Electronic Journal of Differential Equations, Vol. **2000**(2000), No. 72, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

NONCLASSICAL RIEMANN SOLVERS AND KINETIC RELATIONS III: A NONCONVEX HYPERBOLIC MODEL FOR VAN DER WAALS FLUIDS

Philippe G. LeFloch & Mai Duc Thanh

Abstract. This paper deals with the so-called p-system describing the dynamics of isothermal and compressible fluids. The constitutive equation is assumed to have the typical convexity/concavity properties of the van der Waals equation. We search for discontinuous solutions constrained by the associated mathematical entropy inequality. First, following a strategy proposed by Abeyaratne and Knowles and by Hayes and LeFloch, we describe here the whole family of *nonclassical Riemann solutions* for this model. Second, we supplement the set of equations with a *kinetic relation* for the propagation of nonclassical undercompressive shocks, and we arrive at a uniquely defined solution of the Riemann problem. We also prove that the solutions depend L^1-continuously upon their data. The main novelty of the present paper is the presence of *two inflection points* in the constitutive equation. The Riemann solver constructed here is relevant for fluids in which viscosity and capillarity effects are kept in balance.

1. Introduction

We consider the Riemann problem for a compressible and isothermal fluid described by the following two conservation laws of mass and momentum:

$$\begin{aligned}\partial_t u + \partial_x p(v) &= 0,\\ \partial_t v - \partial_x u &= 0.\end{aligned} \tag{1.1}$$

Here $v > 0$ and u denote the specific volume and the velocity of the fluid, respectively, while the pressure $p = p(v)$ is a given smooth function depending on the fluid under consideration. The initial datum has the form:

$$(u, v)(x, 0) = \begin{cases} (u_l, v_l) & \text{for } x < 0,\\ (u_r, v_r) & \text{for } x > 0,\end{cases} \tag{1.2}$$

where (u_l, v_l) and (u_r, v_r) are constants. In typical models of (liquid-vapor) phase transformation, the pressure p admits two inflection points and tends to $+\infty$ at

2000 Mathematics Subject Classifications: 35L65, 76N10, 76L05.
Key words: compressible fluid dynamics, phase transitions, Van der Waals, entropy inequality, hyperbolic conservation law, kinetic relation, nonclassical solutions, Riemann solver.

Submitted June 7, 2000. Published December 1, 2000.
P.L.F. was supported in part by the Centre National de la Recherche Scientifique.
This work was done while M.D.T. was visiting the Ecole Polytechnique (1999-2000) and was partially supported by the French-Vietnamese Institute "ForMathVietnam".

$v = 0$. That is, for some constants a and b, we have

$$\begin{aligned} &p''(v) > 0 \quad \text{for } v \in (0, a) \cup (b, +\infty), \\ &p''(v) < 0 \quad \text{for } v \in (a, b), \\ &p'(a) < 0, \\ &\lim_{v \to 0} p(v) = +\infty, \quad \lim_{v \to +\infty} p(v) \geq 0. \end{aligned} \tag{1.3}$$

As a consequence, the first derivative p' attains a maximum value at $v = a$ and, since $p'(a) < 0$,

$$p'(v) < 0 \quad \text{for } v > 0.$$

Of course, the case where v is restricted to remain above some threshold v_* is covered also by the theory in this paper, provides one changes v into $v - v_*$.

The system (1.1) under consideration has the general form of a system of conservation laws,

$$\partial_t U + \partial_x F(U) = 0, \qquad U := (u, v), \quad F(U) = \big(p(v), -u\big). \tag{1.4}$$

Since $p' < 0$, the matrix $DF(U)$ admits two real and distinct eigenvalues, depending only on v,

$$\lambda_1(v) := -\sqrt{-p'(v)} < 0 < \sqrt{-p'(v)} := \lambda_2(v).$$

Therefore, (1.1) is strictly hyperbolic. Setting $c(v) := \sqrt{-p'(v)}$, which is called the sound speed, right-eigenvectors of $DF(U)$ may be chosen to be $r_1(v) := (c(v), 1)$ and $r_2(v) := (-c(v), 1)$.

As is customarily, all of the weak solutions of the system (1.1) are required to fullfil the following entropy inequality

$$\begin{aligned} &\partial_t \mathcal{U}(u, v) + \partial_x \mathcal{F}(u, v) \leq 0, \\ &\mathcal{U}(u, v) := \frac{u^2}{2} + \Sigma(v), \quad \mathcal{F}(u, v) = u\, p(v), \\ &\Sigma(v) := -\int_0^v p(w)\, dw, \end{aligned} \tag{1.5}$$

where (U, F) is a mathematical entropy-entropy flux pair for the system of conservation laws (1.1) (Lax [11]). Under the assumption (1.3), the entropy U is strictly convex in the conservative variables (u, v).

The present paper is based on recent work by Abeyaratne and Knowles [1, 2], LeFloch et al. [8–10, 12–14], and Shearer et al. [18, 19] on nonclassical undercompressive shock waves of hyperbolic and hyperbolic-elliptic systems of conservation laws. We also rely on earlier contributions on propagating phase boundaries in van der Waals fluids, especially the pioneering work by Slemrod [20–22] and the papers [3–7].

First of all, in Section 2, we provide a precise description of the set of *all Riemann solutions* consistent with the two conservation laws (1.1) and the entropy inequality (1.5). In Section 3, we recall that the Riemann problem admits a unique (classical) solution characterized by the so-called Liu entropy criterion [17]. This is the solution usually described in the engineering literature. However the solutions generated by zero viscosity-capillarity limits associated with the system (1.1) do not coincide with the (classical) Riemann solution. Therefore, in Section 4, we construct solutions that only satisfy the entropy ionequality (1.5). For the sake of uniqueness, it is known that the so-called *kinetic relation* should be added. Our main result (Theorem 4.3) establishes the existence and uniqueness of the weak solution of the Riemann problem (1.1)-(1.2)-(1.5) satisfying a prescribed kinetic relation. This represents an extension of previous results by the authors [15-16] on nonclassical Riemann solvers and kinetic relation. Comparing with our earlier study [15] of a nonconvex hyperbolic model of elastodynamics, the major novelty is the existence of two inflexion points in the equation of state (1.3), which significantly complicates the analysis of the Riemann problem.

2. Entropy Dissipation Function

We are going to investigate the properties of the entropy dissipation function associated with the entropy inequality (1.5). First, we need to point out basic properties of the pressure function and introduce some useful notation. Virtually all of the properties stated in this section can be checked *geometrically* from the graph of the function p. In the following we consider points on this graph and refer to them simply by their v-coordinate.

We rely here on the assumptions (1.3) made on the pressure function. In the interval (a, b), the function p is concave, and thus remains above its tangent at the inflection point b. This tangent intersects the graph of p at some other point, outside the interval (a, b), whose coordinate will be denoted by $b^{-\natural} < a$. Similarly the tangent to the curve at the other inflection point a also intersects the graph of p at some point $a^{-\natural} > b$. (This notation will become clear as we will introduce shortly some functions $\varphi^{-\natural}$ and $\psi^{-\natural}$.)

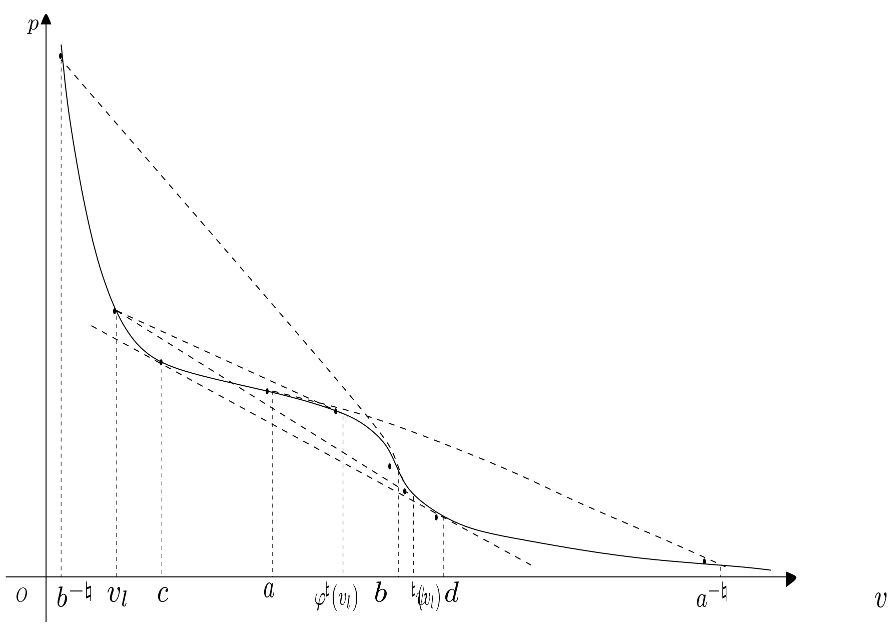

Figure 2.1: Pressure function.

Geometrically on the graph of p, we see that for any $v \in (b^{-\natural}, a^{-\natural})$ there exists exactly two lines which are passing through the point with the coordinate v and are also tangent to the graph. Call these two points $\psi^\natural(v)$ and $\varphi^\natural(v)$ with the convention that $\varphi^\natural(v) < \psi^\natural(v)$. In other words we have

$$\begin{aligned} p'\big(\varphi^\natural(v)\big) &= \frac{p(v) - p\big(\varphi^\natural(v)\big)}{v - \varphi^\natural(v)}, \\ p'\big(\psi^\natural(v)\big) &= \frac{p(v) - p\big(\psi^\natural(v)\big)}{v - \psi^\natural(v)}. \end{aligned} \tag{2.1}$$

The definition extends to the end points of the interval under consideration by setting

$$\varphi^\natural(b^{-\natural}) = \psi^\natural(b^{-\natural}) = b \quad \text{and} \quad \varphi^\natural(a^{-\natural}) = \psi^\natural(a^{-\natural}) = a.$$

No tangent can be draw from a point outside the interval $[b^{-\natural}, a^{-\natural}]$ as the function p "resembles" a convex function in that region. The two tangent functions $\varphi^\natural$ and $\psi^\natural : [b^{-\natural}, a^{-\natural}] \to \mathbb{R}$ are going to play a central role in the forthcoming constructions in Sections 3 and 4.

The following properties are elementary:

Proposition 2.1.

(i) *The values v and $\psi^\natural(v)$ always lie on different sides with respect to b, and the values v and $\varphi^\natural(v)$ always lie on different sides with respect to a, in the sense that:*

$$\begin{aligned} (\varphi^\natural(v) - a)(v - a) < 0 \quad &\text{for } v \neq a, \qquad \varphi^\natural(a) = a, \\ (\psi^\natural(v) - b)(v - b) < 0 \quad &\text{for } v \neq b, \qquad \psi^\natural(b) = b. \end{aligned}$$

(ii) *Considering the convex hull of the epigraph of p, we see that there exist two points c and d such that (Figure 2.1)*

$$b^{-\natural} < c < a < b < d < a^{-\natural}$$

and

$$\psi^\natural(c) = d \quad \text{and} \quad \varphi^\natural(d) = c.$$

(iii) *The function $\psi^\natural$ is increasing for $v \in [b^{-\natural}, c]$ and decreasing for $v \in [c, a^{-\natural}]$. The function $\varphi^\natural$ is decreasing for $v \in [b^{-\natural}, d]$ and increasing for $v \in [d, a^{-\natural}]$. Moreover $\varphi^\natural$ maps $[b^{-\natural}, a^{-\natural}]$ onto $[c, b]$, while $\psi^\natural$ maps $[b^{-\natural}, a^{-\natural}]$ onto $[a, d]$.*

Consider next the graph of p from a somewhat different perspective. We are interested in the intersection points of any tangent line with the graph itself. Observe first that the convex hull of the epigraph of p coincides with the epigraph except in the interval $[c, d]$, defined in Proposition 2.1. Equivalently, the points c and d can be characterized by the conditions $c < a < d < d$ and

$$p'(c) = \frac{p(d) - p(c)}{d - c} = p'(d).$$

We observe that the tangent at any point $v \notin [c, d]$ remains globally below the graph of p. So we focus on values $v \in [c, d]$.

For all $v \in (c, d)$, the tangent at the point with coordinate v intersects the graph of p at exactly two distinct points, say denoted by $\varphi^{-\natural}(v)$ and $\psi^{-\natural}(v)$ with the convention

$$\varphi^{-\natural}(v) < \psi^{-\natural}(v).$$

The functions $\varphi^{-\natural}$ and $\psi^{-\natural}$ are not one-to-one, however one can check geometrically that, by restricting attention to the interval $[a, b]$, their inverses coincide with the functions $\varphi^{\natural}$ and $\psi^{\natural}$ defined above:

$$\varphi^{\natural} \circ \varphi^{-\natural} = \psi^{\natural} \circ \psi^{-\natural} = id \qquad \text{on the interval } [a, b].$$

We are now ready to investigate the sign of the entropy dissipation function associated with shock waves. Consider a shock wave solution of the hyperbolic system (1.1), connecting a left-hand state (u_0, v_0) to a right-hand state (u_1, v_1) and propagating with the speed $s \in \mathbb{R}$. For this shock wave to be a weak solution, the Rankine-Hugoniot relations must hold:

$$s\,(u_1 - u_0) - p(v_1) - p(v_0) = 0, \qquad s\,(v_1 - v_0) + u_1 - u_0 = 0, \tag{2.2}$$

which yield

$$s = \frac{p(v_1) - p(v_0)}{u_1 - u_0} = -\frac{u_1 - u_0}{v_1 - v_0}.$$

Therefore, whenever $(p(v_1) - p(v_0))/(v_1 - v_0) \le 0$, the shock speed

$$s = \mp \overline{c}(v_0, v_1) := \mp \sqrt{-\frac{p(v_1) - p(v_0)}{v_1 - v_0}} \tag{2.3}$$

is well-defined and independent of u_0 and u_1, so we simply set $s = s(v_0, v_1)$. In (2.3), the 1– and 2–shocks correspond to the $\mp$ signs, respectively.

Similarly, the entropy inequality (1.5) holds for the shock wave provided the corresponding entropy dissipation function is negative:

$$\begin{aligned} E(u_0, v_0; u_1, v_1) &:= -s(v_0, v_1)\Big(\frac{u_1^2 - u_0^2}{2} + \Sigma(v_1) - \Sigma(v_0)\Big) + u_1\,p(v_1) - u_0\,p(v_0) \\ &\le 0. \end{aligned} \tag{2.4}$$

An easy calculation based on the Rankine-Hugoniot relations leads to the simpler expression

$$E(v_0, v_1) = -s(v_0, v_1)\Big(\Sigma(v_1) - \Sigma(v_0) + \frac{p(v_1) + p(v_0)}{2}\,(v_1 - v_0)\Big). \tag{2.5}$$

In particular, $E = E(v_0, v_1)$ is independent of u_0 and u_1.

It is not difficult to determine the sign of the function E geometrically. Given some values v_0 and v_1, the straightline connecting the two corresponding points on the graph of p may cut the graph at four points at most, and thus may determine at most three signed areas comprised between the line and the graph, say $A_1(v_0, v_1)$, $A_2(v_0, v_1)$, and $A_3(v_0, v_1)$. By convention, an area is positive when the graph is above the straightline and negative otherwise. The notation can be trivially extended to the situations where only one or two areas are determined by the given line. In view of (2.5) we find that the entropy dissipation is essentially the sum of these three areas:

$$E(v_0, v_1) = -s(v_0, v_1)\Big(A_1(v_0, v_1) + A_2(v_0, v_1) + A_3(v_0, v_1)\Big).$$

In the following we will state various monotonicity properties for the functions associated with zeros of E. Those porperties can be checked immediately from this geometrical interpretation of the entropy dissipation.

For definiteness, from now on, we restrict attention to waves propagating with negative speed. A tedious calculation yields

$$\frac{\partial E(v_0, v_1)}{\partial v_1} = \frac{1}{2}\sqrt{-\frac{v_1 - v_0}{p(v_1) - p(v_0)}}\,\frac{1}{(w - v_0)^2}\,M(v_0, v_1)\,N(v_0, v_1), \tag{2.6}$$

where

$$M(v_0, v_1) := -p(v_1) + p(v_0) + p'(v_1)\,(v_1 - v_0)$$

and

$$N(v_0, v_1) := 2\left(\Sigma(v_0) - \Sigma(v_1)\right) - (3\,p(v_1) - p(v_0))\,(v_1 - v_0).$$

Therefore, the sign of E is given by the signs of M and N, which we now investigate.

The following properties of the function M are immediately obtained geometrically:

(1) If $v_0 \in (0, b^{-\natural}) \cup (a^{-\natural}, +\infty)$, then

$$M(v_0, v_1) > 0 \quad \text{for all } v_1 \neq v_0. \tag{2.7a}$$

(2) If $v_0 \in [b^{-\natural}, a^{-\natural}]$, then

$$\begin{aligned} &M(v_0, v_1) < 0 && \text{if } v_1 \in \big(\varphi^\natural(v_0), \psi^\natural(v_0)\big),\\ &M(v_0, v_1) = 0 && \text{if } v_1 = v_0, \varphi^\natural(v_0) \text{ or } \psi^\natural(v_0),\\ &M(v_0, v_1) > 0 && \text{otherwise.} \end{aligned} \tag{2.7b}$$

On the other hand, the function Σ being convex, N is bounded away from zero. Namely we have

$$\begin{aligned} N(v_0, v_1) &\geq 2p(v_1)(v_1 - v_0) - (3p(v_1) - p(v_0))(v_1 - v_0)\\ &= -(p(v_1) - p(v_0))(v_1 - v_0) > 0, \quad \text{for all } v_1 \neq v_0. \end{aligned} \tag{2.8}$$

We conclude that the functions E and M have the same sign.

If $v_0 \in (0, b^{-\natural}) \cup \big(a^{-\natural}, +\infty\big)$, then, by (2.7a), the entropy dissipation function $E(v_0, .)$ is globally monotone increasing in $v_1 > 0$. If $v_0 \in [b^{-\natural}, a^{-\natural}]$, then, by (2.7b), it is monotone increasing in $\big(0, \varphi^\natural(v_0)\big]$ and in $\big[\psi^\natural(v_0), +\infty\big)$, but is monotone decreasing in $\big[\varphi^\natural(v_0), \psi^\natural(v_0)\big]$. Therefore, in this latter case, the entropy dissipation attains a maximal value $F(v_0) := E(v_0, \varphi^\natural(v_0))$ at $v_1 = \varphi^\natural(v_0)$ and a minimal value $G(v_0) := E(v_0, \psi^\natural(v_0))$ at $v_1 = \psi^\natural(v_0)$. To determine the sign of E, one must know the sign of $F(v_0)$ and $G(v_0)$.

Regarding F and G as functions of $v \in \left[b^{-\natural}, a^{-\natural}\right]$, we obtain

$$\begin{aligned}\frac{dF}{dv}(v) &= -(p(\varphi^\natural(v)) - v)(\varphi^\natural(v) - v) < 0 \quad \text{iff } v \in (c, d),\\ \frac{dG}{dv}(v) &= -(p(\psi^\natural(v)) - v)(\psi^\natural(v) - v) < 0 \quad \text{iff } v \in (c, d).\end{aligned}$$

Thus, both functions F and G are decreasing in each of the intervals $(b^{-\natural}, c)$ and $(d, a^{-\natural})$, and are increasing in the interval (c, d). Moreover we have

$$F(a) = G(b) = 0,$$

which indecate that F and G are both negative at $v = c$ and positive at $v = d$. Also it is not difficult to check that F and G are both positive at $v = b^{-\natural}$ and both negative at $v = a^{-\natural}$. Geometrically, in the interval $(b^{-\natural}, b)$, the graph of p remains below its tangent at $v = b$. In the interval $(a, a^{-\natural})$, the graph remains below its tangent at $v = a$.

For each of the functions F and G, there exist two values denoted by $e < f$ and $e' < f'$, respectively, which satisfy

$$e, e' \in (b^{-\natural}, c), \qquad f, f' \in (d, a^{-\natural})$$

and

$$\begin{aligned}&F(v) < 0 \quad \text{iff } v \in (e', a) \cup (f, a^{-\natural}),\\ &F(e') = F(a) = F(f) = 0,\\ &F(v) > 0 \quad \text{otherwise,}\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}&G(v) < 0 \quad \text{iff } v \in (e, b) \cup (f', a^{-\natural}),\\ &G(e) = G(b) = G(f') = 0,\\ &G(v) > 0 \quad \text{otherwise.}\end{aligned} \tag{2.10}$$

In view of (2.7a)–(2.10), we arrive to the following conclusions:

Theorem 2.2. (Fundamental properties of the entropy dissipation)

For each $v_0 \in (0, b^{-\natural}) \cup (a^{-\natural}, +\infty)$, the entropy dissipation $E(v_0, v_1)$ is a globally monotone decreasing function of $v_1 > 0$. For each $v_0 \in [b^{-\natural}, a^{-\natural}]$, the function $v_1 \mapsto E(v_0, v_1)$ is monotone increasing in the intervals $\left(0, \varphi^\natural(v_0)\right]$ and $\left[\psi^\natural(v_0), +\infty\right)$, but is monotone decreasing in the interval $\left[\varphi^\natural(v_0), \psi^\natural(v_0)\right]$.

More precisely, the entropy inequality (2.4) select the following admissible shock waves:

(i) *If $v_0 \in (0, e] \cup [f, +\infty)$, then the constraint (2.4) is equivalent to*

$$v_1 \leq v_0.$$

(ii) *If $v_0 \in (e, a]$, then we have $E(v_0, \psi^\natural(v_0)) = G(v_0) < 0$ and the entropy dissipation admits three roots. Hence, there exist two values, distinct from v_0 and denoted by $\varphi^\flat_\infty(v_0)$ and $\psi^\flat_\infty(v_0)$, such that*

$$v_0 \leq a \leq \varphi^\natural(v_0) \leq \varphi^\flat_\infty(v_0) < \psi^\natural(v_0) < \psi^\flat_\infty(v_0)$$

and

$$E(v_0, \varphi^\flat_\infty(v_0)) = E(v_0, \psi^\flat_\infty(v_0)) = E(v_0, v_0) = 0.$$

The inequality (2.4) *is equivalent to*

$$v_1 \in (0, v_0] \cup [\varphi^\flat_\infty(v_0), \psi^\flat_\infty(v_0)].$$

(iii) *If* $v_0 \in (a, b)$, *there exist two values, distinct from* v_0 *and denoted by* $\varphi^\flat_\infty(v_0)$ *and* $\psi^\flat_\infty(v_0)$, *such that*

$$\varphi^\flat_\infty(v_0) < \varphi^\natural(v_0) < a < v_0 < b < \psi^\natural(v_0) < \psi^\flat_\infty(v_0)$$

and

$$E(v_0, \varphi^\flat_\infty(v_0)) = E(v_0, \psi^\flat_\infty(v_0)) = E(v_0, v_0) = 0.$$

The inequality (2.4) *is equivalent to*

$$v_1 \in (0, \varphi^\flat_\infty(v_0)] \cup [v_0, \psi^\flat_\infty(v_0)].$$

(iv) *If* $v_0 \in [b, f)$, *then we have* $E(v_0, \varphi^\natural(v_0)) = F(v_0) > 0$. *There exist two values, distinct from* v_0 *and denoted by* $\varphi^\flat_\infty(v_0)$ *and* $\psi^\flat_\infty(v_0)$, *such that*

$$\varphi^\flat_\infty(v_0) < \varphi^\natural(v_0)) < \psi^\flat_\infty(v_0) \le \psi^\natural(v_0) \le b \le v_0$$

and

$$E(v_0, \varphi^\flat_\infty(v_0)) = E(v_0, \psi^\flat_\infty(v_0)) = E(v_0, v_0) = 0.$$

The inequality (2.4) *is equivalent to*

$$v_1 \in (0, \varphi^\flat_\infty(v_0)] \cup [\psi^\flat_\infty(v_0), v_0].$$

The two functions $\varphi^\flat_\infty$ and $\psi^\flat_\infty : [e, f] \to \mathbb{R}$ introduced in Theorem 2.2 play a central role in the construction of the Riemann solutions. Indeed they determine some important boundaries of the set of right-hand states that can be reached by an (admissible) shock wave satisfying the entropy inequality (1.5). Their monotonicity properties are summarized in the following proposition:

Corollary 2.3. *The function* $\varphi^\flat_\infty$ *is monotone decreasing in the interval* $[e, \psi^\natural(e)]$ *with*

$$\varphi^\flat_\infty(\varphi^\flat_\infty(v)) = v, \quad v \in [e, \psi^\natural(e)], \tag{2.12a}$$

and is monotone increasing in the interval $[\psi^\natural(e), f]$ *with*

$$\psi^\flat_\infty(\varphi^\flat_\infty(v)) = v, \quad v \in [\psi^\natural(e), f]. \tag{2.12b}$$

The function $\psi^\flat_\infty$ *is monotone decreasing in the interval* $[\varphi^\natural(f), f]$ *with*

$$\psi^\flat_\infty(\psi^\flat_\infty(v)) = v, \quad v \in [\varphi^\natural(f), f], \tag{2.13a}$$

and is monotone increasing in the interval $[e, \varphi^\natural(f)]$ *with*

$$\varphi^\flat_\infty(\psi^\flat_\infty(v)) = v, \quad v \in [e, \varphi^\natural(f)]. \tag{2.13b}$$

Moreover,

$$\varphi^\flat_\infty(e) = \psi^\flat_\infty(e) = \psi^\natural(e), \qquad \varphi^\flat_\infty(f) = \psi^\flat_\infty(f) = \varphi^\natural(f),$$

and

$$\varphi^\flat_\infty(a) = a, \qquad \psi^\flat_\infty(b) = b.$$

Proof. The last conclusion is an immediate consequence of the values e, f in (2.9) and (2.10). First of all, we claim that

$$\varphi^\flat_\infty(v) \le \psi^\natural(e), v \in [e, f]. \tag{2.14}$$

Actually, the values $v \ge a$ satisfy

$$\varphi^\flat_\infty(v) \le \varphi^\natural(v) \le a \le a < b < \psi^\natural(e).$$

We are left to considering values $v \in (e, a)$. The line between e and $\psi^\natural(e)$ crosses the graph of p at a middle point e^*. Since $\psi^\natural(e)$ lies on the convex part of p, the line connecting any $v \in (e^*, \psi^\natural(\psi^\natural(e))), \psi^\natural(\psi^\natural(e)) > a$ and $\psi^\natural(e)$ cut the graph of p at some middle point v_1 such that

$$p'(v_1) < \frac{p(v) - p(\psi^\natural(e))}{v - \psi^\natural(e)} < p'(\psi^\natural(e)).$$

By a continuity argument, we deduce that there exists a (unique) point $v^* \in (v_1, \psi^\natural(e))$ such that

$$\frac{p(v) - p(v^*)}{v - v^*} = p'(v^*),$$

i.e.,

$$\psi^\natural(e) > v^* = \psi^\natural(v) > \varphi^\flat_\infty(v),$$

satisfying (2.14). If $v \in (e, e^*)$, it is easy to see that the line connecting v and $\psi^\natural(e)$ lies below the line connecting e and $\psi^\natural(e)$. The convexity and concavity properties of the pressure function then guarantees that

$$E(v, \psi^\natural(e)) < 0.$$

In view of the item (ii) of Lemma 2.2, we deduce (2.14). Besides, it is not difficult to check that

$$\psi^{-\natural}(v) > \psi^\natural(v), \qquad \text{for all } v \in (b, d). \tag{2.15}$$

Now, let $v \in [e, a)$, so that $\varphi^\flat_\infty(v) > a$. If $\varphi^\flat_\infty(v) \in (a, b]$, then

$$\psi^\flat_\infty(\varphi^\flat_\infty(v)) > b > v,$$

which, by the skew-symmetry property of E, yields (2.12a). Assume that $\varphi^\flat_\infty(v) \in (b, \psi^\natural(e)]$. We have, since $v \in (a, b)$

$$v_1 := \varphi^\flat_\infty(v) < \varphi^{-\natural}(v) := v_2 \in (b, d).$$

In view of (2.15) and the monotonicity of the function $\psi^{-\natural}$ on the interval (a,d), it holds that

$$\psi^\flat_\infty(\varphi^\flat_\infty(v)) = \psi^\flat_\infty(v_1) > \psi^{-\natural}(v_1) > \psi^{-\natural}(v_2) > \varphi^\natural(v_2) = v.$$

The last inequality and the skew-symmetry of E, yields (2.12a) as well. Let $v \in (a,b)$, then $\varphi^\flat_\infty(v) < a$. The inequalities (ii) of Lemma 2.2 yield

$$\psi^\flat_\infty(\varphi^\flat_\infty(v)) > \psi^\natural(\varphi^\flat_\infty(v)) \geq b > v,$$

which again yields (2.12a). Let $v \in (b, \psi^\natural(e))$, then $\varphi^\flat_\infty(v) < \varphi^\natural(v) < a < \psi^\flat_\infty(v)$. Therefore,

$$E(v', \psi^\natural(e)) = -E(\psi^\natural(e), v') < 0, \quad \text{for all } v' \in (e,a) \subset (e, \psi^\flat_\infty(\psi^\natural(e))),$$

which, in particular for $v' = \varphi^\flat_\infty(v)$, leads us to

$$\psi^\flat_\infty(v') > \psi^\natural(e).$$

Hence, (2.12a) is again a consequence of the skew-symmetry of E and the last inequality. Finally (2.14) yields (2.12b).

The proof of (2.13) is entrirely similar. The monotonicity properties are consequences of (2.12a)-(2.13b). The proof of Corollary 2.3 is complete. □

Recall finally that an arbitrary solution of the Riemann problem (1.1)-(1.2) may also contain rarefaction waves. Given a left-hand state (u_0, v_0), the integral curve associated with the vector field $r_1(v)$ is:

$$\mathcal{O}_1(u_0, v_0) := \big\{(u,v)/u - u_0 = \int_{v_0}^{v} c(w)\, dw\big\}. \tag{2.16}$$

Based on the property that the characteristic speed be increasing in a rarefaction fan, we find easily:

Lemma 2.4. (1–Rarefaction waves)
Given some left-hand state (u_0, v_0), the set of all right-hand states (u_1, v_1) attainable through a 1-rarefaction wave is the portion of the integral curve $\mathcal{O}_1(w_0)$ determined by the following constrains:

(i) *If $v_0 \in (0,a]$, then $v_1 \in [v_0, a]$.*
(ii) *If $v_0 \in (a,b)$, then $v_1 \in [a, v_0]$.*
(iii) *If $v_0 \in [b, +\infty)$, then $v_1 \in [v_0, +\infty)$.*

3. Classical Riemann Solver

To begin with the construction of Riemann solutions, in this section we restrict attention to shock waves satisfying the so-called Liu entropy condition, which is much stronger than our condition (1.5). The solutions constructed now are referred to as the *classical Riemann solutions.* Recall that a 1–*shock wave* connecting (u_0, v_0) to (u_1, v_1) satisfies the *Liu entropy condition* (see (2.3)) iff

$$-\overline{c}(v_0, v) \geq -\overline{c}(v_0, v_1) \qquad \text{for all } v \text{ between } v_0 \text{ and } v_1. \tag{3.1}$$

Note that the condition (3.1) implies the *Lax shock inequalities*

$$\lambda_1(v_0) = -c(v_0) \geq -\bar{c}(v_0, v_1) \geq -c(v_1) = \lambda_1(v_1). \tag{3.2}$$

The Liu condition can be interpreted geometrically, since it is equivalent to

$$\frac{p(v) - p(v_0)}{v - v_0} \geq \frac{p(v_1) - p(v_0)}{v_1 - v_0} \quad \text{for all } v \text{ between } v_0 \text{ and } v_1.$$

In other words, for all v between v_0 and v_1, the graph of p is below (respectively above) the line connecting v_0 to v_1 when $v_1 < v_0$ (resp. $v_1 > v_0$).

Given some left-hand state (u_0, v_0), we now determine the 1–*wave curve* made of all right-hand states that can be arrived at by combining one or several elementary waves. That is, we try to combine together rarefaction fans and shocks satisfying the Hugoniot relations and the Liu condition. Observe that, in view of (2.2)-(2.3), the Hugoniot curve for the first wave family is given by

$$\mathcal{H}_1(u_0, v_0) := \Big\{(u, v) \,/\, u - u_0 = \bar{c}(v_0, v)\,(v - v_0)\Big\}. \tag{3.3}$$

The following lemma singles out those shock waves that are admissible for the Liu criterion.

Lemma 3.1. (Liu admissible shock waves)
Given a left-hand state (u_0, v_0)*, the set of right-hand states* (u_1, v_1) *attainable by a* 1*-shock satisfying the Liu entropy condition* (3.1) *is characterized as follows:*

(i) *If* $v_0 \in (0, c) \cup (a^{-\natural}, +\infty)$*, then* $v_1 \in (0, v_0]$*.*
(ii) *If* $v_0 \in [c, a]$*, then* $v_1 \in (0, v_0] \cup [\varphi^{-\natural}(v_0), \psi^{\natural}(v_0)]$*.*
(iii) *If* $v_0 \in (a, b)$*, then* $v_1 \in (0, \varphi^{-\natural}(v_0)] \cup [v_0, \psi^{\natural}(v_0)]$*.*
(iv) *If* $v_0 \in [b, a^{-\natural}]$*, then* $v_1 \in (0, \varphi^{-\natural}(\psi^{\natural}(v_0))] \cup [\psi^{\natural}(v_0), v_0]$*.*

We are ready to construct the classical 1–wave curve $\mathcal{W}_1^c(u_l, v_l)$ consisting of all right-hand states (u_m, v_m) that can be arrived at by a combination of of Liu admissible shocks and rarefactions. We rely here on Lemma 3.1 for the shocks and Lemma 2.4 for the rarefactions. The solution is actually directly determined from the convex hull and the concave hull of the graph of the function p.

First, assume that $v_l \in (0, c)$. According to Lemma 3.1, all the states (v_m, u_m) having $v_m \in (0, v_l)$ can be arrived at by a single Liu admissible 1–shock. By Lemma 2.4, all of the points (v_m, u_m) with $v_m \in (v_l, a]$ can be arrived at by a single 1-rarefaction. If now $v_m \in [a, d]$, we have $\varphi^{\natural}(v_m) \in [c, a]$. In that case, the solution is thus a rarefaction wave from v_l to $\varphi^{\natural}(v_m)$ followed by a shock from $\varphi^{\natural}(v_m)$ to v_m. Finally, if $v_m > d$, the solution is made of three elementary waves: a rarefaction wave from v_l to c, followed by a shock from c to d, and followed by a rarefaction wave from d to v_m.

Second, assume that $v_l \in [c, a]$. If $v_m \in (0, v_l)$, the Riemann solution is a single Liu-admissible 1–shock. The states (v_m, u_m) with $v_m \in (v_l, a]$ can be arrived at by a single 1-rarefaction. If $v_m \in [a, \varphi^{-\natural}(v_l)]$, then $\varphi^{\natural}(v_m) \in [v_l, a]$ and the Riemann solution is a rarefaction wave from v_l to $\varphi^{\natural}(v_m)$ followed by a shock from $\varphi^{\natural}(v_m)$ to v_m. If $v_m \in (\varphi^{-\natural}(v_l), \psi^{\natural}(v_l)]$, the solution is a single shock. Finally, if $v_m > \psi^{\natural}(v_l)$, the solution is a shock from v_l to $\psi^{\natural}(v_l)$ followed with a rarefaction wave connecting $\psi^{\natural}(v_l)$ to v_m.

Third, assume that $v_l \in (a, b)$. The points (v_m, u_m) with $v_m \in (0, \varphi^{-\natural}(v_l)] \cup [v_l, \psi^{\natural}(v_l)]$ can be arrived at by a single shock. The points w_m with $v_m \in [a, v_l]$ can be arrived at by a single rarefaction wave. If $v_m \in (\varphi^{-\natural}(v_l), a)$, then there exists a unique value $v^* \in (a, v_l)$ such that $\varphi^{-\natural}(v^*) = v_m$. That is $v^* = \varphi^{\natural}(v_m)$. In that case the Riemann solution is a rarefaction wave connecting v_l to v^* followed by a shock connecting v^* to v_m. Finally, if $v_m > \psi^{\natural}(v_l)$, the Riemann solution is a shock connecting v_l to $\psi^{\natural}(v_l)$ followed with a rarefaction wave from $\psi^{\natural}(v_l)$ to v_m.

Fourth, assume that $v_l \in [b, a^{-\natural}]$. The states w_m with $v_m \in (0, \varphi^{-\natural}(\psi^{\natural}(v_l))] \cup [\psi^{\natural}(v_l), v_l]$ can be arrived at by a single shock. The states w_m with $v_m \in [v_l, +\infty)$ can be reached by a single rarefaction wave. If $v_m \in [a, \psi^{\natural}(v_l))$, the Riemann solution is a shock from v_l to $\psi^{\natural}(v_l)$ followed by a rarefaction from $\psi^{\natural}(v_l)$ to v_m. If $v_m \in (\varphi^{-\natural}(\psi^{\natural}(v_l)), a)$, the solution contained three waves: a shock from v_l to $\psi^{\natural}(v_l)$, followed by a rarefaction from $\psi^{\natural}(v_l)$ to $\varphi^{\natural}(v_m)$, and followed by a shock connecting $\varphi^{\natural}(v_m)$ to v_m.

Finally, assume that $v_l \in (a^{-\natural}, +\infty)$. In that case the Riemann solution is simply a shock if $v_m < v_l$ and a rarefaction wave otherwise.

¿From now on, in addition to (1.3) we also assume that

$$\int_b^{\infty} \sqrt{-p'(v)}dv = +\infty. \tag{3.4}$$

It is not difficult to check that the wave curve described above is smooth and monotone increasing and covers the whole range of values $u \in (-\infty, +\infty)$. A similar construction can be given for the 2–wave curve $\mathcal{W}_2^c(u_r, v_r)$ made of all left-hand states attainable through a combination of 2–rarefaction fans or Liu-admissible 2-shocks, starting from the right-hand state (u_r, v_r). Additionally, it can be seen from the explicit formulas of the Hugoniot and rarefaction curves that the two wave curves are globally transverse and intersect at a single point.

We arrive at the following main result in this section.

Theorem 3.2. (Classical Riemann solver)
Under the assumption (1.3)*, the Riemann problem* (1.1)-(1.2) *admits a unique classical solution in the class of piecewise smooth self-similar functions made of rarefaction fans and shock waves satisfying the Liu entropy criterion.*

4. Nonclassical Riemann Solvers

We return to the general conditions in Theorem 2.2. A shock wave is said to be *nonclassical* if the entropy condition (1.5) holds but the Liu entropy condition (3.1) does not. Determining the set of all right-hand states (u_1, v_1) attainable through nonclassical shocks from a given left-hand state (u_0, v_0) is immediate from Theorem 2.2 and Lemma 3.1.

Corollary 4.1. *Given a left-hand state (u_0, v_0), the set of all right-hand states (u_1, v_1) that can be connected to w_0 by a nonclassical shock wave is determined as follows:*

(i) *If $v_0 \in (e, c]$, then $v_1 \in [\varphi^{\flat}_{\infty}(v_0), \psi^{\flat}_{\infty}(v_0)]$.*
(ii) *If $v_0 \in (c, a]$, then $v_1 \in [\varphi^{\flat}_{\infty}(v_0), \varphi^{-\natural}(v_0)) \cup (\psi^{\natural}(v_0), \psi^{\flat}_{\infty}(v_0)]$.*
(iii) *If $v_0 \in (a, b)$, then $v_1 \in (\varphi^{-\natural}(v_0), \varphi^{\flat}_{\infty}(v_0)] \cup (\psi^{\natural}(v_0), \psi^{\flat}_{\infty}(v_0)]$.*
(iv) *If $v_0 \in [b, f)$, then $v_1 \in (\varphi^{-\natural}(\psi^{\natural}(v_0)), \varphi^{\flat}_{\infty}(v_0)] \cup [\psi^{\flat}_{\infty}(v_0), \psi^{\natural}(v_0))$.*
(v) *If $v_0 \in [f, a^{-\natural})$, then $v_1 \in (\varphi^{-\natural}(\psi^{\natural}(v_0)), \psi^{\natural}(v_0))$.*

When $v_0 \in (0, e) \cup (a^{-\natural}, \infty)$*, no such shock exists.*

Denote by $\mathcal{N}(v_0)$ the closure of the set of all values attainable by nonclassical shocks, as described in Corollary 4.1. The *kinetic function* $\varphi^\flat$ is defined to be a decreasing function defined in the interval $[e, b]$ and such that

$$\varphi^\flat(v) \in \mathcal{N}(v_0), \qquad v \in [e, b]. \tag{4.1a}$$

We also impose the condition

$$\varphi^\flat(b) = b^{-\natural} \tag{4.1b}$$

which, as we will see, guarantees the continuity of the Riemann solution with respect to its end states.

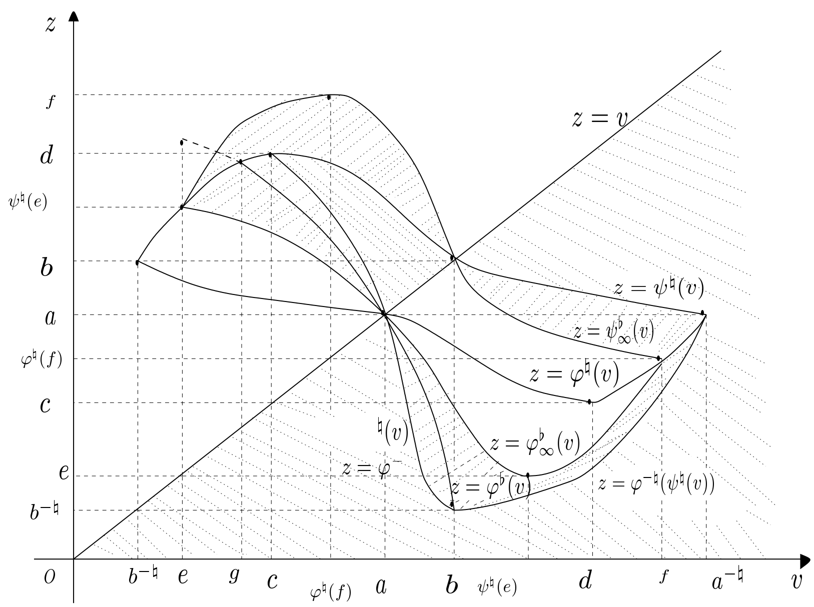

FIGURE 4.1: KINETIC FUNCTION.

The graph of the kinetic function then intersects the one of the function $\psi^\natural$ at a unique point, denoted by $g \in [e, c]$. The straightline connecting v and $\varphi^\flat(v)$ intersects the graph of p at three points when $v \in [g, b]$, limiting two finite areas, and four points when $v \in [e, g)$, limiting three finite areas. Motivated by the derivation of the model made in phase transition dynamics (only the first inflection point is actually physically meaningful), we propose to restrict attention to the interval $[g, b]$, as far as nonclassical shocks are concerned. The *kinetic relation* is the requirement that, for any nonclassical shock connecting some left-hand state (u_0, v_0) to a right-hand state (u_1, v_1), we have

$$v_1 = \varphi^\flat(v_0). \tag{4.2}$$

By the results in Section 3, the Riemann problem (1.1)-(1.2) always admits a solution satisfying the Liu entropy criterion (3.1). Since classical shocks are still admissible in the nonclassical construction to be discussed in the present section, the classical solution is in principle admissible. We are going to allow nonclassical shocks as well and, therefore, to ensure uniqueness, it is clear that one must exclude the classical solution. We postulate here that

$$\text{Nonclassical shock waves are prefered, whenever available.} \tag{P}$$

We now proceed with the construction of the 1-wave curve $\mathcal{W}_1(u_l, v_l)$.

Suppose first that $v_l \in (0, g)$. Any point $v_m \in (0, v_l)$ can be achieved by a single classical shock. Any point $v_m \in (v_l, a]$ is attainable by a single rarefaction wave. If $v_m \in (a, \varphi^\flat(g)]$, there exists a unique point $v_* \in [g, a)$ such that $v_m = \varphi^\flat(v_*)$. The solution is then the composite of a rarefaction wave from v_l to v_* followed by a nonclassical shock from v_* to v_m. If $v_m \in (\varphi^\flat(g), +\infty)$, the solution consists of three parts: A rarefaction wave from v_l to g followed by a nonclassical shock from g to $\varphi^\flat(g)$, followed by a rarefaction wave from $\varphi^\flat(g)$ to v_m.

Second, suppose that $v_l \in [g, a)$. A point $v_m \in (0, v_l)$ can be attained by a single classical shock. A point $v_m \in (v_l, a]$ is attainable by a single rarefaction wave. If $v_m \in (a, \varphi^\flat(v_l)]$, there exists a unique point $v_* \in [v_l, a)$ such that $v_m = \varphi^\flat(v_*)$. The solution is then the composite of the rarefaction wave from v_l to v_* followed by a nonclassical shock from v_* to v_m. If $v_m \in (\varphi^\flat(v_l), \varphi^\flat(g)]$, there exists a unique point $v^* \in [g, v_l)$ such that $v_m = \varphi^\flat(v^*)$. For this construction to make sense, one must here check whether the classical shock from v_l to v^* is slower than the nonclassical shock from v^* to v_m. So, consider the function

$$\tilde{p}(v) := \begin{cases} p(v) & \text{if } v \in (0, v_l], \\ p(v_l) + p'(v_l)(v - v_l) & \text{if } v \in (v_l, +\infty). \end{cases} \tag{4.3}$$

If $v_m \in (\varphi^\flat(v_l), h)$, where

$$h := \min\{\varphi^\flat(g), \varphi^{-\natural}(v_l)\},$$

the function $\tilde{p}$ is convex on $(0, +\infty)$ and the points v^* and v_m belong to its epigraph. Therefore, the straightline connecting v^* and v_m should lie above the graph of $\tilde{p}$ in the interval $(v^*, v_m) \ni v_l$. This is to say

$$\frac{\tilde{p}(v_l) - \tilde{p}(v^*)}{v_l - v^*} < \frac{p(v_m) - p(v^*)}{v_m - v^*},$$

i.e.,

$$s(v_l, v^*) < s(v^*, v_m). \tag{4.4}$$

The latter inequality means that the classical shock from v_l to v^* can be followed by the nonclassical shock from v^* to v_m. In the latter construction, if $v_l \in [g, \varphi^\natural(\varphi^\flat(g)),$ then

$$h = \varphi^\flat(g),$$

and we have completed the argument when $v_m \in (\varphi^\flat(v_l), \varphi^\flat(g))$.

For $v_m \in (\varphi^\flat(g), +\infty)$, the Riemann solution consists of three parts: A classical shock from v_l to g followed by a nonclassical shock from g to $\varphi^\flat(g)$, followed by a rarefaction wave from $\varphi^\flat(g)$ to v_m.

Suppose next that $v_l \in [\varphi^\natural(\varphi^\flat(g)), a)$, then

$$h = \varphi^{-\natural}(v_l).$$

If $v_m \in [\varphi^{-\natural}(v_l), \psi^\natural(v_l)]$, the solution can be a classical shock connecting v_l to v^* followed by a nonclassical shock from v^* to v_m provided (4.4) holds, or else a single classical shock. For $v_m \in (\psi^\natural(v_l), +\infty)$, the solution consists of a classical shock from v_l to $\psi^\natural(v_l)$, followed by a rarefaction wave from $\psi^\natural(v_l)$ to v_m.

Third, suppose that $v_l \in [a, b)$. The points $v_m \in [a, +\infty)$ are reached by the classical construction described in Section 3. If $v_m \in [\varphi^\flat(v_l), a]$, there exists a unique point $v^* \in [a, v_l]$ such that $v_m = \varphi^\flat(v^*)$. The solution then consists of a rarefaction wave connecting v_l to v^* followed by a nonclassical shock from v^* to v_m. If $v_m \in [\varphi^{-\natural}(v_l), \varphi^\flat(v_l))$, then there exists a unique point $v_* \in [v_l, b)$ such that $v_m = \varphi^\flat(v_*)$. Since both v_l and v_* belong to $[a, b]$ and the function p is concave in this interval, we have

$$\frac{p(v_l) - p(v_*)}{v_l - v_*} < \frac{p(\varphi^\flat(v_l)) - p(v_*)}{\varphi^\flat(v_l) - v_*} < \frac{p(v_m) - p(v_*)}{v_m - v_*}.$$

This means the shock speed $s(v_l, v_*)$ is less than the shock speed $s(v_*, v_m)$. Therefore the Riemann solution can be a classical shock from v_l to v_* followed by a nonclassical shock from v_* to v_m. If $v_m \in [\varphi^{-\natural}(v_l), b^{-\natural})$, there exists a unique point $v^* \in [v_l, b)$ such that $v_m = \varphi^\flat(v^*)$. The solution then consists of a classical shock from v_l to v^* followed by a nonclassical shock from v^* to v_m provided

$$-\bar{c}(v_l, v^*) < -\bar{c}(v^*, v_m), \tag{4.5}$$

or else a single classical shock. The states $v_m \in (0, b^{-\natural}]$ are reached by single classical shocks.

Finally, when $v_l \in [b, +\infty)$, we also use the classical construction described in Section 3.

Denote by $\varphi^{-\flat} : [b^{-\natural}, \varphi^\flat(g)] \to [g, b]$, the inverse of the kinetic function $\varphi^\flat$, which is also a monotone decreasing mapping. The arguments presented above are summarized as follows:

Theorem 4.2. (Construction of the 1-wave curve)

Fix some left-hand state (u_l, v_l). Under the assumptions (1.3) and (3.4), we have the following description of the 1-wave curve $\mathcal{W}_1(u_l, v_l)$ consisting of all of the right-hand states (u_m, v_m) that can be reached by a combination of rarefaction fans and shock waves, satisfying the entropy inequality (1.5), the kinetic relation (4.2) (for nonclassical shocks), and the condition (P):

Case 1: $v_l \in (0, g)$.

1. *If $v_m \in (0, v_l)$, the solution is a single classical shock.*
2. *If $v_m \in (v_l, a]$, the solution is a single rarefaction wave.*
3. *If $v_m \in (a, \varphi^\flat(g)]$, the solution is the composite of a rarefaction wave connecting v_l to $v_* := \varphi^{-\flat}(v_m)$ followed by a nonclassical shock from v_* to v_m.*
4. *If $v_m \in (\varphi^\flat(g), +\infty)$, the solution consists of three parts: A rarefaction wave from v_l to g followed by a nonclassical shock from g to $\varphi^\flat(g)$, followed by a rarefaction wave from $\varphi^\flat(g)$ to v_m.*

Case 2: $v_l \in [g, a)$.

1. *If $v_m \in (0, v_l)$, the solution is a single classical shock.*

(2) *If* $v_m \in (v_l, a]$, *the solution is a single rarefaction wave.*
(3) *If* $v_m \in (a, \varphi^\flat(v_l)]$, *the solution is the composite of a rarefaction wave from* v_l *to* $v_* := \varphi^{-\flat}(v_m)$ *followed by a nonclassical shock from* v_* *to* v_m.
(4) *If* $v_l \in [g, \varphi^\natural(\varphi^\flat(g)))$ *and* $v_m \in (\varphi^\flat(v_l), \varphi^\flat(g))$, *then the solution consists of a classical shock from* v_l *to* $v^* := \varphi^{-\flat}(v_m)$ *followed by a nonclassical shock from* v^* *to* v_m.
(5) *If* $v_l \in [g, \varphi^\natural(\varphi^\flat(g)))$ *and* $v_m \in (\varphi^\flat(g), +\infty)$, *the solution consists of three waves: A classical shock from* v_l *to* g *followed by a nonclassical shock from* g *to* $\varphi^\flat(g)$, *followed by a rarefaction wave from* $\varphi^\flat(g)$ *to* v_m.
(6) *If* $v_l \in [\varphi^\natural(\varphi^\flat(g)), a)$ *and* $v_m \in (\varphi^\flat(v_l), \varphi^{-\natural}(v_l))$, *the solution consists of the classical shock from* v_l *to* $v^* := \varphi^{-\flat}(v_m)$ *followed by a nonclassical shock from* v^* *to* v_m.
(7) *If* $v_l \in [\varphi^\natural(\varphi^\flat(g)), a)$ *and* $v_m \in [\varphi^{-\natural}(v_l), \psi^\natural(v_l)]$, *the solution is a classical shock from* v_l *to* v^* *followed by a nonclassical shock from* v^* *to* v_m *if (4.3) holds, or else a single classical shock.*
(8) *If* $v_l \in [\varphi^\natural(\varphi^\flat(g)), a)$ *and* $v_m \in (\psi^\natural(v_l), +\infty)$, *the solution consists of a classical shock from* v_l *to* $\psi^\natural(v_l)$ *followed by a rarefaction wave from* $\psi^\natural(v_l)$ *to* v_m.

Case 3: $v_l \in [a, b)$.

(1) *If* $v_m \in [a, +\infty)$, *the solution is classical (Section 3).*
(2) *If* $v_m \in [\varphi^\flat(v_l), a]$, *the solution consists of the rarefaction wave from* v_l *to* $v^* := \varphi^{-\flat}(v_m)$ *followed by a nonclassical shock from* v^* *to* v_m.
(3) *If* $v_m \in [\varphi^{-\natural}(v_l), \varphi^\flat(v_l))$, *the solution consists of a classical shock from* v_l *to* $v_* := \varphi^{-\flat}(v_m)$ *followed by a nonclassical shock from* v_* *to* v_m.
(4) *If* $v_m \in [\varphi^{-\natural}(v_l), b^{-\natural})$, *the solution consists of the classical shock wave from* v_l *to* $v^* := \varphi^\flat(v_m)$ *followed by a nonclassical shock from* v^* *to* v_m *provided (4.3) holds, or else a single classical shock.*
(5) *The states* $v_m \in (0, b^{-\natural}]$ *are reached by a single classical shock.*

Case 4: $v_l \in [b, +\infty)$.
The construction is classical (Section 3).

A similar result holds for the 2-wave curve. We are now in the position to state the main result of this paper.

Theorem 4.3. *Under the assumptions* (1.3) *and* (3.4), *the Riemann problem* (1.1)-(1.2) *admits a unique piecewise smooth, self-similar solution made of rarefaction fans and shock waves, satisfying the entropy inequality* (1.5), *the kinetic relation* (4.2), *and the condition* (P). *Moreover, the Riemann solution depends* L^1_{loc} *continuously upon its data.*

Proof. We only need to check that the 1-wave curve $\mathcal{W}_1(u_l, v_l)$ constructed earlier is continuous, monotone increasing and extends from $(u_m, v_m) = (-\infty, 0)$ to $(u_m, v_m) = (+\infty, +\infty)$. To begin with, the continuity is easy checked from our construction. For large values of v_m, any right-hand wave in the 1-wave fan connecting v_l and v_m should be a rarefaction wave. The formulation (2.16) and the assumption (3.4) yield

$$u_m \to +\infty \quad \text{as} \quad v_m \to +\infty.$$

Any 1-wave pattern connecting v_l to v_m with $v_m < b^{-\natural}$ must be a single classical shock, by construction. The hypotheses (1.3) and the formulation (3.3) then yield

$$u_m \to -\infty \quad \text{as} \quad v_m \to 0.$$

Finally, since the shock speed $-\overline{c}(v_l, v_m)$ is a continuous function in both variables v_m and v_l, we conclude that the Riemann solution depends $L^1_{\rm loc}$-continuously on the data.

It remains only to check the monotonicity of the wave curve. The classical parts are easily seen to be monotone increasing, so we omit the details. We observe that, in the construction of Theorem 4.2, besides the classical ones, four distinct wave patterns can be distinguished:

(i) A rarefaction wave followed by a nonclassical shock. This happens for instance when $v_l \in (0, g)$ and $v_m \in (a, \varphi^\flat(g))$.

(ii) A classical shock followed by a nonclassical one, say $v_l \in (g, \varphi^\natural(\varphi^\flat(g)))$ and $v_m \in (\varphi^\flat(v_l), \varphi^\flat(g))$.

(iii) In for instance the interval $v_l \in [a, b)$ and $v_m \in [\varphi^{-\natural}(v_l), b^{-\natural})$, a classical shock followed by a nonclassical one if (4.3) holds true, or a single classical shock elsewhere.

Consider first the case (iii). For any fixed $v_l \in [a, b)$, the set of $v_m \in [\varphi^{-\natural}(v_l), b^{-\natural})$ satisfying the condition (4.5) is open, and therefore is a countable union of intervals. In each subinterval, we are back to the case (ii) or to the classical construction. Thus, we only need to treat Cases (i) and (ii). In the rest of the proof, we consider a specific situation arising in these cases, as other possibilities are similar. Recall that

$$c(v) := \sqrt{-p'(v)}$$

and

$$\overline{c}(v_0, v_1) := \sqrt{-\frac{p(v_1) - p(v_0)}{v_1 - v_0}}.$$

Consider the pattern (i). The solution is made of a rarefaction wave followed by a nonclassical shock. In other words, with the notation introduced earlier,

$$\begin{aligned} u_m(v_m) - u_m(\varphi^{-\flat}(v_m)) &= \overline{c}(\varphi^{-\flat}(v_m), v_m)\,(v_m - \varphi^{-\flat}(v_m)), \\ u_m(\varphi^{-\flat}(v_m)) - u_l &= \int_{v_l}^{\varphi^{-\flat}(v_m)} c(z)\,dz. \end{aligned} \tag{4.6}$$

For v_m in the interval $(a, \varphi^\flat(g))$, we deduce from (4.6) that

$$\begin{aligned} \frac{du_m}{dv_m} = &- \frac{d\varphi^{-\flat}(v_m)}{2dv_m} \frac{\theta}{\overline{c}(\varphi^{-\flat}(v_m), v_m)} \Big(c(\varphi^{-\flat}(v_m)) - \overline{c}(\varphi^{-\flat}(v_m), v_m)\Big)^2 \\ &+ c^2(v_m) + \overline{c}^2(\varphi^{-\flat}(v_m), v_m) > 0, \end{aligned}$$

which yields the desired monotone property of the wave curve.

Consider next the pattern (ii). The solution is a composite of a classical shock connecting v_l to $\varphi^{-\flat}(v_m)$ followed by a nonclassical shock connecting $\varphi^{-\flat}(v_m)$ with v_m. ¿From (2.16) and (3.3) we deduce that

$$\begin{aligned} u_m(v_m) - u_m(\varphi^{-\flat}(v_m)) &= \overline{c}(\varphi^{-\flat}(v_m), v_m)(v_m - \varphi^{-\flat}(v_m)), \\ u_m(\varphi^{-\flat}(v_m)) - u_l &= \overline{c}(v_l, \varphi^{-\flat}(v_m))(\varphi^{-\flat}(v_m) - v_l). \end{aligned} \tag{4.7}$$

This yields

$$\begin{aligned}\frac{du_m}{dv_m} = &- \frac{d\varphi^{-\flat}(v_m)}{2dv_m}\Big(\overline{c}(\varphi^{-\flat}(v_m), v_l) - \overline{c}(\varphi^{-\flat}(v_m), v_m)\Big)\\ &\times\Big(\frac{c^2(\varphi^{-\flat}(v_m))}{\overline{c}(\varphi^{-\flat}(v_m), v_l)\,\overline{c}(\varphi^{-\flat}(v_m), v_m)} - 1\Big) + \frac{c^2(v_m)+1}{2\overline{c}(\varphi^{-\flat}(v_m), v_m)}.\end{aligned} \tag{4.8}$$

Since the function p is convex in the interval $(0, a) \ni v_l, \varphi^{-\flat}(v_m)$ and since $v_l > \varphi^{-\flat}(v_m)$, we have

$$\frac{p(\varphi^{-\flat}(v_m)) - p(v_l)}{\varphi^{-\flat}(v_m) - v_l} > p'(\varphi^{-\flat}(v_m)).$$

Hence we obtain

$$c(\varphi^{-\flat}(v_m)) > \overline{c}(\varphi^{-\flat}(v_m), v_l) > \overline{c}(\varphi^{-\flat}(v_m)v_m), \tag{4.9}$$

where the last inequality follows from the fact that the shock speed is increasing and the classical shock is followed by the nonclassical one. The inequalities (4.9) used in (4.8) yield

$$\frac{du_m}{dv_m} > 0,$$

which implies the monotonicity of the wave curve. The proof of Theorem 4.3 is complete. □

References

[1] Abeyaratne R. and Knowles J.K., On the dissipative response due to discontinuous strains in bars of unstable elastic materials, Int. J. Solids Structures 24 (1988), 1021–1044.

[2] Abeyaratne R. and Knowles J.K., Kinetic relations and the propagation of phase boundaries in solids, Arch. Rational Mech. Anal. 114 (1991), 119–154.

[3] Fan H.T., A vanishing viscosity approach on the dynamics of phase transitions in van der Waals fluids, J. Differential Equations 103 (1993), 179–204.

[4] Fan H.T., One-phase Riemann problem and wave interactions in systems of conservation laws of mixed type, SIAM J. Math. Anal. 24 (1993), 840–865.

[5] Fan H.T. and Slemrod M., The Riemann problem for systems of conservation laws of mixed type, in "Shock induces transitions and phase structures in general media", R. Fosdick, E. Dunn, and H. Slemrod ed., IMA Vol. Math. Appl. 52, Springer-Verlag, 1993, pp. 61–91.

[6] Hattori H., The Riemann problem for a van der Waals fluid with the entropy rate admissibility criterion: isothermal case, Arch. Rational Mech. Anal. 92 (1986), 246–263.

[7] Hattori H., The Riemann problem for a van der Waals fluid with the entropy rate admissibility criterion: Non-isothermal case, J. Differential Equations 65 (1986), 158–174.

[8] Hayes B.T. and LeFloch P.G., Nonclassical shocks and kinetic relations: Scalar conservation laws, Arch. Rational Mech. Anal. 139 (1997), 1–56.

[9] Hayes B.T. and LeFloch P.G., Nonclassical shocks and kinetic relations : Finite difference schemes, SIAM J. Numer. Anal. 35 (1998), 2169-2194.

[10] Hayes B.T. and LeFloch P.G., Nonclassical shocks and kinetic relations: Strictly hyperbolic systems, SIAM J. Math. Anal. 31 (2000), 941–991.

[11] Lax P.D., Shock wave and entropy, in "Contributions to Functional Analysis", ed. E.A. Zarantonello, Academic Press, New York, 1971, pp. 603–634.

[12] LeFloch P.G., Propagating phase boundaries: Formulation of the problem and existence via the Glimm method, Arch. Rational Mech. Anal. 123 (1993), 153–197.

[13] LeFloch P.G., An introduction to nonclassical shocks of systems of conservation laws, Proceedings of the "International School on Theory and Numerics for Conservation Laws", Freiburg/Littenweiler (Germany), 20–24 October 1997, ed. D. Kröner, M. Ohlberger and C. Rohde, Lecture Notes in Computational Science and Engineering, 1998, pp. 28–72.

[14] LeFloch P.G., *Hyperbolic Systems of Conservation Laws: The theory of classical and nonclassical shock waves,* ETH Lecture Notes Series, to appear.

[15] LeFloch P.G. and Thanh M.D., Nonclassical Riemann solvers and kinetic relations I. An hyperbolic model of elastodynamics, Z.A.M.P., to appear.

[16] LeFloch P.G. and Thanh M.D., Nonclassical Riemann solvers and kinetic relations II. An hyperbolic-elliptic model of phase transitions, in preparation.

[17] Liu T.-P., The Riemann problem for general 2×2 conservation laws, Trans. Amer. Math. Soc. 199 (1974), 89–112.

[18] Schulze S. and Shearer M., Undercompressive shocks for a system of hyperbolic conservation laws with cubic nonlinearity, J. Math. Anal. Appl. 229 (1999), 344–362.

[19] Shearer M., The Riemann problem for a class of conservation laws of mixed type, J. Differential Equations 46 (1982), 426–443.

[20] Slemrod M., Admissibility criteria for propagating phase boundaries in a van der Waals fluid, Arch. Rational Mech. Anal. 81 (1983), 301–315.

[21] Slemrod M., The viscosity-capillarity criterion for shocks and phase transitions, Arch. Rational. Mech. Anal. 83 (1983), 333–361.

[22] Slemrod M., Dynamic phase transitions in a van der Waals fluid, J. Differential Equations 52 (1984), 1–23.

Philippe G. LeFloch

Centre de Mathématiques Appliquées & Centre National de la Recherche Scientifique, U.M.R. 7641, Ecole Polytechnique, 91128 Palaiseau Cedex, France.

E-mail address: `lefloch@cmapx.polytechnique.fr`

Mai Duc Thanh

Hanoi Institute of Mathematics, P.O. Box 631 BoHo, 10.000 Hanoi, Vietnam.

E-mail address: `mdthanh@hanimath.ac.vn, thanh@cmapx.polytechnique.fr`