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Exact multiplicity results for quasilinear boundary-value problems with cubic-like nonlinearities *

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Abstract

We consider the boundary-value problem

$$-(\varphi_p(u'))' = \lambda f(u) \text{ in } (0,1)$$
$$u(0) = u(1) = 0\,,$$

where $p > 1$, $\lambda > 0$ and $\varphi_p(x) = |x|^{p-2}x$. The nonlinearity f is cubic-like with three distinct roots $0 = a < b < c$. By means of a quadrature method, we provide the exact number of solutions for all $\lambda > 0$. This way we extend a recent result, for $p = 2$, by Korman et al. [17] to the general case $p > 1$. We shall prove that when $1 < p \leq 2$ the structure of the solution set is exactly the same as that studied in the case $p = 2$ by Korman et al. [17], and strictly different in the case $p > 2$.

1 Introduction

We consider the question of determining the exact number of solutions of the quasilinear boundary-value problem

$$-(\varphi_p(u'))' = g(\lambda, u), \text{ in } (0,1) \tag{1}$$
$$u(0) = u(1) = 0\,,$$

where $p > 1$, $\lambda > 0$ and $\varphi_p(u) = |u|^{p-2}u$ for all $u \in \mathbb{R}$ and $g(\lambda, u) = \lambda f(u)$. Here the nonlinearity $f \in C^2(\mathbb{R}, \mathbb{R})$ is cubic-like satisfying

$$f(0) = f(b) = f(c) = 0 \text{ for some constants } 0 < b < c\,, \tag{2}$$

$$f(x) > 0 \text{ for } x \in (-\infty, 0) \cup (b, c) \tag{3}$$
$$f(x) < 0 \text{ for } x \in (0, b) \cup (c, +\infty)\,,$$

$$f''(u) \text{ changes sign exactly once when } u \in (0, c)\,, \tag{4}$$

$$F(c) > 0, \text{ where } F(s) = \int_0^s f(u)du,\ s \in \mathbb{R}\,. \tag{5}$$

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Beside conditions (2)-(5) we shall assume in the case where $p \neq 2$, the following additional conditions: There exists $u_0 \in (0,c)$ such that

$$(p-2)f'(u) - uf''(u) \leq 0 \text{ for } u \in (0,u_0] \tag{6}$$

with strict inequality in an open interval $I \subset (0,u_0)$, and

$$(p-2)f'(u) - uf''(u) \geq 0 \text{ for } u \in [u_0,c)\,. \tag{7}$$

When $p = 2$, we prove in Section 3, that (6) and (7) are consequences of (2)-(4).

During this last decade, many articles dealing with boundary-value problems with cubic-like nonlinearities have been published. (See for instance; [12]-[26]). However, all the related results have been obtained for the case $p = 2$; that is, for the Laplacian operator. The case of cubic-like nonlinearities when the differential operator is the p-Laplacian with $p \neq 2$ has yet to be studied.

When $p = 2$ and f satisfies conditions (2)-(5), the solution set of problem (1) was studied recently by Korman et al. [17]. They provide exactness results. They show (among other interesting things) that there exists a critical number $\lambda_0 > 0$ such that problem (1) has no nontrivial solution for $0 < \lambda < \lambda_0$, has a unique nontrivial solution for $\lambda = \lambda_0$ and has exactly two nontrivial solutions for all $\lambda > \lambda_0$. So, a natural question arises; how does the solution set of (1) look like when $p \neq 2$? The purpose of this work is to answer this question. We shall give an exactness result with respect to $p > 1$; we prove, in particular, that when $1 < p \leq 2$ the structure of the solution set of (1) is exactly the same as that studied in the case $p = 2$ by Korman et al. [17], and is strictly different in the case $p > 2$.

It is known that exactness results are more difficult to derive than a lower bound of the number of solutions to boundary value problems such as (1).

The main tool used here is the so-called quadrature method. The delicate part in the process of the proof corresponding to the exactness part of the main results is the study of the exact variations of the time map under consideration over its *entire* definition domain (Lemma 4).

Notice that here, the cubic-like nonlinearity f has three distinct roots $a < b < c$ with $a = 0$. Recently, together with A. Benmezaï [2] (see also, [3]), we considered the case $a < b = 0 < c = -a$ and f is odd for the p-Laplacian case with $p > 1$. Also, we have considered in [8] a more general case where $a < b = 0 < c$, $p > 1$, and f is not necessary odd; there we have defined a new kind of functions we called: half-odd. However, the main results of the present paper are directly related to those of Korman et al. [17] and not to those of [2] and [8]. That is why we do not describe them here. (Also, this would require a large space).

The paper is organized as follows. The main results are stated in Section 2. Next, in Section 3 we shall state and prove some properties of the nonlinearity f. These are of importance in the sequel. Some preliminary lemmas are the aim of Section 4; the first lemma (Lemma 1) is technical and in the second one (Lemma 2) we locate all the eventual nontrivial solutions of problem (1). The proof of Lemma 2 is postponed to the appendix. After describing the quadrature method used in order to look for the solutions, we devote two lemmas (Lemmas 3, 4) to study the limits and variations of the time-map. In Section 5, the main results are proved. Finally, in Section 6 we ask two questions.

2 Notation and main results

In order to state the main results, let us first define the subsets of $C^1([0,1])$ which contain the solutions of the problem (1).
Let A_1^+ be the subset of $C^1([0,1])$ consisting of the functions u satisfying

- $u(x) > 0$, for all $x \in (0,1)$, $u(0) = u(1) = 0 < u'(0)$.
- u is symmetrical with respect to $1/2$.
- The derivative of u vanishes once and only once.

Let $\tilde{A}_1^+$ be the subset of $C^1([0,1])$ consisting of the functions u satisfying

- $u(x) > 0$, for all $x \in (0,1)$, $u(0) = u(1) = 0 < u'(0)$.
- u is symmetrical with respect to $1/2$.
- There exists a compact interval $K \subset (0,1)$ such that for all $x \in (0,1)$,
$$u'(x) = 0 \text{ if and only if } x \in K\,.$$

Definition Let $u \in C([\alpha,\beta])$ be a function with two consecutive zeros $x_1 < x_2$. We call the I-hump of u the restriction of u to the open interval $I = (x_1, x_2)$. When there is no confusion we refer to a hump of u.
Let $B^+(k)$, $(k \geq 1)$ be the subset of $C^1([0,1])$ consisting of the functions u satisfying

- For all $i \in \{0,\cdots,k\}$ there exist $a_i = a_i(u)$, $b_i = b_i(u)$ in $[0,1]$ such that
$$\begin{gathered} 0 = a_0 \leq b_0 < \cdots < a_i \leq b_i < \cdots < a_k \leq b_k = 1 \\ u > 0 \text{ in } (b_i, a_{i+1}),\ \text{for all } i \in \{0,\cdots,k-1\} \\ u \equiv 0 \text{ in } [a_i, b_i],\ \text{for all } i \in \{0,\cdots,k-1\}\,. \end{gathered}$$
- Every hump of u is symmetrical with respect to the center of the interval of its definition.
- The derivative of each hump of u vanishes once and only once.
- Each hump is a translated copy of the first one.

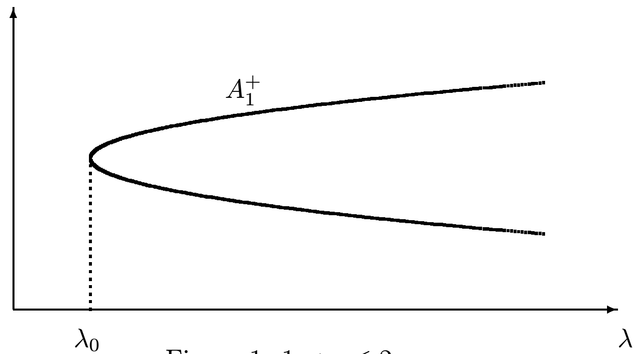

Figure 1: $1 < p \leq 2$.

Let B_k^+ be the subset of $B^+(k)$ consisting of the functions u satisfying

$$a_i(u) = b_i(u) \text{ for all } i \in \{0, \cdots, k\}.$$

If there exists $i_0 \in \{0, \cdots, k\}$ such that $a_{i_0}(u) < b_{i_0}(u)$ we say that $u \in \tilde{B}_k^+$. Therefore, $B^+(k) = B_k^+ \cup \tilde{B}_k^+$ and $B_k^+ \cap \tilde{B}_k^+ = \emptyset$.

We call two functions u_1, u_2 in $\tilde{B}_k^+$ $(k \geq 1)$, equivalent if for all $i \in \{0, \cdots, k\}$, the i-th hump of u_1 is a translated copy of the i-th hump of u_2, or equivalently, u_2 is obtained from u_1 by translating some (maybe all) of its humps. It is clear that this is an equivalence relation. For all $u \in \tilde{B}_k^+$ we denote $\mathrm{Cl}(u)$ the equivalence class of u.

Notice that when f satisfies (2), (3) and (5), there exists a unique $r \in (b, c)$ such that

$$F(r) = 0\,. \tag{8}$$

For $p > 1$ and $x \in [r, c]$, define

$$S_+(x) = \int_0^x \{F(x) - F(\xi)\}^{-1/p} d\xi\,.$$

We shall prove in Lemma 3 that $S_+(r)$ (resp. $S_+(c)$) is infinite if and only if $1 < p \leq 2$. So, for $p > 2$ we can define $\nu = (2S_+(c))^p/p'$, where $p' = p/(p-1)$, and for all integer $k \geq 0$ we define $\lambda_k = (2kS_+(r))^p/p'$ and notice that

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k = k^p \lambda_1 \ldots \text{ for all } k \geq 1, \text{ and } \lim_{k \to +\infty} \lambda_k = +\infty\,.$$

For $\lambda > 0$, denote S_λ the solution set of problem (1).

The main results are worth being described by means of diagrams. The first result (Theorem 2.1) concerns the case where $1 < p \leq 2$. The corresponding diagram (Figure 1) indicates the existence of a unique branch and it is $\subset$-shaped. So, there is no nontrivial solution for $0 < \lambda < \lambda_0$, a unique nontrivial solution for $\lambda = \lambda_0$, and exactly two nontrivial solutions for $\lambda > \lambda_0$. All these solutions are in A_1^+.

When $p > 2$, we have to consider the sequence $(\lambda_k)_{k \geq 0}$ and the number $\nu > 0$. This number maybe smaller than λ_1, equal to λ_1, or greater than λ_1.

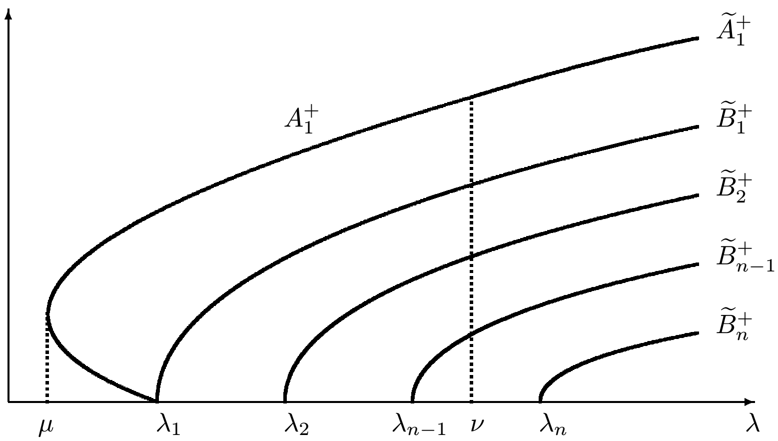

Figure 2: $p > 2$, $\lambda_{n-1} < \nu < \lambda_n$, $1 < n$.

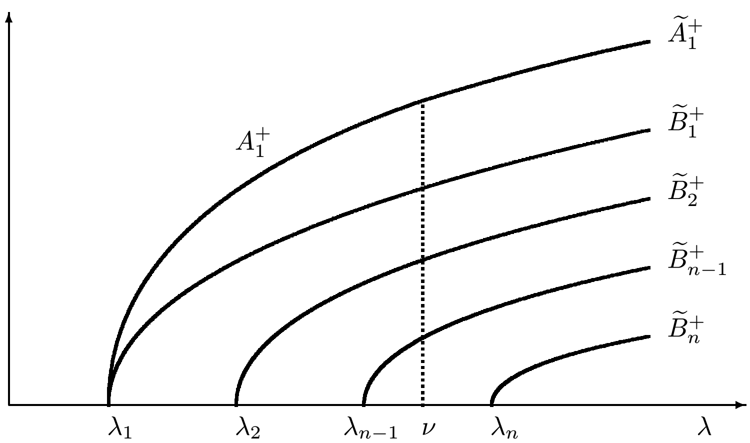

Figure 3: $p > 2$, $\lambda_{n-1} < \nu < \lambda_n$, $1 < n$.

In this later case, it may lie between two consecutive points of the sequence: $\lambda_{n-1} < \nu < \lambda_n$, with $n > 1$ (Figures 2 and 3), or it maybe equal to some λ_n with $n > 1$.

An immediate examination of these bifurcation diagrams, shows that when ν moves from zero to infinity, the upper branch changes but not the others, i.e., beside the upper branch which is different from a diagram to an other, the remaining branches are the same in all these diagrams.

Now consider any one of figures 2 or 3 and let us describe each kind of its branches. The λ-axis designates the trivial solutions, and at each λ_k, $k \geq 1$, there is a bifurcation point which indicates a pair (u_k, λ_k) such that $u_k \in B_k^+$.

The upper branch contains a point which indicates a pair (u_1, ν) such that $u_1 \in A_1^+$. All points lying on this branch which are on the left of (u_1, ν) are in A_1^+, and those lying at the right are in $\tilde{A}_1^+$.

The remaining branches are in some sense "singular". Usually a point (u, λ)

lying on any branch designates a couple where u is a solution of some kind and λ is a real number. This is the case in our diagrams as far as the upper branch or the lower one (λ-axis) are concerned. However, a point on the remaining branches indicates $(\mathrm{Cl}(u), \lambda)$, i.e., the equivalence class of a certain solution u lying in some $\tilde{B}_k^+$, $k \geq 1$, and λ is a real number. So, if u is a solution in some $\tilde{B}_k^+$, with $k \geq 1$, for some $\lambda > 0$ then any $v \in \mathrm{Cl}(u)$ is also a solution in the same $\tilde{B}_k^+$.

The singularity of these branches maybe removed. In fact, consider the same equivalence relation but defined on B_k^+, (for all $k \geq 1$). Then it is clear that

$$\mathrm{Cl}(u) = \{u\}, \text{ for all } u \in B_k^+ .$$

So, the bifurcation points on the λ-axis maybe considered as indicating $(\mathrm{Cl}(u_k), \lambda_k) = (\{u_k\}, \lambda_k)$ instead of (u_k, λ_k). Also, consider on $A_1^+ \cup \tilde{A}_1^+$ the same equivalence relation in essence (which maybe formulated differently). It is clear that

$$\mathrm{Cl}(u) = \{u\}, \text{ for all } u \in A_1^+ \cup \tilde{A}_1^+ .$$

This way, any point on any branch shall designates a couple $(\mathrm{Cl}(u), \lambda)$ and the elements in $\mathrm{Cl}(u)$ are solutions of the problem (1) for the same λ. Therefore, there is coherence in the diagrams and the singularity mentioned above is removed.

The statements of the main results below indicate that for $\nu \leq \lambda_1$ the upper branch contains a turning point, but when $\nu > \lambda_1$, either it still contains a turning point (Figure 2) or there is no such point (Figure 3).

The main results read as follows

Theorem 2.1 *Assume that $1 < p \leq 2$ and f satisfies conditions (2)-(5), and (6), (7). Then there exists $\lambda_0 > 0$ such that*

(i) *If $0 < \lambda < \lambda_0$, $S_\lambda = \{0\}$.*

(ii) *If $\lambda = \lambda_0$, there exists $v_\lambda \in A_1^+$ such that $S_\lambda = \{0\} \cup \{v_\lambda\}$.*

(iii) *If $\lambda > \lambda_0$, there exists v_λ, $w_\lambda \in A_1^+$ such that $v_\lambda \neq w_\lambda$ and $S_\lambda = \{0\} \cup \{v_\lambda, w_\lambda\}$.*

Theorem 2.2 *Assume that $p > 2$ and f satisfies conditions (2)-(5), and (6), (7). Moreover, assume that $\nu < \lambda_1$. Then there exists $\mu \in (0, \nu)$ such that*

(i) *If $0 < \lambda < \mu$, $S_\lambda = \{0\}$.*

(ii) *If $\lambda = \mu$, there exists $v_\lambda \in A_1^+$ such that $S_\lambda = \{0\} \cup \{v_\lambda\}$.*

(iii) *If $\mu < \lambda \leq \nu$, there exists v_λ, $w_\lambda \in A_1^+$ such that $v_\lambda \neq w_\lambda$ and $S_\lambda = \{0\} \cup \{v_\lambda, w_\lambda\}$.*

(iv) *If $\nu < \lambda < \lambda_1$, there exists $v_\lambda \in A_1^+$ and $u_\lambda \in \tilde{A}_1^+$ such that $S_\lambda = \{0\} \cup \{v_\lambda\} \cup \{u_\lambda\}$.*

(v) *If* $\lambda = \lambda_1$, *there exists* $u_\lambda \in \tilde{A}_1^+$ *and* $u_{\lambda,1} \in B_1^+$ *such that* $S_\lambda = \{0\} \cup \{u_\lambda\} \cup \{u_{\lambda,1}\}$.

(vi) *If* $\lambda_k < \lambda < \lambda_{k+1}$, $k \geq 1$, *there exists* $u_\lambda \in \tilde{A}_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$ *and* $S_\lambda = \{0\} \cup \{u_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k})$.

(vii) *If* $\lambda = \lambda_{k+1}, k \geq 1$, *there exists* $u_\lambda \in \tilde{A}_1^+$ *and* $u_{\lambda,1}, \cdots u_{\lambda,k+1}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$, $u_{\lambda,k+1} \in B_{k+1}^+$ *and* $S_\lambda = \{0\} \cup \{u_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k}) \cup \{u_{\lambda,k+1}\}$.

Theorem 2.3 *Assume that* $p > 2$ *and* f *satisfies conditions (2)-(5), and (6), (7). Moreover, assume that* $\nu = \lambda_1$. *Then there exists* $\mu \in (0, \lambda_1)$ *such that*

(i) *If* $0 < \lambda < \mu$, $S_\lambda = \{0\}$.

(ii) *If* $\lambda = \mu$, *there exists* $v_\lambda \in A_1^+$ *such that* $S_\lambda = \{0\} \cup \{v_\lambda\}$.

(iii) *If* $\mu < \lambda < \lambda_1$, *there exists* v_λ, $w_\lambda \in A_1^+$ *such that* $v_\lambda \neq w_\lambda$ *and* $S_\lambda = \{0\} \cup \{v_\lambda, w_\lambda\}$.

(iv) *If* $\lambda = \lambda_1$, *there exists* $v_\lambda \in A_1^+$ *and* $u_{\lambda,1} \in B_1^+$ *such that* $S_\lambda = \{0\} \cup \{v_\lambda\} \cup \{u_{\lambda,1}\}$.

(v) *If* $\lambda_k < \lambda < \lambda_{k+1}$, $k \geq 1$, *there exists* $u_\lambda \in \tilde{A}_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$ *and* $S_\lambda = \{0\} \cup \{u_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k})$.

(vi) *If* $\lambda = \lambda_{k+1}, k \geq 1$, *there exists* $u_\lambda \in \tilde{A}_1^+$ *and* $u_{\lambda,1}, \cdots,$ $u_{\lambda,k+1}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$, $u_{\lambda,k+1} \in B_{k+1}^+$ *and* $S_\lambda = \{0\} \cup \{u_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k}) \cup \{u_{\lambda,k+1}\}$.

Theorem 2.4 *Assume that* $p > 2$ *and* f *satisfies conditions (2)-(5), and (6), (7). Moreover, assume that there exists* $n > 1$ *such that* $\lambda_{n-1} < \nu < \lambda_n$. *Then one and only one of the following possibilities occurs:*

Possibility **A**. *There exists* $\mu \in (0, \lambda_1)$ *such that*

(i) *If* $0 < \lambda < \mu$, $S_\lambda = \{0\}$.

(ii) *If* $\lambda = \mu$, *there exists* $v_\lambda \in A_1^+$ *such that* $S_\lambda = \{0\} \cup \{v_\lambda\}$.

(iii) *If* $\mu < \lambda < \lambda_1$, *there exist* v_λ, $w_\lambda \in A_1^+$ *such that* $v_\lambda \neq w_\lambda$ *and* $S_\lambda = \{0\} \cup \{v_\lambda, w_\lambda\}$.

(iv) *If* $\lambda = \lambda_1$, *there exist* $v_\lambda \in A_1^+$ *and* $u_{\lambda,1} \in B_1^+$ *such that* $S_\lambda = \{0\} \cup \{v_\lambda\} \cup \{u_{\lambda,1}\}$.

(v) *If* $\lambda_k < \lambda < \min\{\lambda_{k+1}, \nu\}$, $1 \leq k \leq n-1$, *there exist* $v_\lambda \in A_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$ *and* $S_\lambda = \{0\} \cup \{v_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k})$.

(vi) *If* $\lambda = \lambda_{k+1}, 1 \le k \le n-2$, *there exist* $v_\lambda \in A_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k+1}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$, $u_{\lambda,k+1} \in B_{k+1}^+$ *and* $S_\lambda = \{0\} \cup \{v_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k}) \cup \{u_{\lambda,k+1}\}$.

(vii) *If* $\lambda = \nu$, *there exist* $v_\lambda \in A_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,n-1}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, n-1$, *and* $S_\lambda = \{0\} \cup \{v_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,n-1})$.

(viii) *If* $\max\{\lambda_k, \nu\} < \lambda < \lambda_{k+1}$, $k \ge n-1$, *there exist* $u_\lambda \in \tilde{A}_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$, *and* $S_\lambda = \{0\} \cup \{u_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k})$.

(ix) *If* $\lambda = \lambda_{k+1}$, $k \ge n-1$, *there exist* $u_\lambda \in \tilde{A}_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k+1}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$, $u_{\lambda,k+1} \in B_{k+1}^+$*and* $S_\lambda = \{0\} \cup \{u_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k}) \cup \{u_{\lambda,k+1}\}$.

Possibility **B**.

(i) *If* $0 < \lambda < \lambda_1$, $S_\lambda = \{0\}$.

(ii) *If* $\lambda = \lambda_1$, *there exists* $u_{\lambda,1} \in B_1^+$ *such that* $S_\lambda = \{0\} \cup \{u_{\lambda,1}\}$.

(iii) *If* $\lambda_k < \lambda < \min\{\lambda_{k+1}, \nu\}$, $1 \le k \le n-1$, *there exist* $v_\lambda \in A_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$ *and* $S_\lambda = \{0\} \cup \{v_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k})$.

(iv) *If* $\lambda = \lambda_{k+1}, 1 \le k \le n-2$, *there exist* $v_\lambda \in A_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k+1}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$, $u_{\lambda,k+1} \in B_{k+1}^+$ *and* $S_\lambda = \{0\} \cup \{v_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k}) \cup \{u_{\lambda,k+1}\}$.

(v) *If* $\lambda = \nu$, *there exist* $v_\lambda \in A_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,n-1}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, n-1$, *and* $S_\lambda = \{0\} \cup \{v_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,n-1})$.

(vi) *If* $\max\{\lambda_k, \nu\} < \lambda < \lambda_{k+1}$, $k \ge n-1$, *there exist* $u_\lambda \in \tilde{A}_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$, *and* $S_\lambda = \{0\} \cup \{u_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k})$.

(vii) *If* $\lambda = \lambda_{k+1}$, $k \ge n-1$, *there exist* $u_\lambda \in \tilde{A}_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k+1}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$, $u_{\lambda,k+1} \in B_{k+1}^+$*and* $S_\lambda = \{0\} \cup \{u_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k}) \cup \{u_{\lambda,k+1}\}$.

Theorem 2.5 *Assume that* $p > 2$ *and* f *satisfies conditions (2)-(5), and (6), (7). Moreover, assume that there exists* $n > 1$ *such that* $\nu = \lambda_n$. *Then one and only one of the following possibilities occurs:*

Possibility **C**. *There exists* $\mu \in (0, \lambda_1)$ *such that*

(i) *If* $0 < \lambda < \mu$, $S_\lambda = \{0\}$.

(ii) *If* $\lambda = \mu$, *there exists* $v_\lambda \in A_1^+$ *such that* $S_\lambda = \{0\} \cup \{v_\lambda\}$.

(iii) *If* $\mu < \lambda < \lambda_1$, *there exist* $v_\lambda,\ w_\lambda \in A_1^+$ *such that* $v_\lambda \neq w_\lambda$ *and* $S_\lambda = \{0\} \cup \{v_\lambda, w_\lambda\}$.

(iv) *If* $\lambda = \lambda_1$, *there exist* $v_\lambda \in A_1^+$ *and* $u_{\lambda,1} \in B_1^+$ *such that* $S_\lambda = \{0\} \cup \{v_\lambda\} \cup \{u_{\lambda,1}\}$.

(v) *If* $\lambda_k < \lambda < \lambda_{k+1}$, $1 \leq k \leq n-1$, *there exist* $v_\lambda \in A_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$ *and* $S_\lambda = \{0\} \cup \{v_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k})$.

(vi) *If* $\lambda = \lambda_{k+1}, 1 \leq k \leq n-1$, *there exist* $v_\lambda \in A_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k+1}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$, $u_{\lambda,k+1} \in B_{k+1}^+$ *and* $S_\lambda = \{0\} \cup \{v_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k}) \cup \{u_{\lambda,k+1}\}$.

(vii) *If* $\lambda_k < \lambda < \lambda_{k+1}$, $k \geq n$, *there exists* $u_\lambda \in \tilde{A}_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$, *and* $S_\lambda = \{0\} \cup \{u_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k})$.

(viii) *If* $\lambda = \lambda_{k+1}$, $k \geq n$, *there exist* $u_\lambda \in \tilde{A}_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k+1}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$, $u_{\lambda,k+1} \in B_{k+1}^+$ *and* $S_\lambda = \{0\} \cup \{u_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k}) \cup \{u_{\lambda,k+1}\}$.

Possibility **D**.

(i) *If* $0 < \lambda < \lambda_1$, $S_\lambda = \{0\}$.

(ii) *If* $\lambda = \lambda_1$, *there exists* $u_{\lambda,1} \in B_1^+$ *such that* $S_\lambda = \{0\} \cup \{u_{\lambda,1}\}$.

(iii) *If* $\lambda_k < \lambda < \lambda_{k+1}$, $1 \leq k \leq n-1$, *there exist* $v_\lambda \in A_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$ *and* $S_\lambda = \{0\} \cup \{v_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k})$.

(iv) *If* $\lambda = \lambda_{k+1}, 1 \leq k \leq n-1$, *there exist* $v_\lambda \in A_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k+1}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$, $u_{\lambda,k+1} \in B_{k+1}^+$ *and* $S_\lambda = \{0\} \cup \{v_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k}) \cup \{u_{\lambda,k+1}\}$.

(v) *If* $\lambda_k < \lambda < \lambda_{k+1}$, $k \geq n$, *there exists* $u_\lambda \in \tilde{A}_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$, *and* $S_\lambda = \{0\} \cup \{u_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k})$.

(vi) *If* $\lambda = \lambda_{k+1}$, $k \geq n$, *there exist* $u_\lambda \in \tilde{A}_1^+$ *and* $u_{\lambda,1}, \cdots, u_{\lambda,k+1}$ *such that* $u_{\lambda,i} \in \tilde{B}_i^+$ *for all* $i = 1, \cdots, k$, $u_{\lambda,k+1} \in B_{k+1}^+$ *and* $S_\lambda = \{0\} \cup \{u_\lambda\} \cup \mathrm{Cl}(u_{\lambda,1}) \cup \cdots \cup \mathrm{Cl}(u_{\lambda,k}) \cup \{u_{\lambda,k+1}\}$.

The novelty in these results concerns the cases $p > 1$ with $p \neq 2$. The case $p = 2$ was proved by Korman et al. [17]. Of course, the case $p = 2$ is also studied here.

3 Some properties of the nonlinearity f

In this section we establish some properties of f. These are used in the sequel and are of importance in our analysis. We first state the properties and next we give the proofs.

Statement of properties

Assume that f satisfies (2)-(4), then there exists $u_0 \in (0,c)$ such that

$$f'' \geq 0 \text{ in } (0,u_0]\,, \tag{9}$$
$$f'' \leq 0 \text{ in } [u_0,c)\,. \tag{10}$$

Moreover, there exist two open intervals I and J with $I \subset (0,u_0)$ and $J \subset (u_0,c)$ such that

$$f'' > 0 \text{ in } I\,, \tag{11}$$
$$f'' < 0 \text{ in } J\,. \tag{12}$$

Hence

$$f' \text{ is increasing in } (0,u_0], \text{ and strictly increasing in } I\,, \tag{13}$$
$$f' \text{ is decreasing in } [u_0,c), \text{ and strictly decreasing in } J\,. \tag{14}$$

Furthermore,

$$f'(0) < 0, f'(u_0) > 0, f'(c) < 0. \tag{15}$$

Hence, there exist $u_0^- \in (0,u_0)$ and $u_0^+ \in (u_0,c)$ such that

$$f' \leq 0 \text{ in } [0,u_0^-) \cup (u_0^+,c] \tag{16}$$
$$f'(u_0^-) = f'(u_0^+) = 0 \tag{17}$$
$$f' > 0 \text{ in } (u_0^-,u_0^+)\,. \tag{18}$$

Moreover,

$$f'(b) > 0\,. \tag{19}$$

So,

$$0 < u_0^- < b < u_0^+ < c\,, \text{ and} \tag{20}$$
$$f \text{ attains its minimum (resp. maximum) on } [0,c] \text{ at } u_0^- \text{ (resp. at } u_0^+). \tag{21}$$

Proof of properties

By (2) and (3), it follows that f'' must change sign at least once in $(0,c)$, say at u_0, and by (4), it follows that (9) and (10) hold. If $f'' \equiv 0$ in $(0,u_0]$ it follows that f' is constant in $(0,u_0]$. But, by (2) and (3) it follows that $f'(0) \leq 0$. Thus, $f' \leq 0$ in $(0,u_0]$. On the other hand, by (10) it follows that f' is decreasing in $[u_0,c)$, thus $f' \leq 0$ in $[u_0,c)$ and furthermore, f is decreasing in $[0,c]$. By

$f(0) = f(c) = 0$ it follows that $f \equiv 0$ in $[0, c]$ which contradicts (3), and therefore, since $f \in C^2$, the existence of I is proved and that of J is similar. Thus, (11) and (12) are proved. Immediate consequences are (13) and (14). So, f' attains its maximum value on $(0, c)$ at u_0. Thus, it can easily be proved that $f'(u_0) > 0$. In fact, if the contrary holds, it follows that $f' \leq 0$ in $(0, c)$ and hence f is monotonic decreasing in $(0, c)$, and by $f(0) = f(c) = 0$ it follows that $f \equiv 0$ in $[0, c]$ which contradicts (3), which proves that $f'(u_0) > 0$.

Let us prove that $f'(0) < 0$ (resp. $f'(c) < 0$). If the contrary holds, that is if $f'(0) \geq 0$ (resp. $f'(c) \geq 0$), by (13) (resp. by (14)) it follows that $f'(x) \geq 0$ for all $x \in (0, u_0)$ (resp. $x \in (u_0, c)$) and hence, f is increasing in $(0, u_0)$ (resp. in (u_0, c)). Due to the fact that f vanishes at 0 (resp. at c), it follows that $f \geq 0$ in $(0, u_0)$ (resp. $f \leq 0$ in (u_0, c)), which contradicts (3). Therefore, (15) is proved. By making use of continuity arguments it follows that (13), (14) and (15) imply (16), (17) and (18).

Let us prove (19). First, by (3) it follows that $f'(b) \geq 0$. If $f'(b) = 0$, by (15)-(18) it follows that $b \in (0, u_0^-) \cup (u_0^+, c)$. Assume that $b \in (0, u_0^-)$ (resp. $b \in (u_0^+, c)$). By (16), it follows that $f' \leq 0$ in $(0, b)$ (resp. in (b, c)) and therefore f is decreasing in $(0, b)$ (resp. in (b, c)). By $f(0) = f(b) = 0$ (resp. $f(b) = f(c) = 0$) it follows that $f \equiv 0$ in $[0, b]$ (resp. in $[b, c]$) which contradicts (3). Therefore, (19) is proved, and immediate consequences are (20) and (21).

4 Preliminary lemmas

Lemma 1 is a technical one. The aim of the next lemma is to answer to the question : how does any solution to (1) look like ? We shall prove that if u is a nontrivial solution to (1), then

$$u \in A_1^+ \cup \tilde{A}_1^+ \cup \bigcup_{k \geq 1} B^+(k).$$

The proof of Lemma 1 is the same as that of Lemma 8 in [7] or Lemmas 6 and 8 in [5] (see also analogous lemmas in [1], [4]). So, it is omitted. The proof of Lemma 2 is not complicated but long and tedious. So, it is postponed to the appendix.

Next, we define the time map on its interval of definition, compute its limits at the boundary points of its definition domain in Lemma 3, and then study its exact variations on its entire definition domain in Lemma 4.

Lemma 1 *Assume that $f \in C(\mathbb{R})$ satisfies (3) and (5). Consider the function defined in $\mathbb{R}^{\pm}$ by*

$$s \longmapsto G_{\pm}(\lambda, E, s) := E^p - p'\lambda F(s), \tag{22}$$

where $E, \lambda > 0$ and $p > 1$ are real parameters. Then

(i) *If $E > E_*(p, \lambda) := (p'\lambda F(c))^{1/p}$ (resp. $E > 0$), the function $G_+(\lambda, E, \cdot)$ (resp. $G_-(\lambda, E, \cdot)$) is strictly positive in $\mathbb{R}^+$ (resp. in $\mathbb{R}^-$).*

(ii) *If $E = E_*(\lambda)$, the function $G_+(\lambda, E, \cdot)$ is strictly positive in $(0, c)$ and vanishes at c.*

(iii) *If $0 < E < E_*(\lambda)$, the function $G_+(\lambda, E, \cdot)$ admits in the open interval (b, c) a unique zero $s_+(\lambda, E)$ and is strictly positive in the open interval $(0, s_+(\lambda, E))$. Moreover,*

(a) *The function $E \mapsto s_+(\lambda, E)$ is C^1 in $(0, E_*(\lambda))$ and,*

$$\frac{\partial s_+}{\partial E}(\lambda, E) = \frac{(p-1)E^{p-1}}{\lambda f(s_+(\lambda, E))} > 0, \tag{23}$$

for all $E \in (0, E_*(\lambda))$.

(b) $\lim\limits_{E \to 0^+} s_+(\lambda, E) = r$ *and* $\lim\limits_{E \to E_*} s_+(\lambda, E) = c$, *where r is the unique zero of F in (b, c) (see, (8)).*

The following lemma locate all possible nontrivial solutions.

Lemma 2 *Let u be a nontrivial solution of (1). Then*

$$u \in A_1^+ \cup \tilde{A}_1^+ \cup \bigcup_{k \geq 1} B^+(k), \text{ and } 0 \leq u'(0) \leq E_*(\lambda) = (p'\lambda F(c))^{1/p}.$$

According to this lemma, for all fixed $\lambda > 0$ and $p > 1$, we shall look for the solutions of problem (1) with respect to their derivative at the origin; $u'(0) = E \in [0, E_*(\lambda)]$.

For $\lambda > 0$, $p > 1$ and $E \in [0, E_*(\lambda)]$, let

$$X_+(\lambda, E) = \{s > 0 : E^p - p'\lambda F(\xi) > 0\,,\ \forall \xi \in (0, s)\}\,.$$

By Lemma 1, it follows that

$$X_+(\lambda, E) = \begin{cases} (0, c) & \text{if } \ E = E_* \\ (0, s_+(\lambda, E)) & \text{if } \ 0 < E < E_*, \\ (0, r) & \text{if } \ E = 0 \end{cases}$$

and therefore,

$$r_+(\lambda, E) := \sup X_+(\lambda, E) = \begin{cases} c & \text{if } \ E = E_* \\ s_+(\lambda, E) & \text{if } \ 0 < E < E_*, \\ r & \text{if } \ E = 0\,, \end{cases} \tag{24}$$

and one deduces from Lemma 1 the following

$$\tfrac{\partial r_+}{\partial E}(\lambda, E) > 0, \forall \lambda > 0, \forall E \in (0, E_*(\lambda)), \tag{25}$$

$$\lim_{E \to 0^+} r_+(\lambda, E) = r, \text{ and } \lim_{E \to E_*} r_+(\lambda, E) = c\,. \tag{26}$$

Define, for any $p > 1$, $\lambda > 0$ the time map T_+ by

$$T_+(\lambda, E) := \int_0^{r_+(\lambda,E)} (E^p - p'\lambda F(\xi))^{-1/p} d\xi, \; E \in [0, E_*(\lambda)], \tag{27}$$

with the convention $T_+(\lambda, 0) = +\infty$ (resp. $T_+(\lambda, E_*(\lambda)) = +\infty$) if the integral in (27) diverges.

Arguing as in Guedda and Veron [11], it follows that

- For each $\lambda > 0$ and $E \in (0, E_*(\lambda))$, problem (1) admits a solution $u \in A_1^+$ satisfying $u'(0) = E$ if and only if $T_+(\lambda, E) = 1/2$, and in this case the solution is unique and its sup-norm is equal to $r_+(\lambda, E)$.
- For each $\lambda > 0$, problem (1) admits a solution $u \in A_1^+$ satisfying $u'(0) = E_*(\lambda)$ if and only if $T_+(\lambda, E_*(\lambda)) = 1/2$, and in this case the solution is unique and its sup-norm is equal to c.
- For each $\lambda > 0$, problem (1) admits a solution $u \in \tilde{A}_1^+$ satisfying $u'(0) = E_*(\lambda)$ if and only if $T_+(\lambda, E_*(\lambda)) < 1/2$, and in this case the solution is unique and its sup-norm is equal to c.
- For each $\lambda > 0$ and $n \in \mathbb{N}^*$, problem (1) admits a solution $u \in B_n^+$ if and only if $nT_+(\lambda, 0) = 1/2$, and in this case the solution is unique and its sup-norm is equal to r.
- For each $\lambda > 0$ and $n \in \mathbb{N}^*$, problem (1) admits a solution $u \in \tilde{B}_n^+$ if and only if $nT_+(\lambda, 0) < 1/2$, and in this case v is an other solution in $\tilde{B}_n^+$ if and only if $v \in \mathrm{Cl}(u)$, and the sup-norm of each solution is equal to r.

A simple change of variables shows that,

$$T_+(\lambda, E) = r_+(\lambda, E) \int_0^1 (E^p - p'\lambda F(r_+(\lambda, E)\xi))^{-1/p} d\xi, \tag{28}$$

which can be written as,

$$T_+(\lambda, E) = (r_+(\lambda, E)/E) \int_0^1 (1 - p'\lambda F(r_+(\lambda, E)\xi)/E^p)^{-1/p} d\xi. \tag{29}$$

Also, observe that one has from the definition of $s_+(\lambda, E)$, (Lemma 1, Assertion **(iii)**), $E^p = \lambda p' F(r_+(\lambda, E))$, so, (28) may be written as,

$$T_+(\lambda, E) = (\lambda p')^{-1/p} \int_0^{r_+(\lambda,E)} (F(r_+(\lambda, E)) - F(\xi))^{-1/p} d\xi. \tag{30}$$

For any $p > 1$ and $x \in [r, c]$ let us define $S_+(x)$ by

$$S_+(x) := \int_0^x (F(x) - F(\xi))^{-1/p} d\xi \in [0, +\infty].$$

Thus, (30) may be written as,

$$T_+(\lambda, E) = (\lambda p')^{-1/p} S_+(r_+(\lambda, E)). \tag{31}$$

The limits of the time map $T_+(\lambda, \cdot)$ are the aim of the following.

Lemma 3 **(i)** *$S_+(r) = +\infty$ if and only if $1 < p \le 2$, and $S_+(c) = +\infty$ if and only if $1 < p \le 2$.*

(ii) $\lim_{E\to 0^+} T_+(\lambda, E) = (\lambda p')^{-1/p} S_+(r)$ *and* $\lim_{E\to E_*} T_+(\lambda, E) = (\lambda p')^{-1/p} S_+(c)$.

Proof of (i). By (3) and (8), one has

$$\lim_{x\to r} \frac{F(r) - F(x)}{r - x} = f(r) > 0\,.$$

Thus, there exist $\delta > 0$ and $M > 0$ such that

$$F(r) - F(x) > M(r - x), \text{ for all } x \in (r - \delta, r)\,.$$

Therefore,

$$\int_{r-\delta}^{r} \frac{dx}{(F(r) - F(x))^{1/p}} < M^{-1/p} \int_{r-\delta}^{r} \frac{dx}{(r - x)^{1/p}} < +\infty \text{ for all } p > 1\,.$$

On the other hand, using L'Hopital's rule twice and (15) it follows that

$$\lim_{x\to 0^+} \frac{F(0) - F(x)}{-x^2} = \frac{f'(0)}{2} < 0\,.$$

Thus, there exist $\varepsilon > 0$, $m_- < 0$ and $M_- < 0$ such that

$$m_- \le \frac{F(0) - F(x)}{-x^2} \le M_-, \text{ for all } x \in (0, \varepsilon)\,.$$

Therefore,

$$(-m_-)^{-1/p} \int_0^{\varepsilon} \frac{dx}{x^{2/p}} \le \int_0^{\varepsilon} \frac{dx}{(F(0) - F(x))^{1/p}} \le (-M_-)^{-1/p} \int_0^{\varepsilon} \frac{dx}{x^{2/p}}\,.$$

The first part of Assertion **(i)** follows from $F(r) = F(0) = 0$ and the well-known fact

$$\int_0^{\varepsilon} \frac{dx}{x^{2/p}} < +\infty \text{ if and only if } p > 2.$$

The second part may be proved similarly. In fact, using L'Hopital's rule twice and (15) it follows that

$$\lim_{x\to c^-} \frac{F(c) - F(x)}{(c - x)^2} = -\frac{f'(c)}{2} > 0.$$

Thus

$$M_+^{-1/p}\int_{c-\varepsilon}^{c}\frac{dx}{(c-x)^{2/p}}\le\int_{c-\varepsilon}^{c}\frac{dx}{(F(c)-F(x))^{1/p}}\le m_+^{-1/p}\int_{c-\varepsilon}^{c}\frac{dx}{(c-x)^{2/p}}$$

for some strictly positive constants M_+, m_+ and ε. Therefore, the second assertion of **(i)** follows from the well-known fact

$$\int_{c-\varepsilon}^{c}\frac{dx}{(c-x)^{2/p}}<+\infty \text{ if and only if } p>2.$$

Proof of (ii). The value of the limits follows by passing to the limit in (31) as E tends to 0 and E_* respectively. Then Lemma 3 is proved. ♢

To study the exact number of solutions of (1) we need to know the exact variations of the time map $T_+(\lambda,\cdot)$ over all its definition domain $(0,E_*(\lambda))$. These variations are the aim of the following,

Lemma 4 *If* $1<p\le 2$, *for all* $\lambda>0$ *the time map* $T_+(\lambda,\cdot)$ *admits a unique critical point; a minimum. If* $p>2$, *for all* $\lambda>0$ either *the time map* $T_+(\lambda,\cdot)$ *is strictly increasing* or *admits a unique critical point; a minimum in* $(0,E_*(\lambda))$.

Proof By (31), recall that for all $\lambda>0$ and $E\in(0,E_*(\lambda))$.

$$T_+(\lambda,E)=(\lambda p')^{-1/p}S_+(r_+(\lambda,E)).$$

On the other hand, by (25) and (26), for each fixed $\lambda>0$, the function $E\mapsto r_+(\lambda,E)$ is an increasing C^1-diffeomorphism from $(0,E_*(\lambda))$ onto (r,c), where r is the unique zero of F in (b,c). A differentiation yields

$$\frac{\partial T_+}{\partial E}(\lambda,E)=(\lambda p')^{-1/p}\times\frac{\partial r_+}{\partial E}(\lambda,E)\times S_+'(r_+(\lambda,E)).$$

Thus, to study the variations of $T_+(\lambda,\cdot)$ in $(0,E_*(\lambda))$ it suffices to study those of $S_+(\cdot)$ in (r,c). One has

$$S_+(\rho)=\int_0^{\rho}\{F(\rho)-F(u)\}^{-1/p}du,\ \rho\in(r,c)$$

and

$$S_+'(\rho)=\frac{1}{p\rho}\int_0^{\rho}\frac{H_p(\rho)-H_p(u)}{\{F(\rho)-F(u)\}^{(p+1)/p}}du,\ \rho\in(r,c)\,, \tag{32}$$

where

$$H_p(u)=pF(u)-uf(u),\ \text{for all } u\in[0,c] \text{ and } p>1\,. \tag{33}$$

To study the sign of the derivative $S_+'(\cdot)$ we need to study that of expression

$$H_p(\rho)-H_p(u)\ \text{ for all } 0<u<\rho \text{ and } r<\rho<c\,.$$

So, a careful analysis of the variations of $H_p(\cdot)$ is required. One has

$$H_p'(u) = (p-1)f(u) - uf'(u),\ \text{for all } u \in [0,c] \text{ and } p > 1 \tag{34}$$

and

$$H_p''(u) = (p-2)f'(u) - uf''(u),\ \text{for all } u \in [0,c] \text{ and } p > 1\,. \tag{35}$$

By (2) it follows that

$$H_p(0) = H_p'(0) = 0,\ \text{for all } p > 1\,, \tag{36}$$

and by (2) and (4), it follows that

$$H_p(c) > 0,\ \text{for all } p > 1, \tag{37}$$

and by (15),

$$H_p'(c) > 0,\ \text{for all } p > 1. \tag{38}$$

Now, let us look closely to the special case where $p = 2$. By (9), it follows that

$$H_2''(u) \le 0,\ \text{for all } u \in (0, u_0]\,,$$

and by (36) it follows that

$$H_2'(u) \le 0,\ \text{for all } u \in [0, u_0]\,. \tag{39}$$

By (11) it follows that there exists a unique $\alpha \in [0, u_0)$ such that

$$\begin{gathered} H_2(u) = H_2'(u) = 0 \text{ for all } u \in [0,\alpha] \\ H_2(u) < 0 \text{ and } H_2'(u) < 0 \text{ for all } u \in (\alpha, u_0]\,. \end{gathered} \tag{40}$$

On the other hand, by (10) it follows that $H_2''(u) \ge 0$, for all $u \in [u_0, c)$, and by (38) and (40) there exist β and γ in (u_0, c) such that

$$\begin{gathered} u_0 < \beta \le \gamma < c \\ H_2'(u) < 0,\ \text{for all } u \in [u_0, \beta) \\ H_2'(u) = 0,\ \text{for all } u \in [\beta, \gamma] \\ H_2'(u) > 0,\ \text{for all } u \in (\gamma, c]\,. \end{gathered}$$

Therefore, regarding (36) and (39), there exists a unique $\delta \in (\gamma, c)$ such that

$$\begin{gathered} H_2(u) < 0,\ \text{for all } u \in [u_0, \beta] \\ H_2(u) = H_2(\beta) < 0,\ \text{for all } u \in [\beta, \gamma] \\ H_2(u) < 0,\ \text{for all } u \in [\gamma, \delta) \text{ and } H_2(\delta) = 0 \\ H_2(u) > 0,\ \text{for all } u \in (\delta, c]\,. \end{gathered}$$

Thus, for all fixed $\rho \in (0, \alpha]$,

$$H_2(\rho) - H_2(u) = 0,\ \text{for all } u \in (0, \rho), \tag{41}$$

and for all fixed $\rho \in (\alpha, \gamma]$

$$H_2(\rho) - H_2(u) < 0 \text{ for all } u \in (0, \min(\rho, \beta)), \tag{42}$$

and for all fixed $\rho \in [\delta, c)$,

$$H_2(\rho) - H_2(u) > 0 \text{ for all } u \in (0, \rho). \tag{43}$$

Notice that to obtain (41)-(43) the starting conditions were conditions (9), (10) and (11). In contrast, if $p \neq 2$, H_p'' changes sign in $(0, u_0)$ since by (9), f'' is of constant sign in $(0, u_0)$ and by (16), (17) and (18), f' changes sign in $(0, u_0)$. This leads us to consider the additional conditions (6) and (7) for all $p > 1$, and therefore

$$H_p''(u) \leq 0, \text{ for all } u \in (0, u_0], \text{ and } p > 1 \tag{44}$$

with strict inequality in an open interval $I_p \subset (0, u_0)$, and

$$H_p''(u) \geq 0, \text{ for all } u \in [u_0, c) \text{ and } p > 1\,. \tag{45}$$

Let us emphasize that (6) and (7) are automatically satisfied if $p = 2$. In fact they are reduced to (9)-(11). So, (6) and (7) do not consist as additional conditions for the special case where $p = 2$.

By (36) and (44) it follows that there exists a unique $\alpha_p \in [0, u_0)$ such that

$$H_p(u) = H_p'(u) = 0, \text{ for all } u \in [0, \alpha_p]\,, \tag{46}$$

$$H_p(u) < 0, \; H_p'(u) < 0, \text{ for all } u \in (\alpha_p, u_0]\,. \tag{47}$$

By (38) and (45) it follows that for all $p > 1$, there exist β_p and γ_p in (u_0, c) such that

$$u_0 < \beta_p \leq \gamma_p < c$$

$$H_p'(u) < 0, \text{ for all } u \in [u_0, \beta_p) \tag{48}$$

$$H_p'(u) = 0, \text{ for all } u \in [\beta_p, \gamma_p] \tag{49}$$

$$H_p'(u) > 0, \text{ for all } u \in (\gamma_p, c]\,. \tag{50}$$

Therefore, there exists a unique $\delta_p \in (\gamma_p, c)$ such that

$$H_p(u) < 0, \text{ for all } u \in [u_0, \beta_p] \tag{51}$$

$$H_p(u) = H_p(\beta_p) < 0, \text{ for all } u \in [\beta_p, \gamma_p] \tag{52}$$

$$H_p(u) < 0, \text{ for all } u \in [\gamma_p, \delta_p) \text{ and } H_p(\delta_p) = 0 \tag{53}$$

$$H_p(u) > 0, \text{ for all } u \in (\delta_p, c]. \tag{54}$$

This implies that: for all $p > 1$ and all fixed $\rho \in (0, \alpha_p]$,

$$H_p(\rho) - H_p(u) = 0, \text{ for all } u \in (0, \rho), \tag{55}$$

and for all fixed $\rho \in (\alpha_p, \gamma_p]$,

$$H_p(\rho) - H_p(u) < 0, \text{ for all } u \in (0, \min(\rho, \beta_p))\,, \tag{56}$$

and for all fixed $\rho \in [\delta_p, c)$,

$$H_p(\rho) - H_p(u) > 0, \text{ for all } u \in (0, \rho)\,. \tag{57}$$

Notice that

$$H_p(r) = pF(r) - rf(r) = -rf(r) < 0\,.$$

Thus, by (46), (47), (53) and (54) it follows that

$$\alpha_p < r < \delta_p, \text{ for all } p > 1. \tag{58}$$

By (57) it follows that

$$S'_+(\rho) > 0 \text{ for all } \rho \in [\delta_p, c) \subset (r, c). \tag{59}$$

It remains to study the variations of $S_+(\cdot)$ on the interval (r, δ_p). Notice that one has to distinguish two cases

$$\alpha_p < r < \gamma_p < \delta_p < c\,, \tag{60}$$

$$\gamma_p \le r < \delta_p < c\,. \tag{61}$$

It is easy to see that if (60) holds, then

$$S'_+(\rho) < 0 \text{ for all } \rho \in (r, \gamma_p]. \tag{62}$$

In fact, this follows by (55), (56), (52) and (32). Now, we will show in the case (60) (resp. case (61)) that S'_+ admits at most one zero in (γ_p, δ_p) (resp. in (r, δ_p)).

Easy computations show that for all $\rho \in (r, c)$,

$$S''_+(\rho) = \frac{p+1}{(p\rho)^2}\int_0^\rho \frac{(H_p(\rho) - H_p(u))^2}{\{F(\rho) - F(u)\}^{(2p+1)/p}}\,du + \frac{1}{p\rho^2}\int_0^\rho \frac{\Phi_p(\rho) - \Phi_p(u)}{\{F(\rho) - F(u)\}^{(p+1)/p}}\,du$$

where $\Phi_p(u) := -p(p+1)F(u) + 2puf(u) - u^2 f'(u)$, for all $u \in (0, c)$.

Let K be a real number. Thus, for all $\rho \in (r, c)$,

$$\begin{aligned} &p\rho^2 S''_+(\rho) + p\rho K S'_+(\rho) \\ &= \int_0^\rho \frac{\Psi_p(\rho) - \Psi_p(u)}{\{F(\rho) - F(u)\}^{(p+1)/p}}\,du + \frac{p+1}{p}\int_0^\rho \frac{(H_p(\rho) - H_p(u))^2}{\{F(\rho) - F(u)\}^{(2p+1)/p}}\,du \end{aligned} \tag{63}$$

where $\Psi_p(u) = \Phi_p(u) + KH_p(u)$, for all $u \in (0, c)$. Choose $K = p + 1$. Thus $\Psi_p(u) = uH'_p(u)$, for all $u \in (0, c)$.

Next, in the case (60) (resp. case (61)), we split the first integral in (63) as follows

$$\begin{aligned} &\int_0^\rho \frac{\Psi_p(\rho) - \Psi_p(u)}{\{F(\rho) - F(u)\}^{(p+1)/p}}\,du \\ &= \int_0^{\beta_p} \frac{\rho H'_p(\rho) - uH'_p(u)}{\{F(\rho) - F(u)\}^{(p+1)/p}}\,du + \int_{\beta_p}^{\gamma_p} \frac{\rho H'_p(\rho) - uH'_p(u)}{\{F(\rho) - F(u)\}^{(p+1)/p}}\,du \\ &\quad + \int_{\gamma_p}^{\rho} \frac{\rho H'_p(\rho) - uH'_p(u)}{\{F(\rho) - F(u)\}^{(p+1)/p}}\,du \end{aligned} \tag{64}$$

for all $\rho \in (\gamma_p, \delta_p)$ (resp. for all $\rho \in (r, \delta_p) \subset (\gamma_p, \delta_p)$).

By (50), (46), (47) and (48) the first integral in (64) is strictly positive, and by (49) and (50) the second one is also strictly positive.

By (45) it follows that H'_p is increasing in (γ_p, δ_p). Using (50) it follows that for all $\rho \in (\gamma_p, \delta_p)$ (resp. $\rho \in (r, \delta_p)$),

$$0 < H'_p(u) \le H'_p(\rho) \text{ for all } u \in (\gamma_p, \rho) \subset (\gamma_p, \delta_p).$$

Therefore, $\rho H'_p(\rho) - u H'_p(u) \ge 0$, for all $u \in (\gamma_p, \rho)$. Thus, the third integral in (64) is positive. It follows that

$$\rho S''_+(\rho) + (p+1) S'_+(\rho) > 0, \text{ for all } \rho \in (\gamma_p, \delta_p), \text{ (resp. } \rho \in (r, \delta_p)),$$

which implies that S_+ is convex in a neighborhood of each of its critical points lying in (γ_p, δ_p) (resp. in (r, δ_p)). Thus, S'_+ vanishes at most once in (γ_p, δ_p) (resp. in (r, δ_p)) for all $p > 1$. Therefore; regarding (59), it follows that S_+ is either strictly increasing in (r, c) or strictly decreasing in (r, s_p) for some $s_p \in (r, \delta_p)$ and then strictly increasing in (s_p, c). For the special case $1 < p \le 2$, by Lemma 3 one has

$$\lim_{\rho \to r^+} S_+(\rho) = \lim_{\rho \to c^-} S_+(\rho) = +\infty.$$

Thus, the first possibility above cannot occur, and in this case S_+ admits a unique critical point which is a minimum. Therefore, Lemma 4 is proved. ◊

5 Proofs of main results

Assume that $1 < p \le 2$. By Lemma 3 and 4, for all fixed $\lambda > 0$, the time map $T_+(\lambda, \cdot)$ admits a unique critical point which is a minimum in $(0, E_*(\lambda))$ and satisfies

$$\lim_{E \to 0^+} T_+(\lambda, E) = \lim_{E \to E_*} T_+(\lambda, E) = +\infty.$$

Also, by Lemma 3

$$\lim_{\rho \to r^+} S_+(\rho) = \lim_{\rho \to c^-} S_+(\rho) = +\infty,$$

and by the proof of Lemma 4, S_+ admits a unique critical point, a minimum in (r, c) at r_*, say. Therefore, based upon the fact that for all $\lambda > 0$, $r_+(\lambda, \cdot)$ is strictly increasing from $(0, E_*(\lambda))$ onto (r, c), it follows that there exists a unique $\tilde{E} = \tilde{E}(\lambda) \in (0, E_*(\lambda))$ such that $r_* = r_+(\lambda, \tilde{E}(\lambda))$. Thus, by (31), for all $E \in (0, E_*(\lambda))$

$$\begin{aligned} T_+(\lambda, \tilde{E}(\lambda)) &= (p'\lambda)^{-1/p} S_+(r_*) \\ &\le (p'\lambda)^{-1/p} S_+(r_+(\lambda, E)) = T_+(\lambda, E), \end{aligned}$$

hence, $T_+(\lambda, \cdot)$ attains its unique global minimum value at $\tilde{E}(\lambda) \in (0, E_*(\lambda))$. It follows that

- If $(p'\lambda)^{-1/p}S_+(r_*) > (1/2)$, the equation $T_+(\lambda, E) = (1/2)$ in the variable $E \in (0, E_*(\lambda))$ admits no solution.
- If $(p'\lambda)^{-1/p}S_+(r_*) = (1/2)$, the equation $T_+(\lambda, E) = (1/2)$ in the variable $E \in (0, E_*(\lambda))$ admits a unique solution; $\tilde{E}(\lambda)$.
- If $(p'\lambda)^{-1/p}S_+(r_*) < (1/2)$, the equation $T_+(\lambda, E) = (1/2)$ in the variable $E \in (0, E_*(\lambda))$ admits exactly two solutions.

Hence, Theorem 2.1 is proved if we let $\lambda_0 = (2S_+(r_*))^p/p'$. ♢

Now, assume that $p > 2$ and let us prove Theorem 2.2. By the assumption

$$\nu = (2S_+(c))^p/p' < (2S_+(r))^p/p' = \lambda_1,$$

it follows that, for all fixed $\lambda > 0$,

$$\lim_{E \to E_*} T_+(\lambda, E) < \lim_{E \to 0} T_+(\lambda, E).$$

According to Lemma 4, it follows that $T_+(\lambda, \cdot)$ admits a unique critical point; a minimum. Thus, as in the case where $1 < p \leq 2$, there exists a unique $r_* \in (r, c)$ and a unique $\tilde{E} = \tilde{E}(\lambda) \in (0, E_*(\lambda))$ such that

$$\begin{aligned} \min_{r \leq \rho \leq c} S_+(\rho) &= S_+(r_*), \text{ and} \\ \min_{0 \leq E \leq E_*} T_+(\lambda, E) &= T_+(\lambda, \tilde{E}(\lambda)) = (p'\lambda)^{-1/p} S_+(r_*). \end{aligned}$$

Define

$$\begin{aligned} J_0 &= \{u \in C^1([0,1]) : u \neq 0 \text{ and } u'(0) = 0\} \\ J_1(\lambda) &= \{u \in C^1([0,1]) : 0 < u'(0) < E_*(\lambda)\} \\ J_2(\lambda) &= \{u \in C^1([0,1]) : u'(0) = E_*(\lambda)\}. \end{aligned}$$

According to Lemma 5, each nontrivial solution to (1) belongs to $J_0 \cup J_1(\lambda) \cup J_2(\lambda)$. So, let us look for the nontrivial solutions in $J_1(\lambda)$.

- If $(p'\lambda)^{-1/p}S_+(r_*) > 1/2$, the equation $T_+(\lambda, E) = 1/2$ in the variable $E \in (0, E_*(\lambda))$ admits no solution. Thus, if $0 < \lambda < \mu := (2S_+(r_*))^p/p'$, problem (1) admits no solution in $J_1(\lambda)$.
- If $(p'\lambda)^{-1/p}S_+(r_*) = 1/2$, the equation $T_+(\lambda, E) = 1/2$ in the variable $E \in (0, E_*(\lambda))$ admits a unique solution; $\tilde{E}(\lambda)$. Thus, if $\lambda = \mu$, problem (1) admits a unique solution v_λ in $J_1(\lambda)$, and this solution belongs to A_1^+.
- If $(p'\lambda)^{-1/p}S_+(r_*) < 1/2 < (p'\lambda)^{-1/p}S_+(c)$, the equation $T_+(\lambda, E) = 1/2$ in the variable $E \in (0, E_*(\lambda))$ admits exactly two solutions. Thus, if $\mu < \lambda < \nu$, problem (1) admits exactly two solutions v_λ, w_λ in $J_1(\lambda)$, and they belong to A_1^+.

- If $(p'\lambda)^{-1/p}S_+(c) = 1/2$, the equation $T_+(\lambda, E) = 1/2$ in the variable $E \in (0, E_*(\lambda))$ admits a unique solution; $E_1 < \tilde{E}(\lambda)$. Thus, if $\lambda = \nu$, problem (1) admits a unique solution v_λ in $J_1(\lambda)$, and it belongs to A_1^+.

- If $(p'\lambda)^{-1/p}S_+(c) < 1/2 < (p'\lambda)^{-1/p}S_+(r)$, the equation $T_+(\lambda, E) = 1/2$ in the variable $E \in (0, E_*(\lambda))$ admits a unique solution; $E_1 < \tilde{E}(\lambda)$. Thus, if $\nu < \lambda < \lambda_1$, problem (1) admits a unique solution v_λ in $J_1(\lambda)$, and it belongs to A_1^+.

- If $(p'\lambda)^{-1/p}S_+(r) \geq 1/2$, the equation $T_+(\lambda, E) = 1/2$ in the variable $E \in (0, E_*(\lambda))$ admits no solution. Thus, if $\lambda \geq \lambda_1$, problem (1) admits no solution in $J_1(\lambda)$.

Now, let us look for the nontrivial solutions in $J_2(\lambda)$. For all $\lambda > 0$,

$$T_+(\lambda, E_*(\lambda)) = 1/2 \text{ if and only if } (p'\lambda)^{-1/p}S_+(c) = 1/2\,.$$

Thus, problem (1) admits a solution in $J_2(\lambda) \cap A_1^+$ if and only if $\lambda = \nu$, and in this case the solution is unique. For all $\lambda > 0$,

$$T_+(\lambda, E_*(\lambda)) < 1/2 \text{ if and only if } (p'\lambda)^{-1/p}S_+(c) < 1/2\,.$$

Thus, problem (1) admits a solution in $J_2(\lambda) \cap \tilde{A}_1^+$ if and only if $\lambda > \nu$, and in this case the solution is unique.

Now let us look for the nontrivial solutions in J_0. Let $n \in \mathbb{N}^*$. For all $\lambda > 0$,

$$nT_+(\lambda, 0) = 1/2 \text{ if and only if } n(p'\lambda)^{-1/p}S_+(r) = 1/2\,.$$

Thus, problem (1) admits a solution in $J_0 \cap B_n^+$ if and only if $\lambda = \lambda_n$, and in this case the solution is unique. For all $\lambda > 0$,

$$nT_+(\lambda, 0) < 1/2 \text{ if and only if } n(p'\lambda)^{-1/p}S_+(r) < 1/2\,.$$

Thus, problem (1) admits a solution $u_{\lambda,n}$ in $J_0 \cap \tilde{B}_n^+$ if and only if $\lambda > \lambda_n$, and in this case each function u in $\mathrm{Cl}(u_{\lambda,n})$ is a solution to (1). Therefore, Theorem 2.2 is proved. ♢

To prove Theorems 2.3, 2.4 and 2.5, the same reasoning works. However, for Theorems 2.4 and 2.5, one has

$$\lim_{E\to 0} T_+(\lambda, E) < \lim_{E\to E_*} T_+(\lambda, E).$$

Thus, according to Lemma 4, $T_+(\lambda, \cdot)$ may have a unique critical point; a minimum, or may be strictly increasing. These two alternatives lead for Theorem 2.4 to the possibilities **A** and **B**, and for Theorem 2.5 to the possibilities **C** and **D**. ♢

6 Open questions

1. For $p > 2$, Theorems 2.4 and 2.5 provide alternative results. Do there exist some sufficient conditions ensuring that possibility **A** (resp. **B**, **C**, **D**) holds? Can one find an example of f such that possibility **A** (resp. **B**, **C**, **D**) holds? Or maybe among the two alternatives Theorem 2.4 (resp. Theorem 2.5) provides, the same one holds always?

2. In the literature, there are some examples of nonlinearities $g(\lambda, u)$ such that the structure of the solution set of (1) does change when p varies (as that studied in this paper) but in others it does not change; for example as that studied by Addou and Benmezaï [4] for $g(\lambda, u) = \lambda \exp(u)$.

   Thus, we ask the question of providing sufficient or necessary conditions on g insuring that the structure of (at least) the set of (positive) solutions of problem (1) does not change when p varies.

7 Appendix

In this section, we prove Lemma 2 which is a consequence of the following two lemmas.

Lemma 5 *Let u be a nontrivial solution of (1). Then*

$$u \geq 0 \text{ in } [0,1] \text{ and } 0 \leq u'(0) \leq E_*(\lambda) = (p'\lambda F(c))^{1/p}.$$

Moreover,

- *If* $0 \leq u'(0) < E_*(\lambda)$, *then* $\max_{0\leq x\leq 1} u(x) < c$.
- *If* $u'(0) = E_*(\lambda)$, *then* $\max_{0\leq x\leq 1} u(x) = c$.

Lemma 6 *Let u be a nontrivial solution of (1). Then*

(a) $u'(0) \in (0, E_*(\lambda))$ *implies* $u \in A_1^+$,

(b) $u'(0) = E_*(\lambda)$ *implies* $u \in A_1^+ \cup \tilde{A}_1^+$,

(c) $u'(0) = 0$ *implies* $u \in \bigcup_{k\geq 1} B^+(k)$.

Proof of Lemma 5. Assume that there exists $x_0 \in (0,1)$ such that

$$u(x_0) = \min_{0\leq x\leq 1} u(x) < 0 \text{ and } u'(x_0) = 0\,.$$

The variations of F imply that $F(u(x_0)) < 0$. On the other hand, by the energy relation (see [7, Lemma 7])

$$|u'(x)|^p = |u'(0)|^p - p'\lambda F(u(x)), \text{ for all } x \in [0,1],$$

it follows that

$$0 = |u'(x_0)|^p = |u'(0)|^p - p'\lambda F(u(x_0)).$$

Thus, $F(u(x_0)) = (1/p'\lambda)|u'(0)|^p \geq 0$. A contradiction. Therefore $u \geq 0$ in $[0,1]$. Let $x_* \in (0,1)$ be such that $u(x_*) = \max_{0\leq x\leq 1} u(x)$. By the energy relation, it follows that

$$0 = |u'(x_*)|^p = |u'(0)|^p - p'\lambda F(u(x_*)).$$

Thus, $u(x_*)$ is a positive zero of the function $s \mapsto |u'(0)|^p - p'\lambda F(s)$. By Lemma 1 this function vanishes at least once if and only if $0 \leq |u'(0)| \leq E_*(\lambda)$. Since, $u \geq 0$ in $[0,1]$ it follows that $u'(0) \geq 0$. Thus $|u'(0)| = u'(0)$ and therefore $0 \leq u'(0) \leq E_*(\lambda)$.

Assume that $u'(0) = 0$ and there exists $x_* \in (0,1)$ such that $u(x_*) = \max_{0\leq x\leq 1} u(x) \geq c$. Thus, there exists $x_0 \in (0,1)$ such that $u(x_0) = c$. By the energy relation, it follows that

$$0 \leq |u'(x_0)|^p = |u'(0)|^p - p'\lambda F(u(x_0)) = -p'\lambda F(c).$$

Thus, $F(c) \leq 0$, which contradicts hypothesis (5). Therefore, $u'(0) = 0$ implies that $\max_{0\leq x\leq 1} u(x) < c$.

Assume that $0 < u'(0) < E_*(\lambda) = (p'\lambda F(c))^{1/p}$, and there exists $x_* \in (0,1)$ such that $u(x_*) = \max_{0\leq x\leq 1} u(x) \geq c$. Thus, there exists $x_0 \in (0,1)$ such that $u(x_0) = c$. By the energy relation, it follows that

$$0 \leq |u(x_0)|^p = |u'(0)|^p - p'\lambda F(u(x_0)) = |u'(0)|^p - p'\lambda F(c).$$

Thus, $F(c) \leq (1/p'\lambda)|u'(0)|^p$ which is impossible since $u'(0) < E_*(\lambda)$ implies that $(1/p'\lambda)|u'(0)|^p < F(c)$. Therefore, $0 < u'(0) < E_*(\lambda)$ implies that $\max_{0\leq x\leq 1} u(x) < c$.

Assume that $u'(0) = E_*(\lambda) = (p'\lambda F(c))^{1/p}$. Let $x_* \in (0,1)$ be such that $\max_{0\leq x\leq 1} u(x) = u(x_*)$ and $u'(x_*) = 0$. By the energy relation it follows that

$$0 = |u'(x_*)|^p = |u'(0)|^p - p'\lambda F(u(x_*)) = p'\lambda F(c) - p'\lambda F(u(x_*))\,.$$

Thus, $F(c) = F(u(x_*))$ and therefore $u(x_*) = c$ since $F(c) > F(x)$ for all $x \geq 0$ and $x \neq c$. Lemma 5 is proved. ◊

Proof of Lemma 6. Each assertion is a consequence of several steps. If u is a nontrivial solution of (1) and satisfying $u'(0) \in (0, E_*(\lambda))$, then Assertion **(a)** is an immediate consequence of the following steps:

(a1) For all $x_* \in (0,1)$, $u'(x_*) = 0$ implies $u(x_*) = s_+(u'(0)) \in (r, c)$.

(a2) For all $x_1, x_2 \in (0,1)$, $x_1 < x_2$, $u'(x_1) = u'(x_2) = 0$ implies $u \equiv s_+(u'(0))$ in $[x_1, x_2]$.

(a3) The derivative u' vanishes exactly once in $(0,1)$.

(a4) The solution u is symmetric with respect to $1/2$.

If u is a nontrivial solution of (1) and satisfying $u'(0) = E_*(\lambda)$, then Assertion **(b)** is an immediate consequence of the following steps:

(b1) For all $x_* \in (0,1)$, $u'(x_*) = 0$ implies $u(x_*) = c$.

(b2) For all $x_1, x_2 \in (0,1)$, $x_1 < x_2$, $u'(x_1) = u'(x_2) = 0$ implies $u \equiv c$ in $[x_1, x_2]$.

(b3) There exist $x_1, x_2 \in (0,1)$, such that $x_1 \le x_2$, and for all $x \in (0,1)$,

$$u'(x) = 0, \text{ if and only if } x \in [x_1, x_2].$$

(b4) There exist $x_1, x_2 \in (0,1)$, such that $0 < x_1 \le x_2 < 1$, and $u' > 0$ on $(0, x_1)$, $u' \equiv 0$ on $[x_1, x_2]$, and $u' < 0$ on $(x_2, 1)$.

(b5) The solution u is symmetric with respect to $1/2$.

If u is a nontrivial solution of (1) and satisfying $u'(0) = 0$, then Assertion **(c)** is an immediate consequence of the following steps:

(c1) For all $x_* \in (0,1)$, $u'(x_*) = 0$ implies $u(x_*) = 0$ or $u(x_*) = r$.

(c2) Each local maxima of u is a strict one.

(c3) There are finitely many critical points at which u attains its maximum value; r.

(c4) If u attains its maximum value at the n points of the strictly increasing sequence $(x_i)_{1\le i\le n}$ then for all $i \in \{1, \cdots, n\}$ there exists $a_i \le b_i$ in $[0,1]$ such that

$$x_i < a_i \le b_i < x_{i+1},, \text{ for all } i \in \{1, \cdots, n-1\}, \tag{65}$$
$$0 = a_0 \le b_0 < x_1 \text{ and } x_n < a_n \le b_n = 1\,, \tag{66}$$
$$u \equiv 0 \text{ on } [a_i, b_i] \text{ for all } i \in \{0, \cdots, n\}, \tag{67}$$
$$u' > 0 \text{ on } (b_i, x_{i+1}) \text{ for all } i \in \{0, \cdots, n-1\}, \tag{68}$$
$$u' < 0 \text{ on } (x_i, a_i) \text{ for all } i \in \{1, \cdots, n\}, \tag{69}$$
$$b_i + a_{i+1} = 2x_{i+1}, \text{ for all } i \in \{0, \cdots, n-1\}, \tag{70}$$
$$u|_{[b_i, a_{i+1}]}, \text{ is symmetric with respect to } x_{i+1} \text{ for } i \in \{0, \cdots, n-1\}, \tag{71}$$
$$u|_{[b_i, a_{i+1}]}, \text{ is a translation of } u|_{[b_0, a_1]}, \text{ for all } i \in \{0, \cdots, n-1\}\,. \tag{72}$$

The proofs of all these steps are simple and therefore omitted. Full details can be found in the author's doctoral thesis [9].

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In this addendum we answer the question about alternative results for Theorems 2.4 and 2.5. We shall prove that for $p > 2$, Possibility B of Theorem 2.4 and Possibility D of Theorem 2.5 never happen. Therefore, the diagram in Fig. 3 does not occur. That is, for $p > 2$ the upper branch has always a turning point (which is unique).

The alternatives in Theorems 2.4 and 2.5 come from the alternative situation on the time map $T_+(\lambda, \cdot)$. Indeed, Lemma 4 states that for all $p > 2$ and $\lambda > 0$, *either* the time map $T_+(\lambda, \cdot)$ is strictly increasing *or* it admits a unique critical point; a minimum in $(0, E_*(\lambda))$.

We shall prove that for all $p > 2$ and all $\lambda > 0$, $T_+(\lambda, \cdot)$ admits at least one minimum in $(0, E_*(\lambda))$. Therefore, it admits a unique critical point for all $p > 1$ (according to the first part of Lemma 4) and it is never strictly increasing on $(0, E_*(\lambda))$. As a consequence, for all $p > 2$, Possibility B of Theorem 2.4 and Possibility D of Theorem 2.5 do not occur. To prove this statement, it suffices to show that:

Lemma. $S'_+(r) = -\infty$ and $S'_+(c) = +\infty$ for all $p > 2$.

Proof. Since $F(r) = F(0) = 0$, the integral in the expression

$$S'_+(r) = \frac{1}{pr} \int_0^r \frac{H_p(r) - H_p(u)}{(F(r) - F(u))^{1+\frac{1}{p}}} \, du$$

has two singularities: one at 0 and one at r. So, we shall write

$$S'_+(r) = \frac{1}{pr}(I_0 + I_r),$$

where

$$I_0 = \int_0^{r/2} \frac{H_p(r) - H_p(u)}{(F(r) - F(u))^{1+\frac{1}{p}}} \, du \,, \quad I_r = \int_{r/2}^r \frac{H_p(r) - H_p(u)}{(F(r) - F(u))^{1+\frac{1}{p}}} \, du \,.$$

Next we prove that $I_0 = -\infty$ and $I_r \in [-\infty, +\infty)$, so that $S'_+(r) = -\infty$.

Proof of $I_0 = -\infty$. Using l'Hopital's rule twice it follows that

$$\lim_{u \to 0} \frac{F(u)}{u^2} = \frac{f'(0)}{2} < 0 \,.$$

This last inequality follows from (15). Therefore,

$$\frac{H_p(r) - H_p(u)}{(F(r) - F(u))^{1+\frac{1}{p}}} = \frac{H_p(r) - H_p(u)}{(-\frac{F(u)}{u^2})^{1+\frac{1}{p}}} \cdot \frac{1}{u^{2(1+\frac{1}{p})}} \approx \frac{H_p(r)}{(-\frac{f'(0)}{2})^{1+\frac{1}{p}}} \cdot \frac{1}{u^{2(1+\frac{1}{p})}}$$

for all $u \in (0,\varepsilon)$ and for some $\varepsilon > 0$. Since $2(1+\frac{1}{p}) > 1$ and

$$\frac{H_p(r)}{(-\frac{f'(0)}{2})^{1+\frac{1}{p}}} = \frac{-rf(r)}{(-\frac{f'(0)}{2})^{1+\frac{1}{p}}} < 0$$

it follows that

$$\int_0^{r/2} \frac{H_p(r)}{(-\frac{f'(0)}{2})^{1+\frac{1}{p}}} \cdot \frac{1}{u^{2(1+\frac{1}{p})}} du = -\infty$$

which proves that $I_0 = -\infty$.

Proof of $I_r \in [-\infty, +\infty)$. We distinguish two cases.
Case $H_p'(r) \neq 0$. Since

$$\lim_{u\to r} \frac{H_p(r) - H_p(u)}{r-u} = H_p'(r) \neq 0$$

and

$$\lim_{u\to r} \frac{F(r) - F(u)}{r-u} = f(r) > 0\,,$$

it follows that

$$\frac{H_p(r) - H_p(u)}{(F(r)-F(u))^{1+\frac{1}{p}}} = \frac{(\frac{H_p(r)-H_p(u)}{r-u})}{(\frac{F(r)-F(u)}{r-u})^{1+\frac{1}{p}}} \cdot \frac{1}{(r-u)^{\frac{1}{p}}} \approx \frac{H_p'(r)}{(f(r))^{1+\frac{1}{p}}} \cdot \frac{1}{(r-u)^{\frac{1}{p}}}$$

for all $u \in (r-\varepsilon, r)$ and for some $\varepsilon > 0$. Since $\frac{1}{p} < 1$ and $\frac{H_p'(r)}{(f(r))^{1+(1/p)}} \neq 0$,

$$\int_{r/2}^{r} \frac{H_p'(r)}{(f(r))^{1+\frac{1}{p}}} \cdot \frac{1}{(r-u)^{\frac{1}{p}}} du \in (-\infty, +\infty)$$

which proves that $I_r \in [-\infty, +\infty)$.

Case $H_p'(r) = 0$. From equations (46)-(50), it follows that $\beta_p \leq r \leq \gamma_p$.

First assume that $\beta_p \neq r$. Then in a left neighborhood of r the integrand function is identically zero. That is, there exists $\varepsilon > 0$ such that

$$\frac{H_p(r) - H_p(u)}{(F(r)-F(u))^{1+\frac{1}{p}}} = 0$$

for all $u \in (r-\varepsilon, r)$. Therefore,

$$\int_{r-\varepsilon}^{r} \frac{H_p(r) - H_p(u)}{(F(r)-F(u))^{1+\frac{1}{p}}}\, du = 0\,.$$

So, the integral I_r presents no singularity at r and $I_r \in (-\infty, +\infty)$.

Now assume that $r = \beta_p$. Then, by (46)-(48) $H_p(r) - H_p(u) \le 0$ for all $u \in (0, r)$. Therefore,

$$\int_{r/2}^{r} \frac{H_p(r) - H_p(u)}{(F(r) - F(u))^{1+\frac{1}{p}}} \, du \in [-\infty, 0],$$

which proves that $I_r \in [-\infty, +\infty)$. Therefore, $S'(r) = -\infty$.

Now, we shall prove that $S'(c) = +\infty$. Since

$$\lim_{u \to c} \frac{H_p(c) - H_p(u)}{c - u} = H_p'(c) = -cf'(c) > 0\,,$$

because of (15), and

$$\lim_{u \to c} \frac{F(c) - F(u)}{(c - u)^2} = -\frac{f'(c)}{2} > 0\,,$$

it follows that

$$\frac{H_p(c) - H_p(u)}{(F(c) - F(u))^{1+\frac{1}{p}}} = \frac{(\frac{H_p(c) - H_p(u)}{c-u})}{(\frac{F(c)-F(u)}{(c-u)^2})^{1+\frac{1}{p}}} \cdot \frac{1}{(c-u)^{1+\frac{2}{p}}} \approx \frac{(-cf'(c))}{(-\frac{f'(c)}{2})^{1+\frac{1}{p}}} \cdot \frac{1}{(c-u)^{1+\frac{2}{p}}}$$

for all $u \in (c - \varepsilon, c)$ and for some $\varepsilon > 0$. Since $1 + \frac{2}{p} > 1$ and $\frac{(-cf'(c))}{(-\frac{f'(c)}{2})^{1+\frac{1}{p}}} > 0$, it follows that

$$\int_0^c \frac{(-cf'(c))}{(-\frac{f'(c)}{2})^{1+\frac{1}{p}}} \cdot \frac{1}{(c-u)^{1+\frac{2}{p}}} du = +\infty,$$

which proves that $S'(c) = +\infty$. Therefore, the present proof is complete, and the claim of the addendum is proved. ◊