	60	110

Table 2: Optimal choice of α and number of iterations in [2].

Example 6.2 *We replace the boundary condition (6.1) in Example 6.1 by*

$$-\frac{\partial u}{\partial n} = \left\{2u + \lambda \sin u - 4(x^2 - y^2) - \lambda \sin(x^2 - y^2)\right\} / (x^2 + y^2)^{\frac{1}{2}} \tag{6.3}$$

and choose

$$\begin{aligned} b_1((x,y),u) &:= \{(1-\lambda)\ \sin u\} / (x^2 + y^2)^{\frac{1}{2}}\,, \\ g(x,y) &:= \left\{4(x^2 + y^2) + \lambda \sin(x^2 - y^2)\right\} / (x^2 + y^2)^{\frac{1}{2}}\,. \end{aligned}$$

For $\lambda = 4.0$ the application of the scheme (5.17) required 6 iterations. The complete computation needed 25 steps as described in (5.14). The number of iterations increases with $\lambda > 4.0$ and the scheme is divergent already for $\lambda = 7.0$.

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