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Carleman estimates for the Euler–Bernoulli plate operator *

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Abstract

We present some a–priori estimates of Carleman type for the Euler–Bernoulli plate operator. As an application, we consider a problem of boundary observability for the Euler–Bernoulli plate coupled with the heat equation.

1 Introduction

In [1] Tataru and the author proved some estimates with singular weights for the heat and for the wave equations. These estimates are a powerful tool for the study of observability (and controllability) of pde's. The present paper is concerned with a problem of boundary observability for the following system

$$\begin{aligned} w_{tt} + \Delta^2 w = \alpha\Delta\theta \quad &\text{in} \quad]0,T[\times\Omega \\ \theta_t - \Delta\theta = \beta\Delta w \quad &\text{in} \quad]0,T[\times\Omega \\ w = \partial_\nu w = \Delta w = \theta = 0 \quad &\text{on} \quad [0,T]\times\partial\Omega \end{aligned} \tag{1}$$

here $\alpha, \beta \in \mathbb{R}$, Ω is an open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, T is a positive number and by $\nu = (\nu_0, \dots, \nu_n)$ we denote the unit outer normal vector to $[0,T]\times\partial\Omega$.

We are concerned with the following question. Knowing that $\partial_\nu\Delta w$ and $\partial_\nu\theta$ are equal to 0 on $[0,T]\times\partial\Omega$ can we conclude that w and θ are 0 in $[0,T]\times\Omega$? More precisely, for any solution (w,θ) of problem (1), we want to prove an observability estimate of the form

$$\|(w,\theta)\| \le C\|(\partial_\nu\Delta w, \partial_\nu\theta)\|_\partial$$

where $\|\cdot\|$ and $\|\cdot\|_\partial$ are suitable interior and boundary Sobolev norms. In other words, this would say that the solution (w,θ) can be reconstructed in a stable fashion if one observes some derivatives on the boundary. Note that the initial data cannot be recovered stably due to the parabolic regularizing effect.

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However, the final data at time T can be obtained. By duality this implies an exact null controllability result for the adjoint problem.

The affirmative answer to the previous questions is given using a–priori estimates of Carleman type for the heat equation and for the following problem

$$\begin{aligned} w_{tt} + \Delta^2 w = f \quad &\text{in} \quad]0,T[\times\Omega \\ w = \partial_\nu w = \Delta w = 0 \quad &\text{on} \quad [0,T]\times\partial\Omega\,. \end{aligned} \tag{2}$$

Our goal is to derive an estimate of the form

$$\|e^{\tau\phi}w\| \le C(\|\partial_\nu \Delta w\|_\partial + \|e^{\tau\phi}f\|)$$

for solutions of (2).

The Carleman estimates have been introduced in [3], and were intensively studied in [5] (for hyperbolic and elliptic operators) and in [7] (in the case of anisotropic operators). A general approach to these estimates for boundary value problems was developed in [11, 13]. Carleman estimates for the initial boundary value problem for the heat equations have been independently obtained in [4, 6] and [14] while the analogous results in the case of hyperbolic equations were proved in [12]. We remark that other approaches to the observability problem for hyperbolic equations have been developed in [8] (using multipliers method) and in [2] (using microlocal analysis). For what concerns observability estimates for the heat equations, an observability result was proved in [9] using microlocal analysis and Carleman estimates for elliptic equations. A similar result was proved in [1] in the case of the wave operator instead of the plate operator. The additional difficulty here is due to the higher order of the operators we deal with. We also remark that since both the heat and the plate operators are anisotropic (the weights of the time and space derivatives are 2 and 1 respectively), no lower bound on the observation time, T, is required. Finally, for simplicity we will limit our computations to the constant coefficients case, but similar results hold true also in the case of operators of the same type with C^1 principal parts and L^∞ lower order terms.

We start recalling a Carleman estimate with singular weight for the heat equation. Next, we deduce a similar estimate for the plate operator (decoupling such an operator as the product of two Schrödinger ones). Finally, putting together the previous estimates we get an observability estimate for the coupled system, which, in particular, yields an affirmative answer to our question. We point out that the last step will require a precise control on the constant that appear on the Carleman estimates.

2 Notation and Preliminaries

Denote by (t,x) (or $(x_0,x_1,\ldots,x_n)$) the coordinates in $[0,T]\times\Omega$. We call t the "time" variable, while the other n coordinates are called the "space" variables. By $\langle\cdot,\cdot\rangle$ and $\Re\langle\cdot,\cdot\rangle$ we denote the L^2–scalar product and its real part respectively.

We will use the following shorter notation

$$D_j = \frac{1}{i}\frac{\partial}{\partial x^j}, \quad u_{i_1 \dots i_k} = \partial_{i_1} \dots \partial_{i_k} u.$$

The symbol u_t indicates the derivative of u with respect to t, ∇ represents the gradient with respect to the space variables while $D = i^{-1}\nabla$ is its selfadjoint version and $\nabla^2 u$ is the Hessian matrix of u (w.r.t. the space variables). Given two operators, A and B, as usual we define their commutator as

$$[A, B] := AB - BA.$$

By H^s we denote the classical Sobolev spaces, with norm $\|\cdot\|_s$ while $\|\cdot\|$ stands for the L^2 norm. In the Carleman estimates we use weighted Sobolev norms. Set

$$\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n) = (\alpha_0, \alpha').$$

Given a nonnegative function η define the following anisotropic norm

$$\|u\|^2_{k,\eta} = \sum_{|\alpha|_a \le k} \int \eta^{2(k-|\alpha|_a)} |\partial^\alpha u|^2 \, dt \, dx$$

where $\partial^\alpha = \partial_0^{\alpha_0} \dots \partial_n^{\alpha_n}$ and $|\alpha|_a = 2\alpha_0 + \alpha_1 + \dots + \alpha_n$.

We conclude this section recalling a Carleman estimate for the heat equation. Consider the parabolic initial boundary value problem

$$\begin{gathered} \theta_t(t,x) - \Delta\theta(t,x) = f(t,x) \quad \text{in} \quad]0,T[\times\Omega\,, \\ \theta = 0 \quad \text{on} \quad [0,T]\times\partial\Omega\,, \\ \theta(0,x) = \theta_0(x) \quad \text{in} \quad \Omega\,, \end{gathered} \tag{3}$$

with $(f, \theta_0) \in L^2([0,T]\times\Omega)\times H^1_0(\Omega)$. A Carleman estimate for the solution of the above problem looks like

$$\|e^{\tau\phi}\theta\| \le \|e^{\tau\phi}\partial_\nu\theta\|_\partial + \|e^{\tau\phi} f\|\,,$$

with appropriate norms and a suitable function ϕ, uniformly with respect to the large parameter τ. The obstruction to such an estimate is that no Sobolev norm of the initial data can be controlled by the right hand side. Hence the only hope is to consider a weight function ϕ which approaches $-\infty$ at time 0. Estimates of this type have been proved in ([1, 4, 6] and [14]). Thus, we introduce a function g defined as

$$g(t) = \frac{1}{t(T-t)} \quad (t \in]0,T[)\,. \tag{4}$$

Notice that

$$\Big|\Big(\frac{d}{dt}\Big)^k g(t)\Big| \le C_k g^k(t) \qquad \forall t \in]0,T[\tag{5}$$

for a suitable positive constant C_k. Let $\psi(x)$ be a function such that

$$\nabla\psi(x) \ne 0 \quad \text{for all } x \in \Omega\,^1\,. \tag{6}$$

[1]This is a pseudoconvexity condition with respect to the heat operator.

Define

$$\phi(t,x) := g(t)\big(e^{\lambda\psi(x)} - 2e^{\lambda\Phi}\big), \qquad \Phi = \|\psi\|_{L^\infty(\Omega)}. \tag{7}$$

The weight function ϕ thus defined approaches $-\infty$ at times $0, T$. The additional parameter λ is essential in order to obtain the control of the constants which enables us to handle arbitrarily large coefficients in the coupling terms.

Theorem 2.1 *Let ϕ be given as in (7) with ψ satisfying (6). Let θ be the solution of problem (3). Then there exists λ_0 so that for each $\lambda > \lambda_0$ there exists τ_λ so that for $\tau > \tau_\lambda$ the following estimate holds uniformly in λ, τ:*

$$C_1\lambda\|\tilde\tau^{-\frac{1}{2}}e^{\tau\phi}\theta\|^2_{2,\tilde\tau} \le \|e^{\tau\phi}f\|^2 + \int_{[0,T]\times\partial\Omega} \tilde\tau e^{2\tau\phi}\partial_\nu\psi|\partial_\nu\theta|^2\,d\sigma\,, \tag{8}$$

here $\tilde\tau = \lambda\tau g e^{\lambda\psi}$ and C_1 is a suitable positive constant.

For the reader convenience we give the proof of the above result in the appendix, for a proof in a more general setting see [1].

3 The Plate Operator

Let us consider the following problem

$$\begin{gathered} w_{tt} + \Delta^2 w = f \quad \text{in} \quad]0,T[\times\Omega \\ w = \partial_\nu w = \Delta w = 0 \quad \text{on} \quad [0,T]\times\partial\Omega \\ (w(0,x), w_t(0,x)) \in (H^4(\Omega)\cap H^2_0(\Omega))\times H^2(\Omega)\,, \end{gathered} \tag{9}$$

with $f \in L^2([0,T]\times\Omega)$. Our goal is to prove a Carleman estimate for a solution w of the above problem. The idea of the proof is the following. First, we decompose the plate operator as the product of two Schrödinger ones. Then, we get some Carleman estimates for such operators (see Theorem 3.1 and Theorem 3.3 below). Finally, the result follows putting together these estimates (see Theorem 3.4). It is clear that in order to prove a Carleman estimate for the Schrödinger operator we need to assume a suitable condition on the weight function. More precisely, we suppose a positive constant γ exists so that, for any $x \in \Omega$,

$$\nabla^2\psi(x)\xi\cdot\xi \ge \gamma|\xi|^2 \qquad \forall\xi\in\mathbb{R}^n\,{}^2\,. \tag{10}$$

Lemma 3.1 *Let $u \in C(]0,T[;H^2(\Omega))\cap C^1(]0,T[;L^2(\Omega))$ be a solution of the (ill posed) problem*

$$\begin{gathered} u_t + i\Delta u = f \quad \textit{in} \quad]0,T[\times\Omega \\ u = 0 \quad \textit{on} \quad [0,T]\times\partial\Omega\,. \end{gathered} \tag{11}$$

[2] This is a strong pseudoconvexity condition for the constant coefficients Schrödinger equation.

with $f \in L^2(]0,T[\times\Omega)$ and let ϕ be given as in (7) with ψ satisfying (6) and (10). Then there exists λ_0 so that for each $\lambda > \lambda_0$ there exists τ_λ so that for $\tau > \tau_\lambda$ the following estimate holds uniformly in λ, τ:

$$C_2\|e^{\tau\phi}\tilde{\tau}^{\frac{1}{2}}u\|^2_{1,\tilde{\tau}} \leq \int_{[0,T]\times\partial\Omega} \tilde{\tau}e^{2\tau\phi}\partial_\nu\psi|\partial_\nu u|^2 d\sigma + \|e^{\tau\phi}f\|^2 , \tag{12}$$

here $\tilde{\tau} = \lambda\tau g e^{\lambda\psi}$ and C_2 is a suitable positive constant.

Proof. Let us compute the conjugated operator to $P = \partial_t + i\Delta$. We have that

$$P_\tau = e^{\tau\phi}Pe^{-\tau\phi} = iD_t - \tau\phi_t - iD^2 + i\tau^2|\nabla\phi|^2 + \tau(D\cdot\nabla\phi + \nabla\phi\cdot D) .$$

Clearly, we can split the operator P_τ into its symmetric and antisymmetric parts as follows

$$P_\tau = P^s_\tau + P^a_\tau$$

with

$$P^s_\tau = -\tau\phi_t + \tau(D\cdot\nabla\phi + \nabla\phi\cdot D)$$

and

$$P^a_\tau = iD_t - iD^2 + i\tau^2|\nabla\phi|^2 .$$

Set $v = e^{\tau\phi}u$ and observe that

$$v(0,x) = v(T,x) = 0 \qquad \forall x \in \Omega \tag{13}$$

and

$$v(t,x) = 0 \qquad \forall (t,x) \in [0,T]\times\partial\Omega. \tag{14}$$

So, estimate (12) reduces to

$$C_2\|\tilde{\tau}^{\frac{1}{2}}v\|^2_{1,\tilde{\tau}} \leq 2\int_{[0,T]\times\partial\Omega} \tilde{\tau}\partial_\nu\psi|\partial_\nu v|^2\, d\sigma + \|e^{\tau\phi}f\|^2 . \tag{15}$$

Clearly,

$$\|P_\tau v\|^2 = \|P^s_\tau v\|^2 + \|P^a_\tau v\|^2 + 2\Re\langle P^s_\tau v, P^a_\tau v\rangle \tag{16}$$

and

$$2\Re\langle P^s_\tau v, P^a_\tau v\rangle = 2\tau\Re\langle(-\phi_t + D\cdot\nabla\phi + \nabla\phi\cdot D)v, i(D_t - D^2 + \tau^2|\nabla\phi|^2)v\rangle.$$

In order to complete the proof we must only to compute explicitly the terms of the above scalar product. It is immediate to see that

$$\Re\langle\phi_t v, i\tau^2|\nabla\phi|^2 v\rangle = 0 . \tag{17}$$

Integrating by parts and using (14), we get

$$\begin{aligned} 2\tau\Re\langle(D\cdot\nabla\phi + \nabla\phi\cdot D)v, i\tau^2|\nabla\phi|^2 v\rangle &= 2\tau^3\langle v, \nabla\phi\cdot\nabla|\nabla\phi|^2 v\rangle \\ &= 4\tau^3\langle\nabla^2\phi\nabla\phi\cdot\nabla\phi\, v, v\rangle. \end{aligned} \tag{18}$$

Moreover, recalling (13), we have that

$$2\tau\Re\langle -\phi_t v, iD_t v\rangle = \tau\Re\langle[-\phi_t, iD_t]v, v\rangle = \tau\langle\phi_{tt}v, v\rangle \text{ (= lower order term)}, \quad (19)$$

while (13) and (14) imply that

$$\begin{aligned} &2\tau\Re\langle(D\cdot\nabla\phi+\nabla\phi\cdot D)v, iD_t v\rangle \\ &= 2\tau\Re(\langle\nabla\phi\cdot Dv, iD_t v\rangle + \langle(D\cdot\nabla D_t\phi + D\cdot\nabla\phi D_t)v, iv\rangle) \qquad (20)\\ &= 2\tau\Re(\langle\nabla\phi\cdot Dv, iD_t v\rangle + \langle D\cdot\nabla D_t\phi v, iv\rangle + \langle D_t v, i\nabla\phi\cdot Dv\rangle)\\ &= -2\tau\Re\langle\nabla\phi_t\cdot Dv, v\rangle \text{ (= lower order term)}. \end{aligned}$$

(Here and in the sequel, lower order terms means terms which can be absorbed in the LHS of (15) taking τ large enough.)

Finally, integrating by parts once and recalling (14), we get

$$2\tau\Re\langle -\phi_t v, -iD^2 v\rangle = -2\tau\Re\langle\nabla\phi_t\cdot Dv, v\rangle \text{ (= lower order term)}. \quad (21)$$

Now, it remains to compute the higher order terms. Using once more integrations by parts and (14) we obtain that

$$\begin{aligned} &-2\tau\Re\langle(D\cdot\nabla\phi+\nabla\phi\cdot D)v, iD^2v\rangle\\ &= -2\tau\Re\sum_{k=1}^{n}\langle D_k((D\cdot\nabla\phi+\nabla\phi\cdot D)v), iD_k v\rangle - 4\tau\int_{[0,T]\times\partial\Omega}\partial_\nu\phi|\partial_\nu v|^2\,d\sigma\\ &= -2\tau\Re\sum_{k=1}^{n}\langle(D\cdot\nabla(D_k\phi)+\nabla(D_k\phi)\cdot D)v), iD_k v\rangle\\ &\quad -2\tau\int_{[0,T]\times\partial\Omega}\partial_\nu\phi|\partial_\nu v|^2\,d\sigma \qquad (22)\\ &= 2\tau\Re\sum_{k=1}^{n}\langle(D\cdot\nabla\phi_k+\nabla\phi_k\cdot D)v), D_k v\rangle - 2\tau\int_{[0,T]\times\partial\Omega}\partial_\nu\phi|\partial_\nu v|^2\,d\sigma\\ &= 2\tau\Re\Big(\int_{]0,T[\times\Omega}2\nabla^2\phi Dv\cdot\overline{Dv} + \frac{1}{i}\,v\nabla(\Delta\phi)\cdot\overline{Dv}\,dt\,dx\Big)\\ &\quad -2\tau\int_{[0,T]\times\partial\Omega}\partial_\nu\phi|\partial_\nu v|^2\,d\sigma \end{aligned}$$

Substituting (17)–(22) in (16), we get

$$\begin{aligned} &\|P_\tau v\|^2\\ &\ge 4\tau^3\langle\nabla^2\phi\nabla\phi\cdot\nabla\phi\, v, v\rangle + \tau\langle\phi_{tt}v, v\rangle - 4\tau\Re\langle\nabla\phi_t\cdot Dv, v\rangle\\ &\quad +2\tau\sum_{j=1}^{n}\Re\langle\Delta(D_j\phi)D_j\, v, v\rangle - 2\tau\int_{[0,T]\times\partial\Omega}\partial_\nu\phi|\partial_\nu v|^2\,d\sigma + 4\tau\sum_{j,k=1}^{n}\langle\phi_{jk}D_k v\,,\,D_j v\rangle\\ &\ge \Re\langle(4\tau^3\nabla^2\phi\nabla\phi\cdot\nabla\phi + \tau\phi_{tt} - (4\tau|\nabla\phi_t|)^2 - (2\tau|\nabla(\Delta\phi)|)^2)v, v\rangle\\ &\quad +\int_{]0,T[\times\Omega}4\tau\nabla^2\phi Dv\cdot\overline{Dv} - 2|Dv|^2\,dt\,dx - 2\tau\int_{[0,T]\times\partial\Omega}\partial_\nu\phi|\partial_\nu v|^2\,d\sigma. \end{aligned}$$

Moreover, it is not difficult to verify that

$$4\tau^3\nabla^2\phi\nabla\phi\cdot\nabla\phi+\tau\phi_{tt}-(4\tau|\nabla\phi_t|)^2-(2\tau|\nabla(\Delta\phi)|)^2\approx\lambda\tilde{\tau}^3$$

and

$$4\tau\nabla^2\phi-2\mathrm{I}\approx\tilde{\tau}\mathrm{I}$$

provided that λ and τ are sufficiently large. So,

$$\|P_\tau v\|^2+2\int_{[0,T]\times\partial\Omega}\tilde{\tau}\partial_\nu\psi|\partial_\nu v|^2\,d\sigma\geq C_2\|\tilde{\tau}^{\frac12}v\|^2_{1,\tilde{\tau}},$$

provided that λ and τ are sufficiently large. ♢

Theorem 3.2 *Let u, ϕ and ψ be as in Lemma 3.1. Then there exists λ_0 so that for each $\lambda>\lambda_0$ there exists τ_λ so that for $\tau>\tau_\lambda$ the following estimate holds uniformly in λ,τ:*

$$C_3\|e^{\tau\phi}u\|^2_{1,\tilde{\tau}}\leq\int_{[0,T]\times\partial\Omega}e^{2\tau\phi}\partial_\nu\psi|\partial_\nu u|^2\,d\sigma+\|e^{\tau\phi}\tilde{\tau}^{-\frac12}f\|^2\,,\tag{23}$$

here $\tilde{\tau}=\lambda\tau g e^{\tau\psi}$ and C_3 is a suitable positive constant.

Proof. Set $P=\partial_t+i\Delta$. Applying Lemma 3.1 to $\tilde{\tau}^{-\frac12}u$ we obtain

$$C_2\|e^{\tau\phi}u\|^2_{1,\tilde{\tau}}\leq\int_{[0,T]\times\partial\Omega}e^{2\tau\phi}\partial_\nu\psi|\partial_\nu u|^2d\sigma+\|e^{\tau\phi}([P,\tilde{\tau}^{-\frac12}]+\tilde{\tau}^{-\frac12}P)u\|^2\,,$$

for large enough τ the commutator is small compared to the LHS therefore we get

$$C_3\|e^{\tau\phi}u\|^2_{1,\tilde{\tau}}\leq\int_{[0,T]\times\partial\Omega}e^{2\tau\phi}\partial_\nu\psi|\partial_\nu u|^2d\sigma+\|e^{\tau\phi}\tilde{\tau}^{-\frac12}f\|^2\,,$$

and the conclusion follows. ♢

The next result follows arguing as in Lemma 3.1 and in Theorem 3.2.

Theorem 3.3 *Let $u\in C(]0,T[;H^4(\Omega))\cap C^1(]0,T[;H^2(\Omega))\cap C^2(]0,T[\times L^2(\Omega))$ be a solution of the (ill posed) problem*

$$\begin{aligned}u_t-i\Delta u=f\quad &in\quad]0,T[\times\Omega\\ u=\partial_\nu u=0\quad &on\quad [0,T]\times\partial\Omega\end{aligned}\tag{24}$$

and let ϕ be given as in (7) with ψ satisfying (6) and (10). Then there exists λ_0 so that for each $\lambda>\lambda_0$ there exists τ_λ so that for $\tau>\tau_\lambda$ the following estimate holds uniformly in λ,τ:

$$C_4\|e^{\tau\phi}u\|^2_{1,\tilde{\tau}}\leq\|e^{\tau\phi}\tilde{\tau}^{-\frac12}f\|^2\,,\tag{25}$$

here $\tilde{\tau}=\lambda\tau g e^{\tau\psi}$ and C_4 is a suitable positive constant.

Now, the main result of this section can be easily proved.

Theorem 3.4 *Let w be the solution of the problem* (9) *and let ϕ be given as in* (7) *with ψ satisfying* (6) *and* (10). *Then there exists λ_0 so that for each $\lambda > \lambda_0$ there exists $\tau_\lambda > 0$ so that for $\tau > \tau_\lambda$ the following estimate holds uniformly in λ, τ:*

$$C\|e^{\tau\phi}w\|^2_{2,\tilde\tau} \le \int_{[0,T]\times\partial\Omega} e^{2\tau\phi}\partial_\nu\psi|\partial_\nu\Delta w|^2\,d\sigma + \|e^{\tau\phi}\tilde\tau^{-\frac12}f\|^2\,, \tag{26}$$

here $\tilde\tau = \lambda\tau g e^{\lambda\psi}$ and C is a suitable positive constant.

Proof – Clearly, problem (9) can be recast as follows

$$\begin{gathered} w_t - i\Delta w = u \quad \text{in} \quad]0,T[\times\Omega \\ w = \partial_\nu w = 0 \quad \text{on} \quad [0,T]\times\partial\Omega \end{gathered} \tag{27}$$

and

$$\begin{gathered} u_t + i\Delta u = f \quad \text{in} \quad]0,T[\times\Omega \\ u = 0 \quad \text{on} \quad [0,T]\times\partial\Omega\,. \end{gathered} \tag{28}$$

Now, applying Theorem 3.3 to the solution of problem (27) we get

$$C_4\|e^{\tau\phi}w\|^2_{1,\tilde\tau} \le \|e^{\tau\phi}\tilde\tau^{-\frac12}u\|^2 \tag{29}$$

whilst using Theorem 3.2, we deduce that the solution of (28) satisfies

$$C_3\|e^{\tau\phi}u\|^2_{1,\tilde\tau} \le \int_{[0,T]\times\partial\Omega} e^{2\tau\phi}\partial_\nu\psi|\partial_\nu\Delta w|^2\,d\sigma + \|e^{\tau\phi}\tilde\tau^{-\frac12}f\|^2 \tag{30}$$

for sufficiently large λ and τ. Now, the conclusion follows putting together (29) and (30). ◊

Remark 3.5 We note that a stronger estimate could be proved. In fact, the interior (microlocal) Carleman estimate for the plate operator shows that one can replace the H^2 (anisotropic) norm in the LHS of (26) with the H^3 (anisotropic) norm (see e.g. [12]).

4 The observability estimate

In this section we prove the observability result for the system (1).

Theorem 4.1 *Suppose that (w,θ) is a solution of the system* (1) *and let ϕ be given as in* (7) *with ψ satisfying* (6) *and* (10). *Then there exists λ_0 so that for each $\lambda > \lambda_0$ there exists $\tau_\lambda > 0$ so that for $\tau > \tau_\lambda$ the following estimate holds uniformly in λ, τ*

$$\begin{aligned} \|e^{\tau\phi}w\|^2_{2,\tilde\tau} + \|e^{\tau\phi}\tilde\tau^{-\frac12}\theta\|^2_{2,\tilde\tau} \quad \le \quad & C\Big(\int_{[0,T]\times\partial\Omega} e^{2\tau\phi}\partial_\nu\psi|\partial_\nu\Delta w|^2\,d\sigma \\ & + \int_{[0,T]\times\partial\Omega} \tilde\tau e^{2\tau\phi}\partial_\nu\psi|\partial_\nu\theta|^2\,d\sigma\Big)\,, \end{aligned} \tag{31}$$

for a suitable positive constant C.

Remark 4.2 This estimate shows that solutions to (1) can be reconstructed in a stable manner if one observes their normal derivative on the boundary. As one can see from the above estimate, this observation needs not be on the entire boundary; it suffices to observe (w, θ) in the region Γ, given by

$$\Gamma = [0, T] \times \{x \in \partial\Omega;\ \partial_\nu \psi(x) > 0\}\,.$$

Proof of Theorem 4.1: Applying Theorem 3.4 to w we deduce that

$$C\|e^{\tau\phi} w\|^2_{2,\tilde\tau} \le \int_{[0,T]\times\partial\Omega} e^{2\tau\phi}\partial_\nu\psi|\partial_\nu\Delta w|^2\,d\sigma + \|e^{\tau\phi}\tilde\tau^{-\frac12}\alpha\Delta\theta\|^2. \tag{32}$$

On the other hand, Theorem 2.1 implies that

$$\lambda C_1\|e^{\tau\phi}\tilde\tau^{-\frac12}\theta\|^2_{2,\tilde\tau} \le \|e^{\tau\phi}\beta\Delta w\|^2 + \int_{[0,T]\times\partial\Omega} \tilde\tau e^{2\tau\phi}\partial_\nu\psi|\partial_\nu\theta|^2\,d\sigma\,. \tag{33}$$

Then, the conclusion follows by adding (32) and (33) provided that λ is sufficiently large. ♢

Combining the straightforward energy estimates[3] with (31) we obtain the following consequence:

Theorem 4.3 *For all solutions* (w, θ) *of the system* (1) *we have*

$$\|\nabla\theta(T)\|^2_{L^2(\Omega)} + \|w_t(T)\|^2_{L^2(\Omega)} + \|\Delta w(T)\|^2_{L^2(\Omega)} \le C(\|\partial_\nu\Delta w\|^2_{L^2(\Gamma)} + \|\partial_\nu\theta\|^2_{L^2(\Gamma)})$$

for a suitable positive constant C*.*

The next example shows in the case of Ω a ball of $\mathbb{R}^n$ how one can choose the function ϕ.

Example 4.4 Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ and fix $\overline{x} \in \mathbb{R}^n \setminus \Omega$. Then, define

$$\psi(x) = |x - \overline{x}|^2$$

and

$$\phi(t, x) = \frac{1}{t(T-t)}(e^{\lambda\psi(x)} - 2e^{\lambda\sup_{y\in\Omega}\psi(y)}).$$

It is immediate to verify that ϕ satisfies all the assumption of Theorem 4.1. Moreover, if (w, θ) is a solution of equation (1) and $\partial_\nu\Delta w$ and $\partial_\nu\theta$ are equal to 0 on

$$\Gamma = \{(t, x) \in [0, T] \times \partial\Omega : x \cdot (x - \overline{x}) > 0\},$$

then w and θ are equal to 0 in $]0, T[\times\Omega$.

[3] Here the energy is defined by $E(t) = 2^{-1}\int_\Omega |\nabla\theta(t, x)|^2 + w_t^2(t, x) + |\Delta w(t, x)|^2\,dx$.

A Proof of Theorem 2.1

Set

$$Q(D) = \partial_t - \Delta.$$

As usual, with the substitution $v = e^{\tau\phi}\theta$, the estimate (8) reduces to

$$C_1\lambda\|\tilde{\tau}^{-\frac{1}{2}}v\|^2_{2,\tilde{\tau}} \le \|Q_\tau v\|^2 + \int_{[0,T]\times\partial\Omega} \tilde{\tau}\partial_\nu\psi|\partial_\nu v|^2\, d\sigma\,, \tag{34}$$

where Q_τ is the conjugated operator defined as

$$Q_\tau(t,x,D) := e^{\tau\phi}Q(D)e^{-\tau\phi}.$$

Notice that

$$v(0,x) = v(T,x) = 0 \qquad \forall x \in \Omega$$

and

$$v(t,x) = 0 \qquad \forall (t,x) \in [0,T]\times\partial\Omega.$$

Split Q_τ into

$$Q_\tau(t,x,D) = Q^a_\tau(t,x,D) + Q^s_\tau(t,x,D)$$

where

$$Q^a_\tau(t,x,D) = \partial_t + \tau\left(\nabla\cdot\nabla\phi + \nabla\phi\cdot\nabla\right)$$

is the skew-symmetric part of Q_τ while

$$Q^s_\tau(t,x,D)v = -\Delta - \tau^2|\nabla\phi|^2 - \tau\phi_t$$

is the symmetric part of Q_τ. Since

$$\|Q_\tau v\|^2 = \|Q^s_\tau v\|^2 + \|Q^a_\tau v\|^2 + 2\langle Q^s_\tau v, Q^a_\tau v\rangle\,, \tag{35}$$

the crucial step of the proof will be to estimate from below

$$2\langle Q^s_\tau v, Q^a_\tau v\rangle = -2\langle(\Delta + \tau^2|\nabla\phi|^2 + \tau\phi_t)v, (\partial_t + \tau(\nabla\cdot\nabla\phi + \nabla\phi\cdot\nabla))v\rangle$$

integrating by parts. It is easy to see that $\langle\partial_t v\,,\, Q^s_\tau v\rangle$ is a lower order term compared to the left hand side in (34) since

$$\begin{aligned} 2\langle\partial_t v\,,\, Q^s_\tau v\rangle &= \langle[Q^s_\tau, \partial_t]v, v\rangle \\ &= +\langle(\partial_t(\tau^2|\nabla\phi|^2 + \tau\phi_t)v, v\rangle \end{aligned}$$

therefore

$$|\langle\partial_t v\,,\, Q^s_\tau v\rangle| \le c_\lambda\tau^2\|g^{3/2}v\|^2\,,$$

for a suitable positive constant c_λ. Similarly, for some $c_\lambda > 0$,

$$|\langle\,\tau(\nabla\cdot\nabla\phi + \nabla\phi\cdot\nabla)v, \tau\phi_t v\rangle| \le c_\lambda\tau^2\|g^{3/2}v\|^2\,.$$

Then it remains to estimate

$$-2\tau\,\langle(\nabla\cdot\nabla\phi+\nabla\phi\cdot\nabla\,)v,\,(\Delta+\tau^2|\nabla\phi|^2)v\rangle\,.$$

Compute first the leading zero order terms. Integrating by parts, we get

$$-2\tau\,\langle(\nabla\cdot\nabla\phi+\nabla\phi\cdot\nabla)v\,,\,\tau^2|\nabla\phi|^2v\rangle=2\tau^3\,\langle v,\,\nabla\phi\cdot\nabla|\nabla\phi|^2)v\rangle\,.$$

Since $\nabla\phi=\lambda g e^{\lambda\psi}\nabla\psi$, the highest order terms occur when the derivative falls on the exponential. Hence we get

$$-2\tau\,\langle(\nabla\cdot\nabla\phi+\nabla\phi\cdot\nabla)v\,,\,\tau^2|\nabla\phi|^2v\rangle=4\lambda\langle\tilde\tau^3|\nabla\psi|^4v\,,\,v\rangle+R \tag{36}$$

where R stands for lower order terms,

$$R=O(\|\tilde\tau^{-1/2}v\|^2_{2,\tilde\tau}).$$

Now compute the leading first order terms integrating by parts the expression

$$-2\tau\,\langle(\nabla\cdot\nabla\phi+\nabla\phi\cdot\nabla)v\,,\,\Delta v\rangle\,.$$

Since one operator is symmetric and the other is skew-symmetric, we need to compute their commutator. In fact, we have that

$$\begin{aligned}&-2\tau\,\langle\Delta v,\,(\nabla\cdot\nabla\phi+\nabla\phi\cdot\nabla)v\rangle\\&\quad=\quad\tau\,\langle(\nabla\cdot\nabla\phi+\nabla\phi\cdot\nabla)\Delta v,\,v\rangle-\tau\,\langle\Delta v,\,(\nabla\cdot\nabla\phi+\nabla\phi\cdot\nabla)v\rangle.\end{aligned}$$

Hence, using the equality

$$\begin{aligned}\tau\,\langle\Delta v,\,(\nabla\cdot\nabla\phi+\nabla\phi\cdot\nabla)v\rangle\quad&=\quad-\tau\int_{]0,T[\times\Omega}\nabla v\cdot\nabla((\nabla\cdot\nabla\phi+\nabla\phi\cdot\nabla)v)\,dt\,dx\\&\qquad+2\tau\int_{[0,T]\times\partial\Omega}\partial_\nu\phi|\partial_\nu v|^2\,d\sigma\end{aligned}$$

we deduce that

$$\begin{aligned}&-2\tau\,\langle\Delta v,\,(\nabla\cdot\nabla\phi+\nabla\phi\cdot\nabla)v\rangle\\&\quad=\quad-\tau\langle[\Delta,(\nabla\cdot\nabla\phi+\nabla\phi\cdot\nabla)]v,v\rangle-2\tau\int_{[0,T]\times\partial\Omega}\partial_\nu\phi|\partial_\nu v|^2\,d\sigma.\end{aligned}$$

Then,

$$\begin{aligned}\tau\langle[\Delta,(\nabla\cdot\nabla\phi+\nabla\phi\cdot\nabla)]v,v\rangle&=2\tau\sum_{k=1}^n\langle(\nabla\cdot\nabla\phi_k+\nabla\phi_k\cdot\nabla)v_k,v\rangle\\&=\quad-2\tau\int_{]0,T[\times\Omega}\nabla^2\phi\nabla v\cdot\nabla v+\sum_{k=1}^n v_k\nabla\cdot(\nabla\phi_k v)\,dt\,dx\\&=\quad-4\tau\int_{]0,T[\times\Omega}\nabla^2\phi\nabla v\cdot\nabla v\,dt\,dx+\tau\langle(\Delta^2\phi)v,v\rangle\end{aligned}$$

and

$$-2\tau\,\langle \Delta v,\,(\nabla\cdot\nabla\phi+\nabla\phi\cdot\nabla)v\rangle$$
$$= 4\int_{]0,T[\times\Omega}\tilde\tau\lambda|\nabla\psi\cdot\nabla v|^2 dt\,dx + R - 2\int_{[0,T]\times\partial\Omega}\tilde\tau\partial_\nu\psi|\partial_\nu v|^2\,d\sigma. \quad (37)$$

Hence, if we put together (36) and (37) then we get

$$4\lambda\int_{]0,T[\times\Omega}\tilde\tau^3|v|^2+\tilde\tau|\nabla\psi\cdot\nabla v|^2\,dt\,dx \le 2\langle Q^s_\tau v, Q^a_\tau v\rangle + 2\int_{[0,T]\times\partial\Omega}\tilde\tau\partial_\nu\psi|\partial_\nu\theta|^2\,d\sigma + R\,.$$

Substituting this in (35) we obtain

$$4\lambda\int_{]0,T[\times\Omega}\tilde\tau^3|v|^2+\tilde\tau|\nabla\psi\cdot\nabla v|^2\,dt\,dx + \|Q^s_\tau v\|^2+\|Q^a_\tau v\|^2 \le \|Q_\tau v\|^2 + 2\int_{[0,T]\times\partial\Omega}\tilde\tau\partial_\nu\psi|\partial_\nu\theta|^2\,d\sigma + R\,.$$

In the first term on the left we already control the appropriate weighted L^2 norm of v. The corresponding norm of $\partial_t v$ is easily obtained from the second and the fourth term, while the weighted L^2 norms of ∇v, respectively $\nabla^2 v$ are obtained in an elliptic fashion from the first and the third term to obtain

$$C\lambda\|\tilde\tau^{-\frac12}v\|^2_{2,\tilde\tau} \le \|Q_\tau v\|^2 + 2\int_{[0,T]\times\partial\Omega}\tilde\tau\partial_\nu\psi|\partial_\nu\theta|^2\,d\sigma + R\,.$$

Now the lower order terms in R are negligible (i.e. much smaller than the left hand side) for sufficiently large λ, τ and we obtain (34). ♢

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