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# Multiplicity results for positive solutions to non-autonomous elliptic problems *

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**Abstract**

We are concerned with the multiplicity of positive solutions for non-autonomous elliptic equations with Dirichlet and Neumann boundary conditions. Using Ljusternik-Schnirelmann theory, we show that the number of solutions is affected by the shape of the potential functions.

## 1 Introduction

This paper is devoted to the study of multiplicity results for positive solutions to non-autonomous semilinear elliptic equations with a small diffusion coefficient. Consider the boundary value problem

$$\begin{gathered} -d\Delta u + u = K(x)|u|^{p-2}u, \quad u > 0 \quad \text{in } \Omega, \\ Bu = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.1}$$

where $\Omega$ is a bounded domain; $d$ is a small positive parameter; $K(x) > 0$ in $\bar{\Omega}$ and is a $C^\alpha$ function with $0 < \alpha < 1$; $2 < p < \frac{2N}{N-2}$ if $N \geq 3$, $p > 2$ if $N = 1, 2$; and $Bu$ is the boundary operator which is either Dirichlet, i.e., $Bu = u|_{\partial\Omega}$, or Neumann, i.e., $Bu = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$.

In recent years, singularly perturbed elliptic problems have been studied extensively, [13, 14, 7, 9]. Aiming at applications of mathematical models in biological pattern formations, Lin, Ni and Tagaki discovered the single peakedness of the least-energy solutions for nonlinear autonomous Neumann problems when a small parameter tends to zero. After that, similar phenomena have been revealed in singularly perturbed settings for nonlinear Dirichlet problems and nonlinear Schrödinger equations ([16, 19]). Motivated by the work in [14], Ren [18] studied least-energy solutions for the non-autonomous Problem (1.1) and showed that the least-energy solution of (1.1) will develop single peak as $d$ approaches zero. The location of the peaks is determined by the non-autonomous term of the equation. Therefore, in most situations the effect of $K(x)$ overrides the effect of the geometry of $\Omega$. The goal of this note is to establish some

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multiplicity results on the existence of non-constant positive solutions of (1.1) and to show how the number of positive solutions is affected by the topology of the preimage of $K(x)$, i.e., by the shape of the graph of $K(x)$. Our work is motivated by the above mentioned papers, especially by [18]. Define

$$K_1 = \max_{x\in\bar\Omega} K(x), \quad K_2 = \max_{x\in\partial\Omega} K(x),$$

and

$$K_\Omega = \{x \in \bar\Omega : K(x) = K_1\}, \qquad K_{\partial\Omega} = \{x \in \partial\Omega : K(x) = K_2\},$$

closed subsets of $\bar\Omega$ and $\partial\Omega$ respectively.

In the following, we denote by $\mathrm{cat}_{N_r(K_\Omega)}(K_\Omega)$ (resp. $\mathrm{cat}_{N_r(K_{\partial\Omega})}(K_{\partial\Omega})$) the Ljusternik-Schnirelmann category of $K_\Omega$ in $N_r(K_\Omega)$ (resp. $K_{\partial\Omega}$ in $N_r(K_{\partial\Omega})$), where $N_r(\cdot)$ denotes the closed $r$-neighborhood of a set. $r > 0$ will be chosen and fixed. Our main results are the following theorems.

**Theorem 1.1** *Let $r > 0$ be such that $2r < \mathrm{dist}(K_\Omega, \partial\Omega)$ and assume $K_\Omega \cap \partial\Omega = \emptyset$. Then for $d$ sufficiently small, (1.1) with Dirichlet boundary condition has at least $\mathrm{cat}_{N_r(K_\Omega)}(K_\Omega)$ distinct solutions. Furthermore, each solution $u_d$ has at most one local maximum point $P_d$ on $\bar\Omega$ satisfying*

$$\limsup_{d\to 0} \mathrm{dist}(P_d, K_\Omega) = 0\,.$$

**Theorem 1.2** *Let $r > 0$ be fixed and assume $K_1 > 2^{\frac{p-2}{2}} K_2$. Then for $d$ sufficiently small, (1.1) with Neumann boundary condition has at least $\mathrm{cat}_{N_r(K_\Omega)}(K_\Omega)$ distinct non-constant solutions. Furthermore, each solution $u_d$ has at most one local maximum point $P_d$ on $\bar\Omega$ which satisfies*

$$\limsup_{d\to 0} \mathrm{dist}(P_d, K_\Omega) = 0.$$

**Theorem 1.3** *Let $r > 0$ be fixed and assume $K_1 < 2^{\frac{p-2}{2}} K_2$. Then for $d$ sufficiently small, (1.1) with Neumann boundary condition has at least $\mathrm{cat}_{N_r(K_{\partial\Omega})}(K_{\partial\Omega})$ distinct nonconstant solutions. Furthermore, each solution $u_d$ has at most one local maximum point $P_d$ on $\bar\Omega$ which lies on the boundary of $\Omega$ and satisfies*

$$\limsup_{d\to 0} \mathrm{dist}(P_d, K_{\partial\Omega}) = 0.$$

**Theorem 1.4** *Let $r > 0$ be fixed and assume $K_1 = 2^{\frac{p-2}{2}} K_2$. Then for $d$ sufficiently small, (1.1) with Neumann boundary condition has at least $\mathrm{cat}_{N_r(K_{\partial\Omega})}(K_{\partial\Omega}) + \mathrm{cat}_{N_r(K_\Omega)}(K_\Omega)$ distinct non-constant solutions. Furthermore, each solution $u_d$ has at most one local maximum point $P_d$ on $\bar\Omega$ which satisfies*

$$\limsup_{d\to 0} \mathrm{dist}(P_d, K_{\partial\Omega} \cup K_\Omega) = 0.$$

**Remark** If $K_\Omega$ and $N_r(K_\Omega)$ are homotopically equivalent, then one has $\mathrm{cat}_{N_r(K_\Omega)}(K_\Omega) = \mathrm{cat}_{K_\Omega}(K_\Omega)$. This would be the case when the level sets of $K$ are regular. On the other hand, it is easy to construct examples in which $\mathrm{cat}_{N_r(K_\Omega)}(K_\Omega)$ may depend on $r$ and may tend to $\infty$ as $r \to 0$. In these cases, the number of solutions for (1.1) tends to $\infty$ as $d \to 0$. These features also hold for the Neumann problems.

## 2 Preliminaries

Throughout this discussion, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary. We seek for positive non-constant solutions of (1.1). To this end, let $H$ be the Hilbert space $H_0^1(\Omega)$ if $Bu = u|_{\partial\Omega}$ or $H^1(\Omega)$ if $Bu = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$. It is well known that the solutions of (1.1) correspond to the critical points of the following functional defined on $H$,

$$J_d(u) = \frac{1}{2}\int_\Omega (d|\nabla u|^2 + u^2)\,dx - \frac{1}{p}\int_\Omega K(x)|u|^p\,dx\,. \tag{2.1}$$

By using the Mountain Pass Theorem ([17]), the authors of [14] and [18] proved the existence of a positive non-constant solution $u_d$ of (1.1). Here, in order to establish multiplicity results, we consider a constraint problem for $J_d(u)$ on the Nehari manifold (e.g. [23]),

$$\begin{aligned} V_d &= \{u \in H\backslash\{0\} :< J_d'(u), u >= 0\} \\ &= \{u \in H\backslash\{0\} : \int_\Omega (d|\nabla u|^2 + u^2 - K(x)|u|^p) dx = 0\}. \end{aligned}$$

Clearly, the critical points of $J_d$ are in $V_d$. We define

$$c_d = \inf_{u \in V_d} J_d(u). \tag{2.2}$$

By standard methods (e.g. [23]), $c_d$ is achieved and therefore gives rise to a solution of (1.1). Solutions corresponding to $c_d$ are called least-energy solutions whose behaviors are studied in [18]. We shall prove the existence of multiple critical points of $J_d$ (therefore multiple solutions of (1.1)) with critical values close to $c_d$. Our strategy is to estimate the topology of a certain level set of $J_d$, say

$$J_d^{c_d+\epsilon} = \{u \in V_d : J_d(u) \le c_d + \epsilon\} \tag{2.3}$$

for some appropriate $\epsilon > 0$ depending on $d$. To outline our strategy more precisely, let us consider the Dirichlet problem. We shall prove that for $d$ sufficiently small

$$\operatorname*{cat}_{J_d^{c_d+\epsilon}}(J_d^{c_d+\epsilon}) \ge 2 \operatorname*{cat}_{N_r(K_\Omega)}(K_\Omega). \tag{2.4}$$

Then standard critical point theory yields the existence of at least $2\,\mathrm{cat}_{N_r(K_\Omega)}(K_\Omega)$ critical points in $[c_d, c_d+\epsilon]$. An energy estimate shows that none of these critical points changes sign in $\Omega$. By the maximum principle, these solutions are strictly positive or negative on $\bar{\Omega}$. It follows that there exist at least $\mathrm{cat}_{N_r(K_\Omega)}(K_\Omega)$ positive solutions of (1.1). More precise information will be given in §3.

Now, we give some preliminary results. The ground state solution to the following problem plays an important role in the proof of our main results. First, we summarize known facts about positive solutions to the equation ([6, 10, 11])

$$-\Delta\omega + \omega = \omega^{p-1} \quad \text{in } \mathbb{R}^N. \tag{2.5}$$

**Proposition 2.1** *Equation (2.5) has a solution $\omega$ satisfying*

*i) $\omega \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ and $\omega > 0$ in $\mathbb{R}^N$.*

*ii) $\omega$ is spherically symmetric: $\omega(z) = \omega(r)$ with $r = |z|$ and $d\omega/dr < 0$ for $r > 0$.*

*iii) $\omega$ and its first derivatives decay exponentially at infinity.*

*iv)*

$$m := \frac{\int_{\mathbb{R}^N}(|\nabla\omega|^2 + \omega^2)dx}{(\int_{\mathbb{R}^N}|\omega|^p)^{\frac{2}{p}}} = \inf_{u\in H^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)dx}{(\int_{\mathbb{R}^N}|u|^p)^{\frac{2}{p}}},$$

*and*

$$I(\omega) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla\omega|^2 + \omega^2)dx - \frac{1}{p}\int_{\mathbb{R}^N}\omega^p dx = \frac{p-2}{2p}(m)^{\frac{p}{p-2}}. \tag{2.6}$$

Frequently we rescale the problem (1.1). So that there is a one to one correspondence between the solutions of (1.1) and solutions of

$$\begin{gathered}-\Delta u + u = K(\sqrt{d}x)|u|^{p-2}u \quad \text{in } \Omega_d \\ Bu = 0 \quad \text{on } \partial\Omega_d,\end{gathered} \tag{2.7}$$

where

$$\Omega_d = \{x \in \mathbb{R}^N : \sqrt{d}x \in \Omega\}. \tag{2.8}$$

Then (2.7) is associated with the functional defined by

$$I_d(u) = \frac{1}{2}\int_{\Omega_d}(|\nabla u|^2 + u^2)dx - \frac{1}{p}\int_{\Omega_d}K(\sqrt{d}x)|u|^p dx \quad \text{for } u \in U_d, \tag{2.9}$$

where

$$U_d = \left\{u \in H^1(\Omega_d)\backslash\{0\} : \int_{\Omega_d}(|\nabla u|^2 + u^2)dx = \int_{\Omega_d}K(\sqrt{d}x)|u|^p dx\right\}. \tag{2.10}$$

For $u \in V_d$, define $\sigma(u)(x) = u(\sqrt{d}x)$. Then $\sigma(u)(x) \in U_d$. Moreover, the proof of the following lemma is a simple computation.

**Lemma 2.2** *For each $u \in V_d$, $I_d(\sigma(u)(x)) = d^{-N/2}J_d(u)$, and therefore*

$$\inf_{U_d} I_d = d^{-N/2}\inf_{V_d} J_d.$$

# 3 Asymptotic Estimates

This section is divided into three subsections.

## 3.A. Dirichlet case

We first consider Dirichlet problems in this subsection so that $H = H_0^1(\Omega)$ and we give some asymptotic estimates as $d \to 0$. Assume $K_\Omega \cap \partial\Omega = \emptyset$, i.e., $\max_{x\in\bar\Omega} K(x)$ is attained in the interior of $\Omega$. Let $\eta$ be a smooth non-increasing function on $[0,\infty]$ such that $\eta(t) = 1$, $0 \le t \le 1$; $\eta(t) = 0$, $t \ge 2$ and $|\eta'| \le 2$. Also, let $\eta_r(\cdot) = \eta(\frac{\cdot}{r})$ for $r > 0$ such that $2r < \mathrm{dist}(K_\Omega, \partial\Omega)$ and let $\psi_d(y)$ be the function on $\Omega$ defined by

$$\psi_d(y)(x) = \alpha_y \eta_r(|x-y|)\cdot\omega(\frac{x-y}{\sqrt{d}}) \ \in V_d \tag{3.1}$$

with $y \in K_\Omega$ fixed, where

$$\alpha_y = \left[\frac{\int_\Omega (d|\nabla(\eta_r(|x-y|)\omega(\frac{x-y}{\sqrt{d}}))|^2 + |\eta_r(|x-y|)\omega(\frac{x-y}{\sqrt{d}})|^2)dx}{\int_\Omega K(x)|\eta_r(|x-y|)\omega(\frac{x-y}{\sqrt{d}})|^p dx}\right]^{\frac{1}{p-2}}. \tag{3.2}$$

**Proposition 3.1** *$\psi_d \in C(K_\Omega, V_d)$ and*

$$J_d(\psi_d(y)(x)) = d^{N/2}[K(y)^{-\frac{2}{p-2}} I(\omega) + o(1)] \tag{3.3}$$

*as $d \to 0$ uniformly for $y \in K_\Omega$. Here $\omega$ and $I(\omega)$ are given in Proposition 2.1.*

**Proposition 3.2** *(a) $\lim_{d\to 0} d^{-N/2} c_d = K_1^{-\frac{2}{p-2}} I(\omega)$, where $c_d$ is defined in (2.2).*
*(b) Let $d_n \to 0$ and $u_n \in U_n := U_{d_n}$ be such that*

$$\lim_{n\to\infty} \frac{p-2}{2p}\int_{\Omega_n} \left(|\nabla u_n|^2 + u_n^2\right) dx = K_1^{-\frac{2}{p-2}} I(\omega) := A. \tag{3.4}$$

*Then, there exists $y_n \in \mathbb{R}^N$ with the property that for any $\epsilon > 0, \exists R > 0$ such that*

$$\lim_{n\to\infty} \frac{p-2}{2p}\int_{B_R(y_n)} \left[|\nabla u_n|^2 + u_n^2\right] dx \ge A - \epsilon. \tag{3.5}$$

*Moreover, for every positive and small $\delta$, there exists $C_\delta > 0$ such that*

$$\limsup_{n\to\infty} \mathrm{dist}\left(y_n, \frac{1}{\sqrt{d_n}}(N_\delta(K_\Omega))\right) \le C_\delta. \tag{3.6}$$

Notice that the center of mass of $u \in V_d(\Omega)$ in terms of the $L^p$ norm is

$$\beta(u) = \frac{\int_\Omega |u|^p x dx}{\int_\Omega |u|^p dx} \quad \forall u \in V_d.$$

Also notice that $\beta$ is continuous in $u$ and $\beta(u)$ belongs to the the convex closure of $\Omega$.

**Proposition 3.3** *For $r > 0$ fixed, there exist $\epsilon_1 > 0$ and $d_1 > 0$ such that for any $0 < d \le d_1$ and $0 < \epsilon \le \epsilon_1$ we have*

$$\beta(u) \in N_r(K_\Omega) \quad \forall u \in J_d^{c_d + \epsilon d^{N/2}}.$$

**Proof of Proposition 3.1** Proving that $\psi_d \in C(K_\Omega, V_d)$ is straightforward. To prove (3.3), we proceed as follows. First, let us note that

$$\begin{aligned} J_d(\psi_d(y)(x)) &= \frac{1}{2}\int_\Omega (d|\nabla \psi_d|^2 + \psi_d^2)dx - \frac{1}{p}\int_\Omega K(x)|\psi_d|^p dx \qquad (3.7)\\ &= \frac{p-2}{2p}\alpha_y^2 \int_\Omega (d|\nabla(\eta_r(|x-y|)\omega(\frac{x-y}{\sqrt{d}}))|^2 + |\eta_r(|x-y|)\omega(\frac{x-y}{\sqrt{d}})|^2)dx. \end{aligned}$$

By (3.2), we have

$$\alpha_y^{p-2} = \frac{\int_\Omega (d|\nabla(\eta_r(|x-y|)\omega(\frac{x-y}{\sqrt{d}}))|^2 + |\eta_r(|x-y|)\omega(\frac{x-y}{\sqrt{d}})|^2)dx}{\int_\Omega K(x)|\eta_r(|x-y|)\omega(\frac{x-y}{\sqrt{d}})|^p dx}.$$

Next, we find some estimates for $\alpha_y^{p-2}$. Consider the numerator of $\alpha_y^{p-2}$,

$$N = \int_\Omega (d|\nabla(\eta_r(|x-y|)\omega(\frac{x-y}{\sqrt{d}}))|^2 + |\eta_r(|x-y|)\omega(\frac{x-y}{\sqrt{d}})|^2)dx.$$

By definition,

$$|\nabla(\eta_r\omega)|^2 = |\nabla\eta_r|^2\cdot\omega^2 + 2\eta_r\cdot\omega\cdot\nabla\eta_r\cdot\nabla\omega\cdot\frac{1}{\sqrt{d}} + \frac{1}{d}\eta_r^2\cdot|\nabla\omega|^2.$$

Therefore,

$$\begin{aligned} N &= \int_\Omega \eta_r^2(|\nabla\omega|^2 + \omega^2)dx + \int_\Omega (d|\nabla\eta_r|^2\cdot\omega^2 + 2\eta_r\cdot\omega\cdot\nabla\eta_r\cdot\nabla\omega\cdot\sqrt{d})dx\\ &= I_1 + I_2. \end{aligned}$$

Let $z = x - y$, $h = \frac{z}{\sqrt{d}}$ and $\Omega_{d,y} = \frac{\Omega - \{y\}}{\sqrt{d}}$. Then

$$I_1 = d^{N/2}\int_{\Omega_{d,y}} \eta^2(\frac{|h\sqrt{d}|}{r})\left(|\nabla\omega(h)|^2 + \omega^2(h)\right)dh.$$

By a property of the ground state solution $\omega$, $\forall\epsilon > 0$, $\exists R_1 > 0$ such that

$$\int_{\Omega_{d,y}\cap\{h:|h|\ge R_1\}} \eta^2(\frac{|h\sqrt{d}|}{r})\left(|\nabla\omega(h)|^2(|\nabla\omega(h)|^2 + \omega^2(h)\right)dh < \frac{\epsilon}{2}.$$

For such $R_1$, there exists $d_1 > 0$ such that for each $d \le d_1$,

$$\Big|\int_{\Omega_{d,y}\cap\{h:|h|\ge R_1\}} \eta^2(\frac{|h\sqrt{d}|}{r})\Big(|\nabla\omega(h)|^2(|\nabla\omega(h)|^2 + \omega^2(h)\Big)dh - \int_{\mathbb{R}^N}\left(|\nabla\omega(h)|^2 + \omega^2(h))\right)dh\Big| < \frac{\epsilon}{2},$$

provided $\sqrt{d_1} \le \frac{r}{R_1}$, i.e. $R_1\sqrt{d_1} \le r$ and so $\eta^2(|h\sqrt{d}|/r) = 1$. Therefore,

$$I_1 = d^{N/2}\Big(\int_{\mathbb{R}^N} (|\nabla\omega(h)|^2 + \omega^2(h))dh + o(1)\Big).$$

Notice that

$$\begin{aligned}
I_2 &= \int_\Omega \Big(d|\nabla\eta_r|^2\cdot\omega^2 + 2\eta_r\cdot\omega\cdot\nabla\eta_r\cdot\nabla\omega\cdot\sqrt{d}\Big)dx \\
&= d^{N/2}\int_{\Omega_{d,y}\cap\{h:\frac{r}{\sqrt{d}}\le|h|\le\frac{2r}{\sqrt{d}}\}} \Big(d|\nabla\eta(\frac{|h\sqrt{d}|}{r})|^2\cdot\omega^2(h) \\
&\quad + 2\sqrt{d}\eta(\frac{|h\sqrt{d}|}{r})\cdot\omega(h)\cdot\nabla\eta\cdot\nabla\omega\Big)dh \\
&\le d^{N/2}\int_{\Omega_{d,y}\cap\{h:\frac{r}{\sqrt{d}}\le|h|\le\frac{2r}{\sqrt{d}}\}} \left(d\frac{4\omega^2(h)}{r^2} + 2\sqrt{d}\omega(h)\cdot\frac{2}{r}\cdot|\nabla\omega|\right)dh \\
&\le d^{N/2}\int_{\Omega_{d,y}\cap\{h:\frac{r}{\sqrt{d}}\le|h|\le\frac{2r}{\sqrt{d}}\}} C\cdot\sqrt{d}\left(|\nabla\omega(h)|^2 + \omega^2(h)\right)dh \\
&\le d^{N/2}\cdot o(1),
\end{aligned}$$

where $C = \max\{\frac{4}{r^2}(\sqrt{d}+1), 1\}$. For the denominator of $\alpha_y^{p-2}$, we have

$$\begin{aligned}
D &= \int_\Omega K(x)|\eta_r(|x-y|)\omega(\frac{x-y}{\sqrt{d}})|^p dx \\
&= d^{N/2}\int_{\Omega_{d,y}} K(\sqrt{d}h+y)\cdot|\eta(\frac{\sqrt{d}h}{r})|^p\cdot|\omega(h)|^p dh.
\end{aligned}$$

Now, $\forall\epsilon>0, \exists R_2>0$ such that

$$\begin{aligned}
&\int_{\Omega_{d,y}\cap\{h:|h|\ge R_2\}} K(\sqrt{d}h+y)\cdot|\eta(\frac{\sqrt{d}h}{r})|^p\cdot|\omega(h)|^p dh \\
&\qquad\le \max_{x\in\bar\Omega} K(x)\int_{\Omega_{d,y}\cap\{h:|h|\ge R_2\}} |\omega(h)|^p dh \le \frac{\epsilon}{2}.
\end{aligned}$$

On the other hand, by the continuity of $K(x)$, for the above $\epsilon$ and $R_2$, there exists $\delta>0$ such that $|K(z)-K(y)| < \frac{\epsilon}{2\int_{\mathbb{R}^N}|\omega|^p dh}$ whenever $|z-y|<\delta$; also, there exists $d_2>0$ with $\sqrt{d_2}\le\frac{r}{R_2}$ such that if $d\le d_2$ then $|\sqrt{d}h| \le \min(\delta, \sqrt{d_1}R_2)$. Hence, for $d\le d_2$

$$\begin{aligned}
&\left|\int_{\Omega_{d,y}\cap\{h:|h|\le R_2\}} \Big(K(\sqrt{d}h+y)-K(y)\Big)|\omega(h)|^p dh\right| \\
&\le \int_{\Omega_{d,y}\cap\{h:|h|\le R_2\}} \Big|K(\sqrt{d}h+y)-K(y)\Big||\omega(h)|^p dh \\
&\le \frac{\epsilon}{2\int_{\mathbb{R}^N}|\omega|^p dh}\int_{\mathbb{R}^N}|\omega|^p dh \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\int_{\Omega_{d,y}\cap\{h:|h|\le R_2\}} \Big|K(\sqrt{d}h+y)\Big| \cdot |\eta(\frac{\sqrt{d}h}{r})|^p \cdot |\omega(h)|^p \, dh \\
&\le \int_{\Omega_{d,y}\cap\{h:|h|\le R_2\}} \Big|K(\sqrt{d}h+y)-K(y)\Big| \cdot |\eta(\frac{\sqrt{d}h}{r})|^p \cdot |\omega(h)|^p \, dh \\
&\quad + \int_{\Omega_{d,y}\cap\{h:|h|\le R_2\}} K(y)|\eta(\frac{\sqrt{d}h}{r})|^p \cdot |\omega(h)|^p dh \\
&\le \int_{\Omega_{d,y}\cap\{h:|h|\le R_2\}} |K(\sqrt{d}h+y)-K(y)||\omega(h)|^p dh \\
&\quad + \int_{\Omega_{d,y}\cap\{h:|h|\le R_2\}} K(y)|\omega(h)|^p dh \\
&< \frac{\epsilon}{2} + \int_{\Omega_{d,y}\cap\{h:|h|\le R_2\}} K(y)|\omega(h)|^p \, dh \,.
\end{aligned}
$$

It follows that

$$D = d^{N/2}\left[K(y)\int_{\mathbb{R}^N} |\omega(h)|^p dh + o(1)\right].$$

Hence, using (2.5) we obtain

$$
\begin{aligned}
\alpha_y &= \left[\frac{\int_{\mathbb{R}^N}(|\nabla\omega(h)|^2+\omega^2(h))dh}{K(y)\int_{\mathbb{R}^N}|\omega(h)|^p dh + o(1)}\right]^{\frac{1}{p-2}} \\
&= K(y)^{-\frac{1}{p-2}}[1+o(1)] \quad (\ o(1)\to 0 \quad \text{as } d\to 0).
\end{aligned}
$$

Finally, using (3.7) we get

$$
\begin{aligned}
&J_d(\psi_d(y)(x)) \\
&= \left[K(y)^{-\frac{2}{p-2}}(1+o(1))\right]\left[\frac{p-2}{2p}d^{N/2}\int_{\mathbb{R}^N}(|\nabla\omega(x)|^2+\omega^2(x))dx + o(1)\right] \\
&= d^{N/2}\left(K(y)^{-\frac{2}{p-2}}I(\omega)+o(1)\right),
\end{aligned}
$$

where the last equality follows from (iv) of Proposition 2.1 $\diamondsuit$

To prove Proposition 3.2, we need the following results of P.L. Lions([12]).

**Lemma 3.4 ([12])** *Suppose $\{\mu_n\}$ is a sequence of measures on $\mathbb{R}^N$ such that $\mu_n \ge 0$, $\lim_{n\to\infty}\int_{\mathbb{R}^N}\mu_n dx = A$. Then there is a subsequence $\{\mu_n\}$ (still denoted by $\{\mu_n\}$) such that one of the following three mutually exclusive conditions holds. (1°) (Compactness) There exists a sequence $\{y_n\}\subseteq\mathbb{R}^N$ such that for any $\epsilon>0$ there is $R>0$ with the property that*

$$\lim_{n\to\infty}\int_{B_R(y_n)}\mu_n dx \ge A-\epsilon.$$

*(2°) (Vanishing) For all $R > 0$*

$$\lim_{n\to\infty} (\sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \mu_n dx) = 0.$$

*(3°) (Dichotomy) There exist a number $\tilde{A}$, $0 < \tilde{A} < A$, a sequence $\{R_n\}$ going to infinity, $\{y_n\} \subset \mathbb{R}^N$ and two non-negative measures $\{\mu_n^1\}$, $\{\mu_n^2\}$ such that $0 \le \mu_n^1 + \mu_n^2 \le \mu_n$, $\text{supp}(\mu_n^1) \subset B_{R_n}(y_n)$, $\text{supp}(\mu_n^2) \subset \mathbb{R}^N \backslash B^c_{2R_n}(x_n)$, and as $n \to \infty$*

$$\mu_n^1(\mathbb{R}^N) \to \tilde{A}, \quad \mu_n^2(\mathbb{R}^N) \to A - \tilde{A}.$$

**Lemma 3.5 ([12])** *Let $R > 0$ and $2 \le q \le 2N/N-2$. If $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and if*

$$\sup_{y\in\mathbb{R}^N} \int_{B_R(y)\cap\Omega} |u_n|^q dx \to 0 \quad as \quad n \to \infty,$$

*then $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $2 < p < 2N/N-2$ .*

**Proof of Proposition 3.2** Part (a) is proved in [18][Prop. 3.1] though a different variational formulation was used in there. To prove part (b), we define a family of measures on $\mathbb{R}^N$ by $\mu_n = \frac{p-2}{2p}(|\nabla u_n|^2 + u_n^2)$ (with zero extensions outside $\Omega_n$) and apply (1°) of Lemma 3.1.

**Claim 1** For $\mu_n$, vanishing (2°) in Lemma 3.4 cannot happen. Otherwise, by Lemma 3.5 there exists a subsequence still denoted by $\{u_n\}$ going to zero in $L^p$ for $2 < p < 2N/N-2$. Then, using (3.4) we obtain

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{p-2}{2p} K_1 \int_{\mathbb{R}^N} |u_n|^p dx \\ &\ge \limsup_{n\to\infty} \frac{p-2}{2p} \int_{\mathbb{R}^N} K(\sqrt{d_n}x)|u_n|^p dx \\ &= A > 0. \end{aligned}$$

This contradiction proves Claim 1.

**Claim 2** For $\mu_n$, Dichotomy (3°) in Lemma 3.4 will not occur. Otherwise, let $\phi_n \in C_0^1(\mathbb{R}^N)$ such that $\phi_n \equiv 1$ in $B_{R_n}(y_n)$, $\phi_n \equiv 0$ in $B^c_{2R_n}(y_n)$ and $0 \le \phi_n \le 1, |\nabla\phi_n| \le \frac{2}{R_n}$. Let $u_n = \phi_n u_n + (1-\phi_n)u_n =: u_n^1 + u_n^2$. Then, using (3°) of Lemma 3.4 we have

$$\begin{aligned} I_{d_n}(u_n^1) &\ge \mu_n(B_{R_n}(y_n)) \\ &\ge \mu_n^1(B_{R_n}(y_n)) \\ &= \mu_n^1(\mathbb{R}^N) \to \tilde{A}, \end{aligned}$$

and

$$\begin{aligned} I_{d_n}(u_n^2) &\ge \mu_n(B^c_{2R_n}(y_n)) \\ &\ge \mu_n^2(B^c_{2R_n}(y_n)) \\ &= \mu_n^2(\mathbb{R}^N) \to A - \tilde{A}, \end{aligned}$$

where $I_{d_n}$ is defined in (2.9).

Let $A_n = B_{2R_n}(y_n) \backslash B_{R_n}(y_n)$. Then,

$$\begin{aligned}
&\frac{p-2}{2p}\int_{A_n}(|\nabla u_n|^2+u_n^2)dx \\
&= \mu_n(\mathbb{R}^N)-\mu_n(B_{R_n}(y_n))-\mu_n(B^c_{2R_n}(y_n)) \\
&\le \mu_n(\mathbb{R}^N)-\mu_n^1(\mathbb{R}^N)-\mu_n^2(\mathbb{R}^N)\to 0 \quad \text{as } n\to\infty\,.
\end{aligned} \tag{3.8}$$

Thus, by Sobolev embedding theorem, we have $\int_{A_n}|u_n|^p dx \to 0$ as $d_n \to 0$. Consequently,

$$\begin{aligned}
&\int_{\mathbb{R}^N} K(\sqrt{d_n}x)|u_n|^p dx \\
&= \int_{\mathbb{R}^N} K(\sqrt{d_n}x)|u_n^1+u_n^2|^p dx \\
&= \int_{B_{R_n}(y_n)} K(\sqrt{d_n}x)|u_n^1|^p dx + \int_{B^c_{2R_n}(y_n)} K(\sqrt{d_n}x)|u_n^2|^p dx \\
&\quad + \int_{A_n} K(\sqrt{d_n}x)|u_n|^p dx \\
&= \int_{\mathbb{R}^N} \chi_n^1\cdot K(\sqrt{d_n}x)|u_n^1|^p dx + \int_{\mathbb{R}^N} \chi_n^2\cdot K(\sqrt{d_n}x)|u_n^2|^p dx + o(1)\,,
\end{aligned} \tag{3.9}$$

where $\chi_n^1$ and $\chi_n^2$ are the characteristic functions on $B_{R_n}(y_n)$ and $B^c_{2R_n}(y_n)$ respectively. Next, observe that

$$\int_{\mathbb{R}^N}(|\nabla u_n|^2+u_n^2)dx = \int_{\mathbb{R}^N}(|\nabla u_n^1|^2+(u_n^1)^2)dx + \int_{\mathbb{R}^N}(|\nabla u_n^2|^2+(u_n^2)^2)dx + M_n,$$

where $M_n := 2\int_{\mathbb{R}^N}(\nabla u_n^1\cdot\nabla u_n^2 + u_n^1\cdot u_n^2)dx \to 0$ as $d_n \to 0$ because of (3.8). Now,

$$\begin{aligned}
A &= \liminf_{n\to\infty} I_{d_n}(u_n) \\
&\ge \liminf_{n\to\infty} I_{d_n}(u_n^1) + \liminf_{n\to\infty} I_{d_n}(u_n^2) + o(1) \\
&\ge \tilde{A} + A - \tilde{A} = A,
\end{aligned}$$

(here $u_n^1, u_n^2$ may not be on the manifold $U_n$). Hence,

$$\tilde{A} = \lim_{n\to\infty} I_{d_n}(u_n^1),\, A-\tilde{A} = \lim_{n\to\infty} I_{d_n}(u_n^2). \tag{3.10}$$

Let

$$\gamma_n^1 = \int_{\mathbb{R}^N}\left(|\nabla u_n^1|^2+(u_n^1)^2\right)dx - \int_{\mathbb{R}^N} K(\sqrt{d_n}x)|u_n^1|^p dx,$$

and

$$\gamma_n^2 = \int_{\mathbb{R}^N}\left(|\nabla u_n^2|^2+(u_n^2)^2\right)dx - \int_{\mathbb{R}^N} K(\sqrt{d_n}x)|u_n^2|^p dx.$$

By the fact that $u_n \in U_n$, (3.8) and (3.9) we have

$$\gamma_n^1 = -\gamma_n^2 + o(1). \tag{3.11}$$

Now, we conclude our proof of Claim 2 by showing that (3.11) leads to a contradiction. Let $\alpha_n > 0$ be such that $\alpha_n u_n^1 \in U_n$. That is,

$$\alpha_n^p \int_{\mathbb{R}^N} K(\sqrt{d_n}x)|u_n^1|^p dx = \alpha_n^2 \int_{\mathbb{R}^N} \left(|\nabla u_n^1|^2 + (u_n^1)^2\right) dx.$$

**Case 1:** After passing to a subsequence if necessary, assume $\gamma_n^1 \le 0$. In this case,

$$\begin{aligned}\alpha_n^{p-2} \int_{\mathbb{R}^N} K(\sqrt{d_n}x)|u_n^1|^p dx &= \int_{\mathbb{R}^N} \left(|\nabla u_n^1|^2 + (u_n^1)^2\right) dx \\ &\le \int_{\mathbb{R}^N} K(\sqrt{d_n}x)|u_n^1|^p dx\,.\end{aligned}$$

It follows that $\alpha_n \le 1$. Hence, by the monotonicity of $I_{d_n}$ on $U_n$, (3.10), and by Lemma 2.2, we have

$$d_n^{-N/2} c_{d_n} \le I_{d_n}(\alpha_n u_n^1) \le I_{d_n}(u_n^1) \to \tilde{A} < A,$$

where $c_{d_n}$ is defined in (2.2). This is a contradiction because

$$d_n^{-N/2} c_{d_n} \to A > \tilde{A}.$$

**Case 2:** A similar argument holds for $\gamma_n^2 \le 0$.
**Case 3:** If both $\gamma_n^1$ and $\gamma_n^2$ are positive after passing to a subsequence then, from (3.11) it follows that $\gamma_n^1 = o(1)$ and $\gamma_n^2 = o(1)$. If $\alpha_n \le 1 + o(1)$, we apply similar arguments to those used in Cases 1 and 2. Now, suppose that $\lim_{n\to\infty} \alpha_n = \alpha_0 > 1$. We claim that along a subsequence if necessary, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} K(\sqrt{d_n}x)|u_n^1|^p dx > 0.$$

Otherwise,

$$\lim_{n\to\infty} \gamma_n^1 = \lim_{n\to\infty} \int_{\mathbb{R}^N} \left(|\nabla u_n^1|^2 + (u_n^1)^2\right) dx = 0,$$

which implies that

$$\tilde{A} = \lim_{n\to\infty} \gamma_n^1 = 0,$$

that is impossible. Now, since $\gamma_n^1 = o(1)$ and $\alpha_n u_n^1 \in U_n$ we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \Big[ \int_{\mathbb{R}^N} \left(|\nabla (u_n^1)|^2 + (u_n^1)^2\right) dx - \int_{\mathbb{R}^N} K(\sqrt{d_n}x)|u_n^1|^p dx \Big] \\ &= \lim_{n\to\infty} (\alpha_n^{p-2} \int_{\mathbb{R}^N} K(\sqrt{d_n}x)|u_n^1|^p dx - \int_{\mathbb{R}^N} K(\sqrt{d_n}x)|u_n^1|^p dx) \\ &= (\alpha_0^{p-2} - 1) \lim_{n\to\infty} \int_{\mathbb{R}^N} K(\sqrt{d_n}x)|u_n^1|^p dx > 0, \end{aligned}$$

again a contradiction. Thus, we have proved that dichotomy cannot happen and therefore (3.5) holds.

Next, we turn to proving (3.6). If the conclusion were not true, without loss of generality, we may assume that there is $a>0$ with $\{\frac{x}{\sqrt{d_n}} \mid x\in\Omega,\ K(x)\geq K_1-a\}\subset\frac{1}{\sqrt{d_n}}(N_\delta(K_\Omega))\subset\Omega_n$ such that

$$\lim_{n\to\infty}\operatorname{dist}\left(y_n,\{\frac{x}{\sqrt{d_n}} \mid x\in\Omega,\ K(x)\geq K_1-a\}\right)=\infty.$$

By the first part of the Proposition, there exists $y_n\in\mathbb{R}^N$ such that for any $\epsilon>0,\exists R>0$ with

$$\lim_{n\to\infty}\frac{p-2}{2p}\int_{B_R(y_n)}\left[|\nabla u_n|^2+u_n^2\right]dx\geq A-\epsilon.$$

Taking $\epsilon_m\to 0$ we can find subsequences $d_{n_m}$, $u_{n_m}$ and $R_m\to\infty$ such that

$$\operatorname{dist}\left(y_{n_m},\{\frac{x}{\sqrt{d_{n_m}}} \mid x\in\Omega,\ K(x)\geq K_1-a\}\right)\geq 2R_m$$

and

$$\frac{p-2}{2p}\int_{B_{R_m}(y_{n_m})}\left[|\nabla u_{n_m}|^2+u_{n_m}^2\right]dx\geq A-\epsilon_m,\quad\text{for } m \text{ large}.$$

For simplicity, we denote these subsequences by $d_n$ and $u_n$. Let $w_n(x)=\alpha_n\eta_R(|x-y_n|)u_n(x)$, where $\alpha_n$ is to be chosen such that $w_n(x)\in U_n$. Then, it is easy to see that $\alpha_n\to 1$ as $n\to\infty$ and

$$\lim_{n\to\infty}I_{d_n}(u_n)=\lim_{n\to\infty}I_{d_n}(w_n).$$

We shall show that

$$\lim_{n\to\infty}I_{d_n}(w_n(x))\geq(K_1-a)^{-\frac{2}{p-2}}I(\omega).$$

In fact,

$$\begin{aligned}
I_{d_n}(w_n) &= \frac{1}{2}\int_{\Omega_n}\left(|\nabla w_n|^2+w_n^2\right)dx-\frac{1}{p}\int_{\Omega_n}K(\sqrt{d_n}x)|w_n|^p dx \qquad (3.12)\\
&= \frac{p-2}{2p}\alpha_n^2\int_{\Omega_n}\left(|\nabla(\eta_R\cdot u_n)|^2+|\eta_R\cdot u_n|^2\right)dx\\
&= \frac{p-2}{2p}\left(\frac{\int_{\Omega_n}(|\nabla(\eta_R\cdot u_n)|^2+|\eta_R\cdot u_n|^2)dx}{\int_{\Omega_n}K(\sqrt{d_n}x)|\eta_R\cdot u_n|^p dx}\right)^{\frac{2}{p-2}}\times\\
&\quad \int_{\Omega_n}(|\nabla(\eta_R\cdot u_n)|^2+|\eta_R\cdot u_n|^2)dx\\
&\geq \frac{p-2}{2p}(K_1-a)^{-\frac{2}{p-2}}\left(\frac{\int_{\Omega_n}(|\nabla(\eta_R\cdot u_n)|^2+|\eta_R\cdot u_n|^2)dx}{\int_{\Omega_n}|\eta_R\cdot u_n|^p dx}\right)^{\frac{2}{p-2}}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \int_{\Omega_n} (|\nabla(\eta_R \cdot u_n)|^2 + |\eta_R \cdot u_n|^2)dx \\
= \;& \frac{p-2}{2p}(K_1 - a)^{-\frac{2}{p-2}} \left( \frac{\int_{\mathbb{R}^N} (|\nabla(\eta_R \cdot u_n)|^2 + |\eta_R \cdot u_n|^2)dx}{(\int_{\mathbb{R}^N} |\eta_R \cdot u_n|^p dx)^{\frac{2}{p}}} \right)^{\frac{p}{p-2}} \\
\geq \;& \frac{p-2}{2p}(K_1 - a)^{-\frac{2}{p-2}} \left( \inf_{u \in H^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2)dx}{(\int_{\mathbb{R}^N} |u|^p dx)^{\frac{2}{p}}} \right)^{\frac{p}{p-2}} \\
= \;& \frac{p-2}{2p}(K_1 - a)^{-\frac{2}{p-2}} \cdot m^{\frac{p}{p-2}} \\
= \;& (K_1 - a)^{-\frac{2}{p-2}} \cdot I(\omega),
\end{aligned}
$$

where $m$ is defined in Proposition 2.1. Now we a contradiction follows from

$$
K_1^{-\frac{2}{p-2}} I(\omega) = \lim_{n\to\infty} I_{d_n}(u_n) = \lim_{n\to\infty} I_{d_n}(w_n) \geq (K_1 - a)^{-\frac{2}{p-2}} I(\omega).
$$

This completes the proof of Proposition 3.2. ♢

**Proof of Proposition 3.3** If this proposition were not true, there would exist $d_n \to 0$, $\epsilon_n \to 0$ and $u_n \in V_n$ such that $J_{d_n}(u_n) \leq c_{d_n} + \epsilon_n d_n^{N/2}$, $c_n = \beta(u_n) \notin N_r(K_\Omega)$. By Lemma 2.2, $v_n = u_n(\sqrt{d_n}x) \in U_n$ and

$$
\lim_{n\to\infty} \int_{\Omega_n} K(\sqrt{d_n}x)|v_n|^p dx = K_1^{-\frac{2}{p-2}} I(\omega).
$$

Choose $a > 0$ such that $\{x \in \bar{\Omega} : K(x) \geq K_1 - a\} \subset \Omega$. By Proposition 3.2, there exist a subsequence, still denoted by $v_n$, a sequence $y_n \in \mathbb{R}^N$, and a constant $C_a > 0$, such that for each $\epsilon > 0$, there is $R > 0$ with

$$
\lim_{n\to\infty} \int_{B_R(y_n)\cap\Omega_n} K(\sqrt{d_n}x)|v_n|^p dx \geq K_1^{-\frac{2}{p-2}} I(\omega) - \epsilon
$$

and

$$
\lim_{n\to\infty} \operatorname{dist}(y_n, \{\frac{x}{\sqrt{d_n}} : x \in \Omega,\ K(x) \geq K_1 - a\}) \leq C_a\,.
$$

Therefore, there exists $t_n \in \{x \in \bar{\Omega} \mid\ K(x) \geq K_1 - a\}$ such that

$$
\lim_{n\to\infty} \operatorname{dist}(y_n, \frac{t_n}{\sqrt{d_n}}) \leq C_a.
$$

By passing to a subsequence, we may assume that $t_n \to t \in \Omega$ with $K(t) \geq K_1 - a$. Without loss of generality, we assume that $c_n = \beta(u_n)$ satisfies $c_n \to 0$ in $\mathbb{R}^N$. By a direct computation we have

$$
\int_{\Omega_n} |v_n|^p x dx = \frac{c_n}{\sqrt{d_n}} \int_{\Omega_n} |v_n|^p dx.
$$

By the assumption $c_n = \beta(u_n) \notin N_r(K_\Omega)$, we have $t \neq 0$. From (3), we have

$$\begin{aligned} K_1 \lim_{n\to\infty} \int_{B_R(y_n)\cap\Omega_n} |v_n|^p dx &\geq \lim_{n\to\infty} \int_{B_R(y_n)\cap\Omega_n} K(\sqrt{d_n}x)|v_n|^p dx \quad (3.13) \\ &\geq K_1^{-\frac{2}{p-2}} I(\omega) - \epsilon. \end{aligned}$$

It follows that

$$\lim_{n\to\infty} \int_{B_R(y_n)\cap\Omega_n} |v_n|^p dx \geq \frac{K_1^{-\frac{2}{p-2}} I(\omega)}{K_1} - \epsilon' = \bar{A} - \epsilon'$$

where $\epsilon' = \frac{\epsilon}{K_1}$, $\bar{A} = \frac{K_1^{-\frac{2}{p-2}} I(\omega)}{K_1}$. For simplicity, we assume that $t = (t^1, t^2, \ldots, t^N)$ with $t^1 > 0$. Without loss of generality, assume

$$\lim_{n\to\infty} \int_{\Omega_n} |v_n|^p dx = B \geq \bar{A}.$$

From (3.13), $\forall \epsilon > 0, \exists R_1 > 0$ such that

$$\lim_{n\to\infty} \int_{B_{R_1}(\frac{t_n}{\sqrt{d_n}})\cap\Omega_n} |v_n|^p dx \geq \bar{A} - \epsilon.$$

Let $s = \min\{y^1 \mid (y^1, y^2, \ldots, y^N) \in K_\Omega\}$. Then, for $n$ large we have

$$\begin{aligned} \frac{c_n^1}{\sqrt{d_n}} \int_{\Omega_n} |v_n|^p dx &= \int_{\Omega_n} x^1 |v_n|^p dx \\ &= \int_{B_{R_1}(\frac{t_n}{\sqrt{d_n}})\cap\Omega_n} |v_n|^p x^1 dx + \int_{\Omega_n\setminus B_{R_1}(\frac{t_n}{\sqrt{d_n}})} |v_n|^p x^1 dx \\ &\geq \left(\frac{t^1}{\sqrt{d_n}} - R_1\right)(\bar{A} - \epsilon) - \frac{|s|}{\sqrt{d_n}}\epsilon \end{aligned}$$

where we use (3) so that

$$\int_{\Omega_n\setminus B_{R_1}(\frac{t_n}{\sqrt{d_n}})} |v_n|^p dx < \epsilon.$$

Hence, we get

$$c_n^1 \int_{\Omega_n} |v_n|^p dx \geq (t_n^1 - R_1\sqrt{d_n})(\bar{A} - \epsilon) - |s|\epsilon.$$

Letting $n \to \infty$ and $\epsilon \to 0$, we obtain $0 \geq t^1 > 0$, a contradiction. ◊

## 3.B. Neumann case with $K_1 > 2^{(p-2)/2}K_2$

We shall state three propositions which are analogous to Propositions 3.1-3.3. The proofs of these results require minor changes from the ones of Section 3.A. Thus, we will do only sketches in this and the next subsection. We assume that $H = H^1(\Omega)$, $r > 0$ such that $2r < \operatorname{dist}(K_\Omega, \partial\Omega)$ and

$$\max_{x\in\bar\Omega} K(x) > 2^{\frac{p-2}{2}} \max_{\partial\Omega} K(x).$$

**Proposition 3.6** *Let $\psi_d$ be given in (3.1). Then $\psi_d \in C(K_\Omega, V_d(\Omega))$ and*

$$J_d(\psi_d(y)(x)) = d^{N/2}[K(y)^{-\frac{2}{p-2}} I(\omega) + o(1)].$$

**Proof.** By (3), $K_\Omega \cap \partial\Omega = \emptyset$. Then, the same proof as that of Proposition3.1 works here since $2r < \operatorname{dist}(K_\Omega, \partial\Omega)$. We omit the details. $\diamondsuit$

**Proposition 3.7** *(a) $\lim_{d\to 0} d^{-N/2} c_d = K_1^{-\frac{2}{p-2}} I(\omega)$, where $c_d$ is defined in (2.2).*
*(b) Let $d_n \to 0$ and $u_n \in U_n$ be such that*

$$\lim_{n\to\infty} \frac{p-2}{2p} \int_{\Omega_n} \left(|\nabla u_n|^2 + u_n^2\right) dx = K_1^{-\frac{2}{p-2}} I(\omega) := B.$$

*Then, there exists $y_n \in \mathbb{R}^N$ such that for any $\epsilon > 0, \exists R > 0$ with*

$$\lim_{n\to\infty} \frac{p-2}{2p} \int_{B_R(y_n)} \left[|\nabla u_n|^2 + u_n^2\right] dx \geq B - \epsilon$$

*and such that for any $\delta > 0$ small there exists $C_\delta > 0$ with*

$$\limsup_{n\to\infty} \operatorname{dist}\left(y_n, \frac{1}{\sqrt{d_n}}(N_\delta(K_\Omega))\right) \leq C_\delta.$$

For the proof of of this proposition we modify the proof of Proposition 3.2 and use the lemma from [21], which is analogous to Lemma 3.5.

**Lemma 3.8** *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Let $d_n \to 0$ and $u_n \in H^1(\Omega_n)$ such that $\|u_n\|_{H^1} \leq C$ for some $C > 0$ and for all $n$. If for some $2 \leq q \leq \frac{2N}{N-2}$ and for some $R > 0$,*

$$\lim_{n\to\infty} \left( \sup_{y\in\mathbb{R}^N} \int_{B_R(y)\cap\Omega_n} |u_n|^q dx \right) = 0\ ,$$

*then*

$$\lim_{n\to\infty} \int_{\Omega_n} |u_n|^p dx = 0,$$

*for all $2 < p < \frac{2N}{N-2}$.*

**Proof of Proposition 3.7** By Lemma 3.8, we can easily rule out the possibility of vanishing. Very much similar arguments to that of the proof of Proposition 3.2 show that dichotomy can not happen. Therefore, we get compactness of the sequence $u_n$ of (3.7). To prove (3.7), we first prove that
$\lim_{n\to\infty} \operatorname{dist}(y_n, \frac{1}{\sqrt{d_n}}(\partial\Omega)) \to \infty$. If $\lim_{n\to\infty} \operatorname{dist}(y_n, \frac{1}{\sqrt{d_n}}(\partial\Omega))$ is finite, without loss of generality, we may assume $y_n \in \partial\Omega$. By compactness, there exists $y_n \in \mathbb{R}^N$ such that for any $\epsilon > 0$, $\exists R > 0$ with

$$\lim_{n\to\infty} \frac{p-2}{2p} \int_{B_R(y_n)\cap\Omega_n} \left[|\nabla u_n|^2 + u_n^2\right] dx \geq B - \epsilon.$$

Taking $\epsilon_m \to 0$ we find subsequences $d_{n_m}$, $u_{n_m}$ and $R_m \to \infty$ such that

$$\frac{p-2}{2p} \int_{B_{R_m}(y_{n_m})\cap\Omega_{n_m}} \left[|\nabla u_{n_m}|^2 + u_{n_m}^2\right] dx \geq B - \epsilon_m$$

for $m$ sufficiently large. For simplicity, we still denote those sequences by $d_n$, $u_n$ , $R_n$ and $\Omega_n$. Because

$$B_{R_n}(y_n) \cap \Omega_n \to R^N_+ = \{x \in \mathbb{R}^N : x = (x_1, x_2, \dots, x_N),\ x_N > 0\}$$

in measures as $n \to \infty$ we have

$$\frac{p-2}{2p} \int_{B_R(y_n)\cap\Omega_n} \left[|\nabla u_n|^2 + u_n^2\right] dx \to \frac{1}{2} K_1^{-\frac{2}{p-2}} I(\omega) = \frac{1}{2} B$$

which contradicts (3). Thus, $\lim_{n\to\infty} \operatorname{dist}(y_n, \frac{1}{\sqrt{d_n}}(\partial\Omega)) \to \infty$ as $n \to \infty$. Now the proof of (3.7) is similar to the proof of (3.6). ♢

**Proposition 3.9** *For $r > 0$, there exist $\epsilon_1 > 0$ and $d_1 > 0$ such that for any $0 < d \leq d_1$ and $0 < \epsilon \leq \epsilon_1$ we have*

$$\beta(u) \in N_r(K_\Omega), \quad \forall u \in J_d^{c_d + \epsilon d^{N/2}}.$$

The proof uses the arguments used in proving Proposition 3.3 with minor changes. We omit it.

## 3.C Neumann case with $K_1 < 2^{\frac{p-2}{2}} K_2$

We assume that $H = H^1(\Omega)$ and

$$\max_{x\in\bar\Omega} K(x) < 2^{\frac{p-2}{2}} \max_{\partial\Omega} K(x).$$

Since $\partial\Omega$ is smooth, there is $r > 0$ such that for any $y \in \partial\Omega$, $B_r(y) \cap \Omega$ is diffeomorphic to $B_1^+(0) := \{x \in B_1(0) \mid x^N > 0\}$. Let $r > 0$ be fixed. Then for $y \in K_{\partial\Omega}$, we define $\psi_d(y) \in V_d$ similarly as in (3.1).

**Proposition 3.10** *$\psi_d \in C(K_{\partial\Omega}, V_d(\Omega))$ and*

$$J_d\left(\psi_d(y)(x)\right) = d^{N/2} \left[\frac{1}{2} K(y)^{-\frac{2}{p-2}} I(\omega) + o(1)\right],$$

*as $d \to 0$ uniformly for $y \in K_{\partial\Omega}$.*

**Proof.** We need some modifications to the proof of Proposition 3.1. Note that $y \in \partial\Omega$ implies $\psi(y) \in H^1(\Omega)$ instead of belonging to $H_0^1(\Omega)$ as in Proposition 3.1; and that, for any fixed $R > 0$, $\frac{1}{\sqrt{d}}(\Omega - \{y\}) \cap \{h : |h| \le R\} \to B_R^+(0)$ in measures as $d \to 0$ uniformly for $y \in \partial\Omega$. Then, similar argument used in proving Proposition 3.1 can show that

$$\begin{aligned} J_d(\psi_d(y)) &= (K(y))^{-\frac{2}{p-2}}(1+o(1))\left(\frac{p-2}{2p}d^{N/2}\int_{R_+^N}(|\nabla\omega|^2+\omega^2)dx + o(1)\right) \\ &= d^{N/2}\left(\frac{1}{2}(K(y))^{-\frac{2}{p-2}}I(\omega)+o(1)\right) \end{aligned}$$

where $R_+^N = \{x \in \mathbb{R}^N \mid x = (x^1, x^2, \dots, x^N),\ x^N > 0\}$. $\diamondsuit$

**Proposition 3.11** *(a)* $\lim_{d\to 0} d^{-N/2}c_d = \frac{1}{2}K_2^{-\frac{2}{p-2}}I(\omega)$, *where* $c_d$ *is defined in (2.2).*

*(b) Let* $d_n \to 0$ *and* $u_n \in U_n$ *be such that*

$$\lim_{n\to\infty}\frac{p-2}{2p}\int_{\Omega_n}\left(|\nabla u_n|^2 + u_n^2\right)dx = \frac{1}{2}K_2^{-\frac{2}{p-2}}I(\omega) := C.$$

*Then, there exists* $y_n \in \mathbb{R}^N$ *such that for any* $\epsilon > 0, \exists R > 0$ *with*

$$\lim_{n\to\infty}\int_{B_R(y_n)}\left[|\nabla u_n|^2 + u_n^2\right]dx \ge C - \epsilon, \tag{3.14}$$

*and such that for any* $\delta > 0$ *small there exists* $C_\delta > 0$ *with*

$$\limsup_{n\to\infty}\operatorname{dist}\left(y_n, \frac{1}{\sqrt{d_n}}(N_\delta(K_{\partial\Omega}))\right) \le C_\delta.$$

**Proof.** The same argument as in the proof of Proposition 3.7 gives the compactness of the sequence $u_n$, i.e., there exists $y_n \in \Omega_n$ such that for any $\epsilon > 0$, there exists $R > 0$, and

$$\lim_{n\to\infty}\int_{B_R(y_n)\cap\Omega_n}\left[|\nabla u_n|^2 + u_n^2\right]dx \ge C - \epsilon.$$

This proves (3.14). Now, if $\lim_{n\to\infty}\operatorname{dist}(y_n, \frac{1}{\sqrt{d_n}}(\partial\Omega)) = \infty$, then let $w_n(x) = \alpha_n\eta_R(|x - y_n|)u_n(x)$, where $\alpha_n$ is to be chosen such that $w_n(x) \in U_n$. Then

$$\frac{1}{2}K_2^{-\frac{2}{p-2}}I(\omega) = \lim_{n\to\infty}I_{d_n}(u_n) = \lim_{n\to\infty}I_{d_n}(w_n).$$

A calculation similar to (3.12) yields

$$I_{d_n}(w_n) \ge K_1^{-\frac{2}{p-2}}\cdot I(\omega).$$

Therefore,
$$\frac{1}{2}K_2^{-\frac{2}{p-2}}I(\omega) \geq K_1^{-\frac{2}{p-2}} \cdot I(\omega)$$
which contradicts $K_1 < 2^{\frac{p-2}{2}}K_2$. Thus, we have $\lim_{n\to\infty} \mathrm{dist}(y_n, \frac{1}{\sqrt{d_n}}(\partial\Omega))$is finite so we may assume $y_n \in \frac{1}{\sqrt{d_n}}(\partial\Omega)$. Now if
$$\lim_{n\to\infty} \mathrm{dist}(y_n, \frac{1}{\sqrt{d_n}}(N_\delta(K_{\partial\Omega})) = \infty$$
for some $\delta > 0$, then there is $a > 0$ such that for a fixed $R > 0$, $B_R(y_n)$ belongs to the region where $K(\sqrt{d_n}x) \geq K_2 - a$, for $n$ large. Then, following the arguments used in proving Proposition 3.2 we get
$$\frac{1}{2}K_2^{-\frac{2}{p-2}}I(\omega) = \lim_{n\to\infty} I_{d_n}(u_n) \geq \frac{1}{2}(K_2 - a)^{-\frac{2}{p-2}}I(\omega),$$
a contradiction. Thus, (3.11) is proved. $\diamondsuit$

**Proposition 3.12** *For $r > 0$ fixed, there exist $\epsilon_1 > 0$ and $d_1 > 0$ such that for any $0 < d \leq d_1$ and $0 < \epsilon \leq \epsilon_1$ we have*
$$\beta(u) \in N_r(K_{\partial\Omega}) \quad \forall u \in J_d^{c_d+\epsilon d^{N/2}}.$$

The proof is similar to the one of Proposition 3.3 and therefore omitted.

## 3.D Neumann case with $K_1 = 2^{\frac{p-2}{2}}K_2$

We assume that $H = H^1(\Omega)$ and
$$\max_{x\in\bar{\Omega}} K(x) = 2^{\frac{p-2}{2}} \max_{\partial\Omega} K(x).$$

Then for $y \in K_{\partial\Omega} \cup K_\Omega$ we may still define $\psi_d(y) \in V_d$ similarly as in section 3.A.

**Proposition 3.13** *$\psi_d \in C(K_{\partial\Omega} \cup K_\Omega, V_d(\Omega))$ with*
*(i) $J_d(\psi_d(y)(x)) = d^{N/2}\left[K_1^{-\frac{2}{p-2}}I(\omega) + o(1)\right]$, as $d \to 0$ uniformly for $y \in K_\Omega$, or*
*(ii) $J_d(\psi_d(y)(x)) = d^{N/2}\left[\frac{1}{2}K_2^{-\frac{2}{p-2}}I(\omega) + o(1)\right]$, as $d \to 0$ uniformly for $y \in K_{\partial\Omega}$.*

**Proof.** First, note that $K_{\partial\Omega}$ and $K_\Omega$ are both closed and $K_{\partial\Omega} \cap K_\Omega = \emptyset$. Therefore, $K_{\partial\Omega}$ and $K_\Omega$ can be completely separated by two distinct open sets. If $y \in K_{\partial\Omega} \cup K_\Omega$, then either $y \in K_{\partial\Omega}$ or $y \in K_\Omega$. Choose $r > 0$ such that $2r < \mathrm{dist}(K_{\partial\Omega}, K_\Omega)$ and define $\eta_r(\cdot)$ as in (3.1) with $y \in K_{\partial\Omega} \cup K_\Omega$ fixed. If $y \in K_\Omega$ we can repeat the proof of Proposition 3.A.1, and for $y \in K_{\partial\Omega}$ the proof is identical with that of Proposition 3.10. $\diamondsuit$

**Proposition 3.14** *(a)* $\lim_{d\to 0} d^{-N/2}c_d = \frac{1}{2}K_2^{-\frac{2}{p-2}}I(\omega)$*, where* $c_d$ *is defined in (2.2).*

*(b) Let* $d_n \to 0$ *and* $u_n \in U_n$ *be such that*

$$\lim_{n\to\infty} \frac{p-2}{2p}\int_{\Omega_n} \left(|\nabla u_n|^2 + u_n^2\right) dx = K_1^{-\frac{2}{p-2}}I(\omega) = \frac{1}{2}K_2^{-\frac{2}{p-2}}I(\omega) := D.$$

*Then, there exists* $y_n \in \mathbb{R}^N$ *such that for any* $\epsilon > 0, \exists R > 0$ *with*

$$\lim_{n\to\infty} \int_{B_R(y_n)\cap\Omega_n} \left[|\nabla u_n|^2 + u_n^2\right] dx \geq D - \epsilon,$$

*and such that for any* $\delta > 0$ *small there exists* $C_\delta > 0$ *where either*

$$\limsup_{n\to\infty} \operatorname{dist}\left(y_n, \frac{1}{\sqrt{d_n}}(N_\delta(K_\Omega))\right) \leq C_\delta, \tag{3.15}$$

*or*

$$\limsup_{n\to\infty} \operatorname{dist}\left(y_n, \frac{1}{\sqrt{d_n}}(N_\delta(K_{\partial\Omega}))\right) \leq C_\delta. \tag{3.16}$$

**Proof.** (a) This is [18][Prop. 3.2, part (1)] which is true for $K_1 = 2^{\frac{p-2}{2}}K_2$. (b) The same arguments as in propositions 3.B.2 and 3.C.2 give the compactness of the sequence $u_n$, i.e. there exists $y_n \in \Omega_n$ such that for any $\epsilon > 0$, $\exists R > 0$ with

$$\lim_{n\to\infty} \frac{2}{p-2}\int_{B_R(y_n)\cap\Omega_n} \left[|\nabla u_n|^2 + u_n^2\right] dx \geq D - \epsilon.$$

If (3.15)-(3.16) were not true, that is, both

$$\limsup_{n\to\infty} \operatorname{dist}\left(y_n, \frac{1}{\sqrt{d_n}}(N_\delta(K_\Omega))\right) = \infty,$$

and

$$\limsup_{n\to\infty} \operatorname{dist}\left(y_n, \frac{1}{\sqrt{d_n}}(N_\delta(K_{\partial\Omega}))\right) = \infty,$$

then following the arguments of Propositions 3.7 and 3.11, we get either

$$K_1^{-\frac{2}{p-2}}I(\omega) = \lim_{n\to\infty} I_{d_n}(u_n) = \lim_{n\to\infty} I_{d_n}(w_n) \geq (K_1 - a)^{-\frac{2}{p-2}}I(\omega),$$

or

$$\frac{1}{2}K_2^{-\frac{2}{p-2}}I(\omega) = \lim_{n\to\infty} I_{d_n}(u_n) \geq \frac{1}{2}(K_2 - a)^{-\frac{2}{p-2}}I(\omega),$$

and both lead to a contradiction. ♢

**Proposition 3.15** *For* $r > 0$ *fixed, there exist* $\epsilon_1 > 0$ *and* $d_1 > 0$ *such that for any* $0 < d \leq d_1$ *and* $0 < \epsilon \leq \epsilon_1$ *we have*

$$\beta(u) \in N_r(K_\Omega \cup K_{\partial\Omega}) \quad \forall u \in J_d^{c_d + \epsilon d^{N/2}}.$$

**Proof.** Suppose the conclusion is not true, then there would exist $d_n \to 0$, $\epsilon_n \to 0$ and $u_n \in V_n$ such that $J_{d_n}(u_n) \le c_{d_n} + \epsilon_n d_n^{N/2}$, $c_n = \beta(u_n) \notin N_r(K_\Omega \cup K_{\partial\Omega})$ or equivalently $c_n = \beta(u_n) \notin N_r(K_\Omega)$ and $c_n = \beta(u_n) \notin N_r(K_{\partial\Omega})$. Then repeating the argument used in Proposition 3.A.3 with the aid of Proposition 3.14, we will have a contradiction. $\diamondsuit$

# 4 Proof of theorems

The proofs of these results are quite similar in spirit, and we shall give details for Theorem 1.1 and the sketch for the other results. The basic idea for the existence of multiplicity results has been used in [1, 2, 3, 21, 22], and the basic idea for proving the shape of solutions has been used in [18, 14, 15, 16, 19, 20, 21, 22].

The proof of Theorem 1.1 is carried out in 3 steps. The first step is to obtain the estimate

$$\operatorname*{cat}_{J_d^{c_d+\epsilon_d}}(J_d^{c_d+\epsilon_d}) \ge 2 \operatorname*{cat}_{N_r(K_\Omega)}(K_\Omega)$$

for $d$ small and for some $\epsilon_d > 0$ depending on $d$. Once we have (4), we may use standard variational techniques on the level set $J_d^{c_d+\epsilon_d}$ and obtain the existence of at least $2\operatorname{cat}_{N_r(K_\Omega)}(K_\Omega)$ critical points of $J_d$ on $J_d^{c_d+\epsilon_d}$. Finally, an energy estimate shows that none of these solutions changes sign, and consequently, we find at least $\operatorname{cat}_{N_r(K_\Omega)}(K_\Omega)$ positive solutions of (1.1).

**Lemma 4.1** *Let $\epsilon_1 > 0$ be given as in Proposition 3.3. For any $\epsilon \in (0, \epsilon_1)$, there exists $d_\epsilon > 0$ such that*

$$\operatorname*{cat}_{J_d^{c_d+\epsilon_d}}(J_d^{c_d+\epsilon_d}) \ge 2 \operatorname*{cat}_{N_r(K_\Omega)}(K_\Omega)$$

*for $\epsilon_d = d^{N/2}\epsilon$, $0 < d \le d_\epsilon$.*

**Proof.** By Proposition 3.3, for some $r$ fixed, there exist $\epsilon_1 > 0$ and $d_1 > 0$ such that for any $0 < d \le d_1$ and $0 < \epsilon \le \epsilon_1$ we have

$$\beta: \; J_d^{c_d+\epsilon_d} \to N_r(K_\Omega),$$

where $\epsilon_d = \epsilon d^{N/2}$. By Proposition 3.1, for each $0 < \epsilon < \epsilon_1$ there exists $d_\epsilon > 0$ such that for $0 < d \le d_\epsilon$

$$\psi_d: \; K_\Omega \to J_d^{c_d+\epsilon_d} \cap \{u \in V_d | \; u \ge 0 \quad \text{a.e. in } \Omega\}$$

is well defined. Both $\psi_d$ and $\beta$ are well-defined and continuous maps. By the construction of $\psi_d$, for any $y \in K_\Omega$,

$$\beta \circ \psi_d(y) \in N_r(K_\Omega).$$

Set $A_+ = J_d^{c_d+\epsilon_d} \cap \{u \in V_d(\Omega) : u \ge 0\text{a.e. in } \Omega\}$ and assume $\operatorname{cat}_{A^+} A^+ = k$. Then, there exist $k$ closed and contractible subsets of $A_+$, say, $A_1, A_2, \dots, A_k$, such that $A_+ \subset \bigcup_{i=1}^k A_i$. Let $Y_i = \psi_d^{-1}(A_i) \subset K_\Omega$, $i = 1, 2, \dots, k$. Then $\bigcup_{i=1}^k Y_i = K_\Omega$, and therefore

$$\operatorname*{cat}_{N_r}(K_\Omega)(K_\Omega) \le \sum_{i=1}^k \operatorname*{cat}_{N_r}(K_\Omega)(Y_i).$$

We shall show that if $Y_i \ne \emptyset$, then $Y_i$ is contractible in $N_r(K_\Omega)$ and $\operatorname{cat}_{N_r(K_\Omega)}(Y_i) = 1$. Since $A_i$ is contractible in $A_+$, there exists $\mathcal{H}_i \in C([0,1] \times A_i, A_+)$ such that

$$\begin{aligned} \mathcal{H}_i(0,a) &= a \quad \forall a \in A_i, \\ \mathcal{H}_i(1,a) &= a_i \in A_+ \quad \forall a \in A_i\,. \end{aligned}$$

Define a map $\mathcal{M} : [0,2] \times Y_i \to N_r(K_\Omega)$ by

$$\mathcal{M}(t,y) = \begin{cases} y - t(y - \beta \circ \mathcal{H}_i(0,\cdot) \circ \psi_d(y)) & \text{for } 0 \le t \le 1, y \in Y_i, \\ \beta \circ \mathcal{H}_i(t-1,\cdot) \circ \psi_d(y) & \text{for } 1 \le t \le 2, y \in Y_i. \end{cases}$$

Then, we verify that $\mathcal{M}(0,y) = y$ for all $y \in Y_i$ and $\mathcal{M}(2,y) = \beta(a_i) \in N_r(K_\Omega)$ for all $y \in Y_i$. By (4), $\mathcal{M}$ is well defined and consequently $Y_i$ is contractible in $N_r(K_\Omega)$. So by (4), $\operatorname{cat}_{N_r(K_\Omega)}(K_\Omega) \le k$. Using $-\psi_d$ and the same argument one can show that

$$\operatorname*{cat}_{N_r(K_\Omega)}(K_\Omega) \ge \operatorname*{cat}_{A_-}(A_-)$$

where $A_- = J_d^{c_d+\epsilon_d} \cap \{u \in V_d(\Omega) : u \le 0 \text{ a.e. in } \Omega\}$. Since $A_+$ and $A_-$ are disjoint in $J_d^{c_d+\epsilon_d}$, we get

$$\begin{aligned} \operatorname*{cat}_{J_d^{c_d+\epsilon_d}}(J_d^{c_d+\epsilon_d}) &\ge \operatorname*{cat}_{A_+\cup A_-}(A_+ \cup A_-) \\ &= \operatorname*{cat}_{A_+}(A_+) + \operatorname*{cat}_{A_-}(A_-) \\ &\ge 2 \operatorname*{cat}_{N_r(K_\Omega)}(K_\Omega). \end{aligned}$$

**Lemma 4.2** *Let $u$ be a critical point of $J_d$ with*

$$J_d(u) < 2c_d. \tag{4.1}$$

*Then $u$ does not change sign.*

**Proof.** If the conclusion was not true, we should have $u = u_+ + u_-$ with $u_+ \not\equiv 0$ and $u_- \not\equiv 0$. By the definition of $V_d$ and $c_d$, let $u = u_+$, then $u_+ \in V_d$. Similarly, $u_- \in V_d$. It follows that

$$c_d \le \frac{2p}{p-2} \int_{\Omega_d} \left(d|\nabla u_\pm|^2 + u_\pm^2\right) dx \le J_d(u).$$

In addition,

$$\int_{\Omega_d} \left(d|\nabla u_\pm|^2 + u_\pm^2\right) dx = \int_{\Omega_d} K(x)|u_\pm|^p dx.$$

But,

$$\int_{\Omega_d} K(x)|u_+|^p dx + \int_{\Omega_d} K(x)|u_-|^p dx = \int_{\Omega_d} K(x)|u_\pm|^p dx.$$

It follows that

$$\int_{\Omega_d} \left(d|\nabla u_+|^2 + u_+^2\right) dx + \int_{\Omega_d} \left(d|\nabla u_-|^2 + u_-^2\right) dx = \int_{\Omega_d} \left(d|\nabla u|^2 + u^2\right) dx$$

i.e.

$$J_d(u) = J_d(u_+) + J_d(u_-)$$

for $u, u_+$ and $u_- \in V_d$. Therefore, we reach a contradiction from

$$2c_d \le J_d(u_+) + J_d(u_-) = J_d(u) < 2c_d.$$

**Proof of Theorem 1.1** By Proposition 3.2, $c_d = d^{N/2}(K_1^{-\frac{2}{p-2}} I(\omega) + o(1))$. For $\epsilon_1 > 0$ given in Proposition 3.3, we choose $0 < \epsilon_0 \le \epsilon_1$. Then there exists $d_0 > 0$ such that for all $d \in (0, d_0)$

$$c_d + d^{N/2} \cdot \epsilon_0 < 2c_d.$$

For this $\epsilon_0$, by Lemma 4.1, there exists $d_0' > 0$ such that

$$\operatorname*{cat}_{J_d^{c_d+\epsilon_d}}(J_d^{c_d+\epsilon_d}) \ge 2 \operatorname*{cat}_{N_r(K_\Omega)}(K_\Omega)$$

$\forall d \in (0, d_0')$ with $\epsilon_d = d^{N/2} \cdot \epsilon_0$.

Applying the minimax method ([17]) here we get at least $2\,\mathrm{cat}_{N_r(K_\Omega)}(K_\Omega)$ critical points of $J_d$ on $J_d^{c_d+\epsilon_d}$. By Lemma 4.2, none of these critical points changes sign, and therefore there exist at least $\mathrm{cat}_{N_r(K_\Omega)}(K_\Omega)$ positive critical points and hence $\mathrm{cat}_{N_r(K_\Omega)}(K_\Omega)$ solutions of (1.1) with Dirichlet boundary condition. ♢

To prove the single peakedness of these solutions, we shall prove the following lemma which states that all low energy solutions are single-peaked solutions.

**Lemma 4.3** *There exist $d_0 > 0$ and $\epsilon_0 > 0$ such that any solution $v_d$ of (1.1), with $d < d_0$ and $J_d(v_d) \le c_d + d^{N/2}\epsilon_0$, has only one local maximum point over $\bar\Omega$ (denoted by $P_d$) which satisfies*

$$\lim_{d\to 0} \operatorname{dist}(P_d, K_\Omega) = 0.$$

**Proof.** By an indirect argument, we only need to consider sequences $d_n \to 0$, $\epsilon_n \to 0$ and a sequence of solutions $v_{d_n}$ which satisfies $J_{d_n}(v_{d_n}) \le c_{d_n} + d_n^{N/2}\epsilon_n$.

It suffices to consider $u_n(x) = v_{d_n}(\sqrt{d_n}x)$ and to show that $u_n$ has only one local maximum point over $\bar\Omega_n$ at some $x_n$ satisfying

$$\lim_{n\to\infty} \operatorname{dist}(x_n, \frac{1}{\sqrt{d_n}} K_\Omega) \le C$$

for some constant $C$.

By assumption and Proposition 3.2(1) we have $I_{d_n}(u_n) \to K_1^{-\frac{2}{p-2}} I(\omega)$.

By Proposition 3.2 again there exists $y_n \in \Omega_n$ such that for any $\epsilon > 0$ there is $R > 0$ with

$$\lim_{n\to\infty} \frac{p-2}{2p} \int_{B_R(y_n)} \left[|\nabla u_n|^2 + u_n^2\right] dx \geq A - \epsilon,$$

and for any $\delta > 0$ small there exists $C_\delta > 0$ such that

$$\lim_{n\to\infty} \operatorname{dist}(y_n, \frac{1}{\sqrt{d_n}}(N_\delta(K_\Omega)) \leq C_\delta.$$

Taking $\epsilon_m \to 0$ we have $R_m \to \infty$ such that (4) holds with $\epsilon$ and $R$ replaced by $\epsilon_m$ and $R_m$. Therefore, we have

$$\min K \lim_{m\to\infty} \int_{\Omega_n \backslash B_{R_m}(y_{n_m})} |u_{n_m}|^p dx \leq \lim_{m\to\infty} \int_{\Omega_n \backslash B_{R_m}(y_{n_m})} K(\sqrt{d_n} x)|u_{n_m}|^p = 0.$$

It follows that

$$\lim_{m\to\infty} \int_{\Omega_n \backslash B_{R_m}(y_{n_m})} |u_{n_m}|^p dx = 0,$$

since $u_n$ satisfies $(I)_{d_n}$ and thus is in the manifold $U_n$.

Let $x_n$ be a local maximum point of $u_n$. Then $u_n(x_n) \geq K_1^{-\frac{1}{p-2}} > 0$ by the maximum principle. Based on the ideas in [14][Lemma 4.1], by Harnack's inequality, there exists a positive constant $C_*$ independent of $d_n$ such that for any $x \in \bar{\Omega}$ one has

$$\sup_{B_{\sqrt{d_n}}(P_{d_n}) \cap \Omega} v_{d_n}(x) \leq C_* \inf_{B_{\sqrt{d_n}}(P_{d_n}) \cap \Omega} v_{d_n}(x).$$

Therefore, there is $\lambda_0 > 0$ such that $v_{d_n}(x) \geq \lambda_0$ for $x \in B_{\sqrt{d_n}}(P_{d_n}) \cap \Omega$, where $P_{d_n}$ is the maximum point of $v_{d_n}$. Now, using this and (4), (4) we conclude that there is a $R_0 > 0$ such that $u_{n_m}$ must achieve any maximum value in $B_{R_0}(y_{n_m})$. This implies that (4) must hold, because if not, let $R_m \to \infty$, then $u_{n_m}$ achieves maximum value at $x_{n_m} = \frac{P_{d_{n_m}}}{\sqrt{d_{n_m}}}$ in $\Omega_n \backslash B_{R_m}(y_{n_m})$ and thus $u_{n_m}(x) \geq \frac{\lambda_0}{\sqrt{d_{n_m}}}$ for all $x \in B_1(\frac{P_{d_{n_m}}}{\sqrt{d_{n_m}}}) \cap \Omega_n$. This contradicts (4).

Assume that $v_n$ has two local maximum points $P_n^1$ and $P_n^2$ for the sequence $d_n \to 0$. Passing to a subsequence if necessary, we first claim that there is a constant $C$ independent of $n$ such that

$$\lim_{n\to 0} d_n^{-\frac{1}{2}} \operatorname{dist}(P_n^1, P_n^2) \leq C.$$

If not, we have

$$d_n^{-\frac{1}{2}} \operatorname{dist}(P_n^1, P_n^2) \to \infty \quad as \quad d_n \to 0,$$

or equivalently

$$\operatorname{dist}(x_n^1, x_n^2) \to \infty \qquad as \quad d_n \to 0,$$

where $x_n^1$ and $x_n^2$ are two local maximum points of $u_n$ for the sequence $d_n \to 0$. Let $r_n = \frac{1}{2}\operatorname{dist}(x_n^1, x_n^2)$. Then, using $r_n \to \infty$ and Proposition 3.2(b) we have

$$\begin{aligned} K_1^{-\frac{2}{p-2}} I(\omega) \leftarrow I_{d_n}(u_n) &= \frac{p-2}{2p}\int_{\Omega_n} \left(|\nabla u_n|^2 + u_n^2\right) dx \\ &\geq \frac{p-2}{2p}\int_{B_{r_n}(x_n^1)} \left(|\nabla u_n|^2 + u_n^2\right) dx + \\ &\quad \frac{p-2}{2p}\int_{B_{r_n}(x_n^2)} \left(|\nabla u_n|^2 + u_n^2\right) dx \\ &\geq 2K_1^{-\frac{2}{p-2}} I(\omega) - \epsilon. \end{aligned}$$

This is a contradiction and thus (4) holds. Consider $u_n(\sqrt{d_n}x + P_n^i)$ and $\Omega_n' = \{x \in \mathbb{R}^N | \sqrt{d_n}x + P_n^i \in \Omega\}$ for $i = 1, 2$. Then using similar arguments to [18][Prop. 3.1] together with the fact that $\lim_{n\to 0} \operatorname{dist}(P_n^i, K_\Omega) = 0$ we have

$$u_n(\sqrt{d_n}x + P_n^i) \to K_1^{-\frac{1}{p-2}}\omega \quad i = 1, 2$$

in $C^{2,\alpha}_{loc}(\mathbb{R}^N)$. Without loss of generality, we assume that the only critical point of $\omega$ is 0 which is non-degenerate. Since $K_1^{-\frac{1}{p-2}}\omega$ has only one critical point at 0 which is non-degenerate, $u_n$ can not have any other critical point around $B_R(0)$ for some $R > 0$. This again contradicts (4). This finishes the proof of Lemma 4.3. $\diamondsuit$

With Lemma 4.3, the single peakedness of solutions follows immediately. Hence we complete the proof of Theorem 1.1.

**Proof of Theorem 1.2** The proof of this theorem is nearly identical to the proof of Theorem 1.1 since the assumption of $K_1 > 2^{\frac{p-2}{2}} K_2$ implies that the maximum of $K(x)$ is achieved in the interior of $\Omega$. $\diamondsuit$

To prove Theorem 1.3, we first give the following lemma which can be regarded as analogous to Lemma 4.1.

**Lemma 4.4** *Let $\epsilon_1 > 0$ be given as in Proposition 3.12. For any $\epsilon \in (0, \epsilon_1)$, there exists $d_\epsilon > 0$ such that*

$$\operatorname*{cat}_{J_d^{c_d+\epsilon_d}} (J_d^{c_d+\epsilon_d}) \geq 2 \operatorname*{cat}_{N_r(K_{\partial\Omega})} (K_{\partial\Omega})$$

*for $\epsilon_d = d^{N/2}\epsilon$, $0 < d \leq d_\epsilon$.*

**Proof.** The proof is almost identical with that of Lemma 4.1 by replacing $N_r(K_\Omega)$ with $N_r(K_{\partial\Omega})$ and using Propositions 3.12 and 3.11.

**Proof of Theorem 1.3** The proof of this theorem is similar to the proof of Theorem 1.1, so we do only a sketch. By Proposition 3.11(a) we have

$$\lim_{d\to 0} d^{-N/2}c_d = \frac{1}{2}K_2^{-\frac{2}{p-2}}I(\omega)$$

as $d \to 0$. For $\epsilon_1$ given in Proposition 3.12, we choose $0 < \epsilon_0 \le \epsilon_1$ with the property that $\epsilon_0 < \frac{1}{2}d^{-\frac{1}{2}}K_2^{-\frac{2}{p-2}}I(\omega)$. Then for this $\epsilon_0 > 0$, by Lemma 4.2, there is $d_{\epsilon_0} > 0$ such that

$$\operatorname*{cat}_{J_d^{c_d+\epsilon_d}}(J_d^{c_d+\epsilon_d}) \ge 2 \operatorname*{cat}_{N_r(K_{\partial\Omega})}(K_{\partial\Omega}) \quad \forall d \in (0, d_0)$$

with $\epsilon_d = d^{\frac{2}{N}}\epsilon_0$. Then, the classical minimax method together with Lemmas 4.4 and 4.2 we can deduce that there exist at least $\mathrm{cat}_{N_r(K_{\partial\Omega})}(K_{\partial\Omega})$ positive solutions for (1.1) with Neumann boundary condition.

To prove single peakedness of these solutions, we consider $u_n(x) = v_{d_n}(\sqrt{d_n}x)$. We need to show that $u_n$ has only one local maximum point over $\bar{\Omega}_n$ at some $x_n$ satisfying

$$\lim_{n\to\infty} \mathrm{dist}(x_n, \frac{1}{\sqrt{d_n}}K_{\partial\Omega}) \le C$$

for some finite constant $C$. But the same argument used in proving Lemma 4.4 can be applied here to conclude that

$$\lim_{d\to 0} \mathrm{dist}(P_d, K_{\partial\Omega}) = 0..$$

The above result also implies that, passing to subsequence if necessary, for $d_n \to 0$

$$d_n^{-\frac{1}{2}}\,\mathrm{dist}(P_{d_n}, K_{\partial\Omega}) \le C$$

for some constant $C$ independent of $d_n$. Using this and repeating the argument used in [14][Theorem 1.3] and [18][Thorem 2.1], we get that any local maximum point $P_{d_n}$ must be on the boundary of $\Omega$, provided $d_n$ is small enough.

Next, assume $v_n$ has two local maximum points $P_n^1$ and $P_n^2$. Similar to what we did to prove Theorem 1.1, we first rule out the case $d_n^{-1/2}\,\mathrm{dist}(P_n^1, P_n^2) \to \infty$ as $d_n \to 0$ by concentration-compactness argument. Using the local convergence of the rescaled solutions $u_n(\sqrt{d_n}x + P_n^i)$, $i = 1, 2$, and a property of the ground state solution, we conclude that $P_n^1 = P_n^2$. $\diamondsuit$

Before proving Theorem 1.4, we give the following lemma which can be proved in a way similar to Lemma 4.1 by making use of Propositions 3.15 and 3.14.

**Lemma 4.5** *Let $\epsilon_1 > 0$ be given as in Proposition 3.15. For any $\epsilon \in (0, \epsilon_1)$, there exists $d_\epsilon > 0$ such that*

$$\operatorname*{cat}_{J_d^{c_d+\epsilon_d}}(J_d^{c_d+\epsilon_d}) \ge 2 \operatorname*{cat}_{N_r(K_\Omega \cup K_{\partial\Omega})}(K_\Omega \cup K_{\partial\Omega})$$

*for $\epsilon_d = d^{N/2}\epsilon$, $0 < d \le d_\epsilon$.*

**Proof of Theorem 1.4** By Proposition 3.13

$$\lim_{d\to 0} d^{-N/2} c_d = \frac{1}{2} K_2^{-\frac{2}{p-2}} I(\omega) = K_1^{-\frac{2}{p-2}} I(\omega).$$

For $\epsilon_1 > 0$ given in Proposition 3.15, we choose $0 < \epsilon_0 \leq \epsilon_1$. Then, there exists $d_0 > 0$ such that for all $d \in (0, d_0)$

$$c_d + d^{N/2} \cdot \epsilon_0 < 2c_d.$$

For this $\epsilon_0$, by Lemma 4.5, there exists $d_0' > 0$ such that

$$\operatorname*{cat}_{J_d^{c_d+\epsilon_d}} (J_d^{c_d+\epsilon_d}) \geq 2 \operatorname*{cat}_{N_r(K_\Omega \cup K_{\partial\Omega})} (K_\Omega \cup K_{\partial\Omega})$$

$\forall d \in (0, d_0')$ with $\epsilon_d = d^{N/2} \cdot \epsilon_0$. Then a minimax method gives that there exist at least $2 \operatorname{cat}_{N_r(K_\Omega \cup K_{\partial\Omega})}(K_\Omega \cup K_{\partial\Omega})$ critical points of $J_d$ in $J_d^{c_d+\epsilon_d}$ for $d \in (0, d_0')$. Lemma 4.2 plus the maximum principle imply that there exist $\operatorname{cat}_{N_r(K_\Omega \cup K_{\partial\Omega})}(K_\Omega \cup K_{\partial\Omega})$ positive critical points. On the other hand, because $K_\Omega \cap K_{\partial\Omega} = \emptyset$ and they are both closed, let $r \leq \frac{1}{2} \operatorname{dist}(K_\Omega, K_{\partial\Omega})$. Then, $\operatorname{cat}_{N_r(K_\Omega \cup K_{\partial\Omega})}(K_\Omega \cup K_{\partial\Omega}) = \operatorname{cat}_{N_r(K_\Omega)}(K_\Omega) + \operatorname{cat}_{N_r(K_{\partial\Omega})}(K_{\partial\Omega})$, which can be easily proved by the definition of category. Hence, there exist at least $\operatorname{cat}_{N_r(K_\Omega)}(K_\Omega) + \operatorname{cat}_{N_r(K_{\partial\Omega})}(K_{\partial\Omega})$ positive critical points and thus $\operatorname{cat}_{N_r(K_\Omega)}(K_\Omega) + \operatorname{cat}_{N_r(K_{\partial\Omega})}(K_{\partial\Omega})$ solutions of (1.1) with *Neumann* condition under the condition $K_1 = 2^{\frac{p-2}{2}} K_2$.

The single peakedness of these solutions can be obtained by combining the corresponding parts of the proofs in Theorems 1.1 and 1.3. Therefore, we omit it here. $\diamondsuit$

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