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First-order differential equations with several deviating arguments: Sturmian comparison method in oscillation theory: I *

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Abstract

The Sturmian Comparison Method, elaborated previously by one of the authors, is developed and applied to differential equations with several deviating arguments. For two delays, we obtain oscillation criteria that are explicit and close to be necessary. We also present a comparison of our results with those known in the literature.

1 Introduction

Differential equations with several deviating (not necessarily delayed) arguments have been intensively investigated for many years. One of the problems studied for such equations is the oscillation of solutions. Other related questions which have been studied are the existence and non-existence of positive solutions on a given finite interval or on the semiaxis, and the existence of lower and upper bounds for the length of sign-preserving intervals of a solution.

For autonomous equations where the coefficients and the deviations of the arguments are constant such problems were almost completely solved long time ago. This means that some sufficient conditions which are rather close to being necessary were obtained. These results were based on the study of the characteristic quasi-polynomial, which reduces many problems to the analysis and location of its roots. Using this approach, each coefficient is taken into account according to its contribution to the equation.

When a differential equation with deviating arguments is non-autonomous, this technique cannot be applied and thus the problem becomes much more complicated. Nevertheless, oscillation theory of Delay Differential Equations (DDE) has been developed; see, for example, [7, 9, 13] and their references.

We can compare the complexity of the oscillatory properties of DDE with only one retarded argument,

$$x'(t) + a(t)x[r(t)] = 0, \quad t \geq t_0, \tag{1.1}$$

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with the complexity of the autonomous equation. The equation with several deviations

$$x'(t) + \sum_{k=1}^{n} a_k(t)x[r_k(t)] = 0, \quad t \geq t_0, \tag{1.2}$$

has been much less studied. The present paper deals with several deviations, but only in first-order DDEs. The study of the second order DDEs is much more difficult.

The main shortcoming in a significant number of publications is that the exact contribution of each coefficient $a_k(t)$ and of each deviation $r_k(t)$ to the behavior of the solution of (1.2) is not evaluated. Unfortunately, most of the publications on the oscillation of equations of type (1.2) are based on methods that cut off all the "outstanding" parts of the equation. The only exceptions being [1, 6, 8, 10, 11, 14, 15, 16, 17, 19], we will compare their results with those obtained in the present paper. Cutting off parts of the equation way lead to unsatisfactory results. Although formally valid, they are rough and non-logically built. In fact, (1.2) is roughly reduced to (1.1) and only afterwards is investigated. Moreover, if the coefficients $a_k(t)$ are of different sign, some of the $r_k(t)$'s are delayed and some of them are advanced, such equations have hardly been studied.

However, the situation with the relevant results in this field and the possibilities of its further development is not so pessimistic. The second author of the present paper developed the basis of a rather efficient method for the investigation of the oscillatory properties of DDEs. This method is called the Sturmian Comparison Method (SCM) [2, 3, 4, 5] by whcih some new results on the oscillation of (1.1) and (1.2) were obtained. In particular the results for (1.2) have not been improved or generalized untill now. Moreover, these results catch the contribution of each $a_k(t)$ and $r_k(t)$, can treat the case of a mixed deviating equation, and also provide an estimate for the distance between two consecutive zeros of a solution.

Unfortunately, Monograph [2] was published in Russian and has not been accessible for many scientists working in this field. That is why its results were almost unknown and did not influence the publications in this area, some of which are particular cases of the results obtained in [2, 3], both from the point of view of generality of the equations and sharpness of the statements.

Discussions in recent years brought us to the conclusion that SCM for DDEs with several deviations should be systematically presented, including both new and known results, with an up-to-date review of the literature. We plan to publish several related papers on this topic. The present paper contains the general theory of SCM for (1.2) (non-retarded arguments only). Further, we apply this method to (1.2) with nonnegative coefficients $a_k(t) \geq 0$ and delays $r_k(t) \leq t$. Later, we plan to consider (1.2) with mixed deviating arguments and with coefficients of different sign.

The results on the oscillatory properties of (1.1) with only one deviation argument are not in the framework of the present discussion. Let us only mention that in [4, 5] the SCM is applied to this class of equations.

2 Sturmian comparison theorem

Suppose $-\infty < \alpha,\ \beta < \infty$, $k = 1, 2, \dots, n$, and the following assumptions hold

(a1) Functions $a_k(t)$ are continuous on (α, β)

(a2) Functions $r_k(t)$ are monotone increasing on (α, β) with continuous derivatives.

We can extend the functions $r_k(t)$ without loss of monotonicity and differentiability in such a way that the range of $r_k(t)$ will include $[\alpha, \beta]$. Then there exist continuously differentiable functions $q_k(t)$ such that $r_k[g_k(t)] = t$, $t \in (\alpha, \beta)$.

In this and the next section, we assume that conditions (a1)-(a2) hold. Denote

$$\begin{gathered} \gamma_k(t) := \max\{t; q_k(t)\}, \quad \delta_k(t) := \min\{t; q_k(t)\}, \\ e_k := \{t \notin (\alpha, \beta) : r_k(t) \in (\alpha, \beta)\}, \quad i_k := \{t \in (\alpha, \beta) : r_k(t) \notin (\alpha, \beta)\}, \\ \tilde{e}_k := \{t : t = r_k(s), \quad s \in i_k\}, \quad \tilde{i}_k := r_k[e_k] \subset (\alpha, \beta), \\ E := \cup_1^n e_k; \quad I := \cup_1^n i_k; \quad \tilde{E} := \cup_1^n \tilde{e}_k. \end{gathered}$$

It is easy to see that

$$e_k = [\delta_k(\alpha), \alpha] \cup [\beta, \gamma_k(\beta)], \quad i_k = [\alpha, \gamma_k(\alpha)] \cup [\delta_k(\beta), \beta].$$

On the space of functions which are continuous on $[\alpha, \beta] \cup \tilde{E}$ and have continuous derivatives on (α, β), we define the differential operators

$$(lx)(t) := x'(t) + \textstyle\sum_{k=1}^n a_k(t) x[r_k(t)], \quad t \in (\alpha, \beta), \tag{2.1}$$

$$(\tilde{l}y)(t) := -y'(t) + \textstyle\sum_{k=1}^n q_k'(t) \tilde{a}_k[q_k(t)] y[q_k(t)], \quad t \in (\alpha, \beta). \tag{2.2}$$

Here $\tilde{a}_k(t)$ are continuous on $[e_k \cup (\alpha, \beta)] \setminus i_k$.

Consider now the two corresponding differential inequalities

$$(lx(t) \le 0, \quad t \in (\alpha, \beta), \tag{2.3}$$

$$(\tilde{l}y(t) \ge 0, \quad t \in (\alpha, \beta). \tag{2.4}$$

Definition (see [2]) The interval (α, β) is called *a regular half-cycle* (RHC) for (2.4) or the corresponding equation if

$$r_k(\beta) > \alpha, \quad \beta > r_k(\alpha), \quad k = 1, \dots, n$$

and there exists a solution $y(t)$ of (2.4) such that

$$y(\alpha) = y(\beta), \quad y(t) > 0, \quad t \in (\alpha, \beta), \quad y(t) \le 0, \quad t \in E. \tag{2.5}$$

The definition of RHC for (2.3) and for the corresponding equation is similar.

Definition (see [2])
a) A solution $x(t)$, $t_0 \le t < \infty$ of a differential equation or inequality is called *non-oscillatory* if there exists T such that $x(t) \neq 0$ for $t \ge T$ and *oscillatory* otherwise.
b) An oscillatory solution x of a differential equation or inequality is called a *regular oscillatory solution* if for every T it has RHC (α, β) with $\alpha > T$. Otherwise, it is called *quickly oscillatory.*

The following principal identity will be used for obtaining the main results of this paper.

Lemma 2.1 *For every function $x(t)$, $t \in (\alpha, \beta)$, with continuous derivative and for every function $y(t)$, $t \in [\alpha, \beta] \cup E$, with continuous derivative, we have*

$$\begin{aligned} \int_\alpha^\beta y(t)(lx)(t)\,dt &= \int_\alpha^\beta x(t)(\tilde{l}y)(t)\,dt + [x(\beta)y(\beta) - x(\alpha)y(\alpha)] \\ &\quad + \sum_{k=1}^n \Big\{ \int_{(\alpha,\beta)\setminus i_k} [a_k(t) - \tilde{a}_k(t)]x[r_k(t)]y(t)\,dt \\ &\quad + \int_{i_k} a_k(t)x[r_k(t)]y(t)\,dt - \int_{e_k} \tilde{a}_k(t)x[r_k(t)]y(t)\,dt \Big\}. \end{aligned} \tag{2.6}$$

The statement of this lemma is to check prove by direct calculations using integration by parts.

The following statement is a direct and exact analogue of the classical Sturmian Comparison Theorem for the second-order ordinary differential equation $x''(t) + a(t)x(t) = 0$.

Theorem 2.1 *Let (α, β) is RHC for (2.4) ,*

$$\tilde{a}_k(t) \ge 0, \quad t \in e_k, \quad k = 1, \dots, n, \tag{2.7}$$

$$a_k(t) \ge \begin{cases} 0, & t \in i_k, \\ \tilde{a}_k(t), & t \in (\alpha, \beta) \setminus i_k, \end{cases} \quad k = 1, \dots, n. \tag{2.8}$$

and at least one of (2.7)-(2.8) is strict on some subinterval of (α, β). Then (2.3) has no positive solutions on $(\alpha, \beta) \cup \tilde{E}$.

Proof The proof is based on the principal identity (2.6). Let (2.5) hold for a solution y of (2.4) and let $x(t)$, $t \in (\alpha, \beta) \cup E$ be a positive solution of (2.3). Then the left-hand side of (2.6) is non-positive.

On the other hand (2.7), (2.8), (2.5) imply that all terms on the right hand-side of (2.6) are nonnegative and at least one term is positive. We have a contradiction, and the statement is proved.

Corollary 2.1 *Suppose (2.4) has a regular oscillatory solution and*

$$a_k(t) \ge \tilde{a}_k(t) \ge 0, \quad k = 1, \dots, n, \quad t \ge t_0.$$

Then (2.3) has no positive solution on (t_0, ∞).

Remark Note the important fact that Theorem 2.1 is concerned with the behavior of solutions of DDE and DDI on a finite interval and not on a semiaxis. Therefore one can obtain from Theorem 2.1 not only explicit conditions of oscillation, but also estimates of the length of the sign-preserving intervals of the solutions.

3 Construction of the "testing equations"

¿From this section, we consider (1.2) with two delays only. The reason is the following: all the possible complications appear always already for $n = 2$, while the computations are technically easier in this case.

The essence of the SCM is the following. One needs to construct a family of "testing" (2.4) for which a given interval (α, β) is RHC and condition (2.8) holds. This family must be as rich as possible. In the present paper we are going to do it for the case of two retarded arguments. Further, we can use Theorem 2.1 for (2.3) to obtain explicit oscillation criteria.

Consider (2.3), the corresponding differential equation and (2.4) for the case $n = 2$ and $r_i(t) \le t$, $i = 1, 2$:

$$(lx)(t) := x'(t) + a_1(t)x[r_1(t)] + a_2(t)x[r_2(t)] \le 0, \quad t \in (\alpha, \beta), \tag{3.1}$$

$$(lx)(t) := x'(t) + a_1(t)x[r_1(t)] + a_2(t)x[r_2(t)] = 0, \quad t \in (\alpha, \beta), \tag{3.2}$$

$$(\tilde{l}y)(t) := -y'(t) + \tilde{a}_1[q_1(t)]q_1'(t)y[q_1(t)] + \tilde{a}_2[q_2(t)]q_2'(t)y[q_2(t)] \ge 0, \quad t \in (\alpha, \beta). \tag{3.3}$$

Denote

$$\rho(t) := \min\{r_1(t); r_2(t)\},\ Q(t) := \max\{q_1(t); q_2(t)\},\ R(t) := \max\{r_1(t); r_2(t)\}.$$

It is obvious that functions $\rho(t)$ and $Q(t)$ are the inverse of each other. Besides, in this particular case ($n = 2$):

$$\begin{aligned} &e_j = [\beta, q_j(\beta)], \quad E = [\beta, Q(\beta)], \quad i_j = [\alpha, q_k(\alpha)], \quad I = [\alpha, Q(\alpha)], \\ &\tilde{e}_k = [r_k(\alpha), \alpha], \quad \tilde{E} = [\rho(\alpha), \alpha], \quad \tilde{i}_k = [r_k(\beta), \beta], \quad \tilde{I} = [\rho(\beta), \beta]. \end{aligned} \tag{3.4}$$

Lemma 3.1 *Let $\rho(\beta) > \alpha$ and let $\varphi(t)$, $k(t)$ be continuous functions defined on $(\rho(\alpha), Q(\beta))$ satisfying the following conditions:*

$$0 \le \int_\alpha^t \varphi(s)ds < \pi, \quad t \in (\alpha, \beta); \quad \int_\alpha^\beta \varphi(s)ds = \pi; \tag{3.5}$$

$$0 < \int_{\rho(t)}^t \varphi(s)ds < \tfrac{\pi}{2}, \quad t \in (\alpha, Q(\beta)); \tag{3.6}$$

$$-\frac{\varphi(t)}{\sin\int_t^{q_2(t)} \varphi(s)ds} \le k(t) \le \frac{\varphi(t)}{\sin\int_t^{q_1(t)} \varphi(s)ds}, \quad t \in (\rho(\beta), \beta). \tag{3.7}$$

If the coefficients in (3.3) are defined by

$$q_i'(t)\tilde{a}_i[q_i(t)] := \frac{\varphi(t) - (-1)^i k(t) \sin\int_t^{q_j(t)} \varphi(s)ds}{\sin\int_t^{q_1(t)} \varphi(s)ds + \sin\int_t^{q_2(t)} \varphi(s)ds} \times \tag{3.8}$$

$$\exp\Big\{-\int_t^{q_i(t)}\frac{\varphi(s)\cos\Big[\frac12\int_s^{q_1(s)}\varphi(\xi)d\xi+\frac12\int_s^{q_2(s)}\varphi(\xi)d\xi\Big]+k(s)\sin\frac12\int_{q_1(s)}^{q_2(s)}\varphi(\xi)d\xi}{\sin\Big[\frac12\int_s^{q_1(s)}\varphi(\xi)d\xi+\frac12\int_s^{q_2(s)}\varphi(\xi)d\xi\Big]}ds\Big\},$$

$i,j=1,2$, $i\neq j$, $t\in(\alpha,\beta)$, then the interval (α,β) is RHC for (3.3).

Proof. Conditions (3.5)-(3.7) imply that $\varphi(t)\geq 0$ on $(\rho(\beta),\beta)$. By direct calculations one can check that the function

$$y(t)=\sin\int_\alpha^t\varphi(s)ds\times \tag{3.9}$$
$$\exp\Big\{\int_\alpha^t\frac{\varphi(s)\cos\Big[\frac12\int_s^{q_1(s)}\varphi(\xi)d\xi+\frac12\int_s^{q_2(s)}\varphi(\xi)d\xi\Big]+k(s)\sin\frac12\int_{q_1(s)}^{q_2(s)}\varphi(\xi)d\xi}{\sin\Big[\frac12\int_s^{q_1(s)}\varphi(\xi)d\xi+\frac12\int_s^{q_2(s)}\varphi(\xi)d\xi\Big]}ds\Big\}$$

is a solution of the equation corresponding to (3.3). Conditions (3.5)-(3.7) and $\rho(\beta)>\alpha$ yield that (α,β) is RHC for (3.3).

From Theorem 2.1 and Lemma 3.1 we obtain the following the following theorem.

Theorem 3.1 *Assume Conditions (3.5)-(3.7) hold , $\rho(\beta)>\alpha$, and*

$$a_i(t)\geq\begin{cases}0, & t\in(\alpha,q_i(\alpha)),\\ \tilde a_i(t), & t\in(q_i(\alpha),\beta),\end{cases}\qquad i=1,2, \tag{3.10}$$

where $\tilde a_i(t)$ are given by (3.8) and at least one of the inequalities (3.10) is strict on some sub-interval. Then (3.1) has no positive solution on $(\rho(\alpha),\beta)$.

Corollary 3.1 *Suppose the conditions of Theorem 3.1 hold for a sequence of intervals (α_k,β_k), $\alpha_k\to\infty$. Then all solutions of (3.2) are oscillatory.*

Remark No restrictions are imposed on the coefficients $a_i(t)$ of (3.2) outside the set $\cup^\infty(\alpha_k,\beta_k)$ in Corollary 3.1.

Corollary 3.2 *Let*

$$R(t)\to\infty,\quad \varphi(t)\geq 0,\quad \int_{t_0}^\infty\varphi(s)ds=\infty,$$

and conditions (3.6), (3.7), (3.10) hold on $[t_0,\infty)$. Then all the solutions of (3.2) are oscillatory.

The following statement gives a clear proof of the well-known fundamental oscillation criterion for the autonomous equation

$$x'(t)+a_1x(t-\tau_1)+a_2x(t-\tau_2)=0,\quad \tau_1\neq\tau_2,\quad a_i>0,\quad \tau_i>0,\quad i=1,2. \tag{3.11}$$

On the other hand, this statement demonstrates the sharpness of Theorem 3.1, which in particular case (of autonomous equations) allows to obtain necessary and sufficient conditions for the oscillation of all solutions of (3.2).

Corollary 3.3 *Suppose that for the characteristic quasi-polynomial of (3.11)*

$$F(\lambda) := \lambda + a_1 \exp\{-\lambda\tau_1\} + a_2 \exp\{-\lambda\tau_2\}$$

the following condition holds:

$$F(\lambda) > 0, \quad \forall\lambda \in (-\infty, \infty). \tag{3.12}$$

Then all the solutions of (3.11) are oscillatory.

It is obvious that (3.12) is also necessary for the oscillation of (3.11) as well.

Proof. It is clear that $F''(\lambda) > 0$ for every λ. Thus the equation

$$F'(\lambda) := 1 - \tau_1 a_1 e^{-\lambda\tau_1} - \tau_2 a_2 e^{-\lambda\tau_2} = 0 \tag{3.13}$$

has a unique root λ_0 and $\inf_{-\infty<\lambda<\infty} F(\lambda) = F(\lambda_0)$. Hence (3.12) is equivalent to

$$F(\lambda_0) > 0. \tag{3.14}$$

The equality $\tau_1 a_1 e^{-\lambda_0\tau_1} + \tau_2 a_2 e^{-\lambda_0\tau_2} = 1$ implies the equivalence

$$\begin{aligned} & F(\lambda_0) > 0 \\ \iff\ & (\tau_1+\tau_2)\left(\lambda_0 + a_1 e^{-\lambda_0\tau_1} + a_2 e^{-\lambda_0\tau_2}\right) > 0 \\ \iff\ & 1 + \tau_2 a_1 e^{-\lambda_0\tau_1} + \tau_1 a_2 e^{-\lambda_0\tau_2} + \lambda_0(\tau_1+\tau_2) > 0 \\ \iff\ & \frac{2}{\tau_1+\tau_2} + \frac{\tau_2-\tau_1}{\tau_1+\tau_2}\left(a_1 e^{-\lambda_0\tau_1} - a_2 e^{-\lambda_0\tau_2}\right) > -\lambda_0 \qquad (3.15) \\ \iff\ & a_i \exp\left\{\tau_i\Big[\frac{2}{\tau_1+\tau_2} + \frac{\tau_2-\tau_1}{\tau_1+\tau_2}\left(a_1 e^{-\lambda_0\tau_1} - a_2 e^{-\lambda_0\tau_2}\right)\Big]\right\} > a_i e^{-\lambda_0\tau_i}, \\ & i = 1, 2\,. \end{aligned}$$

In Theorem 3.1, put

$$\varphi(t) := \frac{\nu}{\tau_1+\tau_2}, \quad k(t) := a_1 e^{-\lambda_0\tau_1} - a_2 e^{-\lambda_0\tau_2},$$

where $\nu \in (0, \frac{\pi}{2}|\tau_2 - \tau_1|)$ is a sufficiently small number which will be chosen below.

Let α and β be such that $\beta - \alpha = \pi(\tau_1+\tau_2)/\nu$. Then (3.5)-(3.6) hold and the inequalities

$$\begin{aligned} k\tau_1 &= \tau_1 a_1 e^{-\lambda_0\tau_1} - \tau_1 a_2 e^{-\lambda_0\tau_2} = 1 - \tau_2 a_2 e^{-\lambda_0\tau_2} - \tau_1 a_2 e^{-\lambda_0\tau_2} < 1, \\ k\tau_2 &= \tau_2 a_1 e^{-\lambda_0\tau_1} - \tau_2 a_2 e^{-\lambda_0\tau_2} = \tau_2 a_1 e^{-\lambda_0\tau_1} - (1 - \tau_1 a_1 e^{-\lambda_0\tau_1}) > -1, \end{aligned}$$

imply that $-\frac{1}{\tau_2} < k < \frac{1}{\tau_1}$. Hence (3.7) holds, too. Equation (3.10) can be written in the form

$$\begin{aligned} & a_i \exp\left\{\tau_i\Big[\frac{\nu ctg\frac{\nu}{2}}{\tau_1+\tau_2} + \frac{\sin\frac{\nu(\tau_2-\tau_1)}{2(\tau_1+\tau_2)}}{\sin\frac{\nu}{2}}\left(a_1 e^{-\lambda_0\tau_1} - a_2 e^{-\lambda_0\tau_2}\right)\Big]\right\} \qquad (3.16) \\ & > \left\{a_i e^{-\lambda_0\tau_i} + \frac{(-1)^i}{\nu}\Big[\frac{\nu\tau_j}{\tau_1+\tau_2} - \sin\frac{\nu\tau_j}{\tau_1+\tau_2}\Big]\left(a_1 e^{-\lambda_0\tau_1} - a_2 e^{-\lambda_0\tau_2}\right)\right\}\frac{\nu}{\sin\nu}, \end{aligned}$$

where $i = 1, 2$, $i \neq j$. It is clear that (3.15) is the limit form of (3.16) as $\nu \to 0$. Hence (3.16) holds for sufficiently small ν. Theorem 3.1 implies now that all solutions of (3.11) are oscillatory.

Remark Actually, (3.16) implies something more: each solution of (3.11) has at least one change of sign on every interval of length greater than $(\pi(\tau_1 + \tau_2)/\nu) + \max\{\tau_1, \tau_2\}$.

From Theorem 3.1, we will obtain an explicit condition for the oscillation not only in pointwise terms, but in *the integral average terms* as well. To avoid unwieldy formulations, we will not deal with the estimations of the lengths of sign-preserving intervals of the solutions (see the remark at the end of Section 2). We will be confined to obtain an explicit condition for the oscillation by Corollary 3.2 only. Readers interested in the first problem are referred to [2, Corollary 4.3.5, p.100].

Theorem 3.2 *Let $\rho(t) \to \infty$ and suppose there exist functions $b_j(t)$, $j = 1, 2$ such that*

$$a_j(t) \geq b_j(t) \geq 0, \quad j = 1, 2, \quad t \geq t_0; \tag{3.17}$$

the limits

$$B_{ij} := \lim_{t\to\infty} \int_{\rho_i(t)}^{t} b_j(s)ds, \quad i = 1, 2, \tag{3.18}$$

are finite with

$$B_{11} + B_{22} > 0; \tag{3.19}$$

the system

$$\begin{gathered} (B_{11}B_{22} - B_{12}B_{21})x_1x_2 - B_{11}x_1 - B_{22}x_2 + 1 = 0 \\ \ln x_1 - B_{11}x_1 - B_{12}x_2 < 0 \\ \ln x_2 - B_{21}x_1 - B_{22}x_2 < 0 \end{gathered} \tag{3.20}$$

has a positive solution $\{x_1; x_2\}$.

Then all solutions of (3.2) are oscillatory.

Proof. In view of (3.20), the system

$$\begin{gathered} (1 - x_1B_{11})\alpha_1 - x_2B_{12}\alpha_2 = 0 \\ -x_1B_{21}\alpha_1 + (1 - x_2B_{22})\alpha_2 = 0 \\ \alpha_1 + \alpha_2 = 1 \end{gathered} \tag{3.21}$$

has a solution $\{\alpha_1; \alpha_2\}$, $\alpha_j > 0$, $j = 1, 2$ (we omit the explanation). For $i = 1, 2$, Denote

$$\begin{aligned} P_i(\nu) \quad := \quad & \frac{\nu x_i}{\sin\nu \cos\nu(\alpha_2 - \alpha_1)} \\ & \times \exp\left\{ [-2\nu\alpha_i \cos\nu - (x_1B_{i1} - x_2B_{i2}) \sin\nu(\alpha_2 - \alpha_1)] \frac{1}{\sin\nu} \right\} \end{aligned} \tag{3.22}$$

From (3.20) and (3.21) it follows

$$P_i(0) = \lim_{\nu\to 0} P_i(\nu) = x_i \exp(-x_1 B_{i1} - x_2 B_{i2}) < 1, \quad i = 1, 2.$$

Thus there exists $\nu_0 > 0$ such that for $\nu \in (0, \nu_0)$ the inequalities $P_i(\nu) < 1$, $i = 1, 2$ hold.

In Corollary 3.2, set

$$\varphi(t) := 2\nu\alpha_1 x_1 q_1'(t) b_1[q_1(t)] + 2\nu\alpha_2 x_2 q_2'(t) b_2[q_2(t)], \tag{3.23}$$

$$k(t) := x_1 q_1'(t) b_1[q_1(t)] - x_2 q_2'(t) b_2[q_2(t)]. \tag{3.24}$$

From (3.19) it follows

$$\int_{t_0}^{\infty} \varphi(s)ds = \infty, \tag{3.25}$$

$$\lim_{t\to\infty} \int_t^{q_i(t)} \varphi(s)ds = 2\nu\alpha_1 x_1 B_{i1} + 2\nu\alpha_2 x_2 B_{i2} = 2\nu\alpha_i, \tag{3.26}$$

$$\lim_{t\to\infty} \int_t^{q_i(t)} k(s)ds = x_1 B_{i1} - x_2 B_{i2}. \tag{3.27}$$

Besides, (3.7) holds for $t > T$, where T is sufficiently large. Indeed,

$$\begin{aligned} &\frac{k(t)}{\varphi(t)} \sin \int_t^{q_1(t)} \varphi(s)ds \\ &= \frac{x_1 q_1'(t) b_1[q_1(t)] - x_2 q_2'(t) b_2[q_2(t)]}{2\nu\left(\alpha_1 x_1 q_1'(t) b_1[q_1(t)] + \alpha_2 x_2 q_2'(t) b_2[q_2(t)]\right)} \sin \int_t^{q_1(t)} \varphi(s)ds \\ &\le \frac{1}{2\nu\alpha_1} \sin \int_t^{q_1(t)} \varphi(s)ds \end{aligned}$$

and

$$\lim_{t\to\infty} \int_t^{q_1(t)} \varphi(s)ds = \sin 2\nu\alpha_1.$$

Since $\frac{\sin 2\nu\alpha_1}{2\nu\alpha_1} < 1$, $k(t)\sin\int_t^{q_1(t)} \varphi(s)ds \le \varphi(t)$ for $t > T$. Similarly, for $t > T$, $k(t)\sin\int_t^{q_2(t)} \varphi(s)ds \ge -\varphi(t)$. This implies that (3.7) holds for $t > T$.

For $i = 1, 2$, Denote

$$C_i(t,\nu) := \frac{\nu x_i}{\sin\int_t^{q_1(t)} \varphi(s)ds + \sin\int_t^{q_2(t)} \varphi(s)ds)}$$
$$\times \exp\Big[-\int_t^{q_i(t)} \frac{\varphi(\xi)\cos\Big(\frac12\int_\xi^{q_1(\xi)}\varphi(s)ds + \frac12\int_\xi^{q_2(\xi)}\varphi(s)ds\Big) + k(\xi)\sin(\frac12\int_{q_1(\xi)}^{q_2(\xi)}\varphi(s)ds)}{\sin\Big(\frac12\int_\xi^{q_1(\xi)}\varphi(s)ds + \frac12\int_\xi^{q_2(\xi)}\varphi(s)ds\Big)} d\xi\Big].$$

It is easy to check that

$$\lim_{t\to\infty} C_i(t,\nu) = P_i(\nu). \tag{3.28}$$

From $P_i(\nu) < 1$, it follows

$$C_i(t,\nu) \leq 1, \quad t > T, \quad i = 1,2. \tag{3.29}$$

Condition (3.14) implies

$$q_i'(t)a_i[q_i(t)] \geq q_i'(t)b_i[q_i(t)]C_i(t,\nu), \quad i = 1,2. \tag{3.30}$$

Equalities (3.20) and (3.21) imply

$$\begin{aligned} \varphi(t) + 2k(t)\nu\alpha_2 &= 2\nu x_1 q_1'(t)b_1[q_1(t)], \\ \varphi(t) - 2k(t)\nu\alpha_1 &= 2\nu x_2 q_2'(t)b_2[q_2(t)]. \end{aligned}$$

Hence (3.7) holds for $t > T$. All conditions of Corollary 3.2 hold and the statement is proved.

Corollary 3.4 *Suppose condition (3.14) holds and the limit*

$$\lim_{t\to\infty} \int_{\rho(t)}^t B(s)ds > \frac{1}{e} \tag{3.31}$$

exists, where

$$B(t) := \begin{cases} b_1(t), & if\ q_1(t) \geq q_2(t) \\ b_2(t), & if\ q_1(t) < q_2(t). \end{cases}$$

Then all solutions of (3.2) are oscillatory.

Proof The proof is similar to the one of Theorem 3.1. We denote here $\varphi(t) := \nu Q^{'}(t)B[Q(t)]$ and

$$k(t) := \begin{cases} \frac{\varphi(t)}{\sin \int_t^{q_1(t)} \varphi(s)ds}, & if\ q_1(t) \geq q_2(t), \\ -\frac{\varphi(t)}{\sin \int_t^{q_2(t)} \varphi(s)ds}, & if\ q_1(t) < q_2(t). \end{cases}$$

Remark In most cases for obtaining explicit oscillation conditions one uses an integral $\int_{R(t)}^t$ but not $\int_{\rho(t)}^t$ (see, for example, the condition

$$\lim_{t\to\infty} \sup \int_{R(t)}^t (a_1(s) + a_2(s))\, ds > 1 \tag{3.32}$$

from [10]). It is not optimal, because conditions like (3.32) disregard all retarded arguments except the $t - R(t)$, that is, the nearest to t. But actually the oscillatory properties of the equation are determined by the farthest delay and not by the nearest one.

Example 3.1 Consider the equation

$$x'(t) + a_1 x(t-1) + a_2 x[r(t)] = 0, \quad t \ge 0, \tag{3.33}$$

where $a_1, a_2 > 0$, $r(t) \le t$, $\lim_{t\to\infty}[t - r(t)] = 0$, $r'(t) > 0$. In Theorem 3.2, put $b_1(t) := a_1(t) = a_1$, $b_2(t) := a_2(t) = a_2$. Then

$$\begin{array}{ll} B_{11} = \lim_{t\to\infty} \int_{t-1}^t a_1 ds = a_1, & B_{12} = \lim_{t\to\infty} \int_{t-1}^t a_2 ds = a_2, \\ B_{21} = \lim_{t\to\infty} \int_{r(t)}^t a_1 ds = 0, & B_{22} = \lim_{t\to\infty} \int_{r(t)}^t a_2 ds = 0. \end{array}$$

Hence system (3.20) turns into

$$\begin{gathered} -a_1 x_1 + 1 = 0 \\ \ln x_1 - a_1 x_1 - a_2 x_2 < 0 \\ \ln x_2 < 0 \end{gathered}$$

which is equivalent to the system

$$\begin{gathered} x_1 = \tfrac{1}{a_1} \\ -\tfrac{1}{a_2} \ln(e a_1) < x_2 < 1. \end{gathered}$$

This system has a solution if and only if

$$a_1 e^{a_2} > \frac{1}{e}. \tag{3.34}$$

Theorem 3.2 implies that Condition (3.34) is sufficient for the oscillation of all solutions of (3.33).

This result confirms the intuitive conjecture that the asymptotic behavior of (3.33) is close to the one of the equation

$$y'(t) + a_1 y(t-1) + a_2 y(t) = 0. \tag{3.35}$$

But it is well known that Condition (3.34) is necessary and sufficient for the oscillation of all solutions of (3.35).

Note that it is impossible to obtain this result by Corollary 3.4.

Example 3.2 Consider the equation

$$x'(t) + \frac{a_1}{t} x(\frac{t}{\mu}) + \frac{a_2}{t} x(t-\tau) = 0, \quad t \ge t_0 > 0, \tag{3.36}$$

where $\mu > 1$, $\tau > 0$, $a_1, a_2 > 0$. In Theorem 3.2, put

$$b_1(t) := a_1(t) = \frac{a_1}{t}, \quad b_2(t) := a_2(t) = \frac{a_2}{t}.$$

Then $B_{11} = a_1 \ln\mu$, $B_{12} = a_2 \ln\mu$, $B_{21} = B_{22} = 0$. System (3.20) turns into

$$\begin{gathered} -a_1 x_1 \ln\mu + 1 = 0 \\ \ln x_1 - a_1 x_1 \ln\mu - a_2 x_2 \ln\mu < 0 \\ \ln x_2 < 0 \end{gathered}$$

which is equivalent to

$$\begin{gathered} x_1 = \tfrac{1}{a_1 \ln \mu} \\ -\ln[a_1 \ln \mu] - 1 < x_2 a_2 \ln \mu \\ \ln x_2 < 0 \end{gathered}$$

and equivalent to

$$\begin{gathered} x_1 = \tfrac{1}{a_1 \ln \mu} \\ \tfrac{-\ln[a_1 \ln \mu] - 1}{a_2 \ln \mu} < x_2 < 1\,. \end{gathered}$$

This last system has a solution if and only if

$$\frac{-\ln[a_1 \ln \mu] - 1}{a_2 \ln \mu} < 1.$$

Hence the condition

$$a_1 \mu^{a_2} > \frac{1}{e \ln \mu} \tag{3.37}$$

is sufficient for the oscillation of all solutions of (3.36). Note that (3.37) does not depend on τ.

Example 3.3 Consider the equation

$$x'(t) + \frac{a_1}{t} x(\frac{t}{\mu}) + \frac{a_2}{t^\beta} x(t - \tau) = 0, \quad t \geq t_0 > 0, \tag{3.38}$$

where $a_1, a_2, \tau > 0$, $\mu > 1$, $0 \leq \beta < 1$. In Theorem 3.2, let

$$b_1(t) := a_1(t) = \frac{a_1}{t}, \quad b_2(t) := \frac{A}{t} \leq a_2(t) = \frac{a_2}{t^\beta}, t > t_0,$$

where A is an arbitrarily large and positive constant. Then $B_{11} = a_1 \ln \mu$, $B_{12} = A \ln \mu$, $B_{21} = B_{22} = 0$. One can repeat now all calculations in Example 3.2. Then (3.37) is $a_1 \mu^A > \frac{1}{e \ln \mu}$ which holds for every $a_1 > 0$ for A sufficiently large. Hence if $a_1 > 0$, $a_2 > 0$ then all the solutions of (3.38) are oscillatory.

This result is rather unexpected. Actually, for $\beta = 0$ each one of the conditions $a_1 > \frac{1}{e \ln \mu}$ and $a_2 > \frac{1}{e}$ is necessary and sufficient for the oscillation of all solutions for the "shortened" equations

$$y'(t) + \frac{a_2}{t} y(\frac{t}{\mu}) = 0$$

and

$$z'(t) + a_2 z(t-1) = 0,$$

respectively. For $\beta > 0$ the "shortened" equation

$$z'(t) + \frac{a_2}{t^\beta} z(t-1) = 0$$

has a non-oscillatory solution for every $a_2 > 0$.

Example 3.4 Consider the equation

$$x'(t) + \frac{a_1}{t \ln t} x(t^\alpha) + \frac{a_2}{t^\beta \ln t} x(t-\tau) = 0, \quad t \ge t_0 > 1, \tag{3.39}$$

where $a_1, a_2, \tau > 0$, $1 > \alpha > 0$, $1 \ge \beta \ge 0$. If $\beta = 1$ and $a_1 \alpha^{-a_2} > \frac{1}{e \ln \frac{1}{\alpha}}$ then all solutions of (3.39) are oscillatory. If $0 \le \beta < 1$ then for every $a_1 > 0$, $a_2 > 0$ all solutions of (3.39) are oscillatory. The proofs of these two statements are similar.

4 Non-oscillation criteria

The aim of this section is to show that the oscillation criteria obtained by SCM are close to be necessary. To this end we will use the following recent non-oscillation criterion.

Theorem 4.1 ([12]) *Let (a1)-(a2) hold and suppose that there exist $t_0 \ge 0$ and positive numbers x_1, x_2 such that for $t \ge t_0$ we have*

$$\begin{aligned} \ln x_1 - x_1 \textstyle\int_{r_1(t)}^t a_1(s)ds - x_2 \int_{r_1(t)}^t a_2(s)ds \ge 0 \\ \ln x_2 - x_1 \textstyle\int_{r_2(t)}^t a_1(s)ds - x_2 \int_{r_2(t)}^t a_2(s)ds \ge 0 \end{aligned} \tag{4.1}$$

Then (3.2) has a non-oscillatory solution.

Denote

$$A_{ij} := \lim_{t\to\infty} \sup \int_{r_i(t)}^t a_j(s)ds, \quad i,j = 1,2.$$

Corollary 4.1 *Let $A_{ij} < \infty$, $i, j = 1, 2$ and suppose that there exist positive numbers x_1, x_2 such that*

$$\begin{aligned} \ln x_1 - A_{11}x_1 - A_{12}x_2 > 0 \\ \ln x_2 - A_{21}x_1 - A_{22}x_2 > 0 \end{aligned} \tag{4.2}$$

Then (3.2) has a non-oscillatory solution.

Corollary 4.2 *Suppose*

$$\lim_{t\to\infty} \int_{r_2(t)}^t a_1(s)ds = \lim_{t\to\infty} \int_{r_2(t)}^t a_2(s)ds = 0 \tag{4.3}$$

and denote, as before,

$$A_{11} := \lim_{t\to\infty} \sup \int_{r_1(t)}^t a_1(s)ds, \quad A_{12} := \lim_{t\to\infty} \sup \int_{r_1(t)}^t a_2(s)ds.$$

If

$$A_{11} \exp\{A_{12}\} < \frac{1}{e}, \tag{4.4}$$

then (3.2) has a non-oscillatory solution.

Proof. We will use Corollary 4.1. Here $A_{21} = A_{22} = 0$.Then (4.2) turns into

$$\begin{gathered} \ln x_1 - A_{11}x_1 - A_{12}x_2 > 0 \\ \ln x_2 > 0 \end{gathered} \tag{4.5}$$

Put $x_1 := \frac{1}{A_{11}}$ (in case $A_{11} > 0$). Then (4.5) turns into $A_{12} < A_{12}x_2 < -1 - \ln A_{11}$. By (4.4), there exists $C : A_{12} < C < -1 - \ln A_{11}$. Therefore the pair $\{x_1, x_2\} = \{\frac{1}{A_{11}}; \frac{C}{A_{12}}\}$ will be a solution of the system (4.5) in case $A_{12} > 0$.

In case $A_{12} = 0$ Condition (4.4) turns into $A_{11} < \frac{1}{e}$ and the pair $\{x_1; x_2\}$, $x_1 = e$, $x_2 > 1$ will be a solution of (4.5). The solvability of (4.5) in case $A_{11} = 0$ is an obvious fact.

As in Example 3.1, consider (3.33), where $a_1, a_2 > 0$, $r(t) \le t$, $\lim_{t\to\infty}[t - r(t)] = 0$. We have $A_{11} = a_1$, $A_{12} = a_2$, $A_{21} = A_{22} = 0$. Then the condition $a_1 e^{a_2} < \frac{1}{e}$ implies the existence of a non-oscillatory solution of (3.33). Note that the inequality $a_1 e^{a_2} > \frac{1}{e}$ implies that all solutions of (3.33) are oscillatory.

As in Example 3.2, consider (3.36), where $\mu > 1$, $\tau > 0$, $a_1, a_2 > 0$. We have $A_{11} = a_1 \ln \mu$, $A_{12} = a_2 \ln \mu$, $A_{21} = A_{22} = 0$. Hence if the condition $a_1 \mu^{a_2} < \frac{1}{e \ln \mu}$ holds, then (3.36) has a non-oscillatory solution. If $a_1 \mu^{a_2} > \frac{1}{e \ln \mu}$ holds then all solutions of (3.36) are oscillatory.

5 Discussion

In this section we compare the results obtained in this paper with some known results.

Statement 5.1 [1] Let $r_i(t) := t - \tau_i$, $\tau_i > 0$, and $\lim_{t\to\infty} \inf \int_t^{t+\tau_i} a_i(s)ds > 0$. Suppose that at least one of the following three conditions holds:

1. $p_{ij}^* = \lim_{t\to\infty} \inf \int_{t-\tau_i}^t a_j(s)ds > 1/e$, for some i, j;
2. $\left[\prod_{i=1}^n \sum_{j=1}^n p_{ij}^*\right]^{1/n} > 1/e$;
3. $\sum_{i=1}^n p_{ij}^* + 2\sum_{i<j}^n (p_{ij}^* p_{ji}^*)^{1/2} > n/2$, for some j.

Then all solutions of (1.2) are oscillatory.

Statement 5.2 [10] Let $a_i(t) > 0$, $0 < t - r_i(t) < \sigma$ and

$$\lim_{t\to\infty} \inf \sum_{i=1}^n a_i(t)(t - r_i(t)) > \frac{1}{e}.$$

Then all solutions of (1.2) are oscillatory.

Statement 5.3 [8] Assume that there exist indices $i_l \in \{1,\dots,n\}$ such that

$$\liminf_{t\to\infty}(t - r_{i_l}(t)) > 0, \quad \liminf_{t\to\infty}\sum_{i=1}^{n} a_{i_l}(t) > 0. \tag{5.1}$$

If

$$\liminf_{t\to\infty}\Big[\inf_{\lambda>0}\Big\{\frac{1}{\lambda}\sum_{i=1}^{n} a_i(t)\exp\{\lambda(t-r_i(t))\}\Big\}\Big] > 1,$$

then all solutions of (1.2) are oscillatory.

Corollary 5.1 *Assume that (5.1) holds and*

$$\liminf_{t\to\infty}\Big\{\big[\prod_{i=1}^{n} a_i(t)\big]^{1/n}\big[\sum_{i=1}^{n}(t-r_i(t))\big]\Big\} > \frac{1}{e}. \tag{5.2}$$

Then all solutions of (1.2) are oscillatory.

Statement 5.4 [19] *Suppose that for sufficiently large T and for some $\lambda > 0$,*

$$-\lambda + \sup_{t\geq T}\sum_{i=1}^{n} a_i(t)\exp\{\lambda(t-r_i(t))\} \leq 0.$$

Then there exists a non-oscillatory solution of (1.2).

Statement 5.5 [17] Suppose there exist a nonempty set $I \subset \{1,\dots,n\}$ and constants τ_0, τ_1, $\tau_0 > \tau_1 > 0$, such that

$$t - r_i(t) \geq \tau_0, \quad i \in I, \quad \liminf_{t\to\infty}\int_t^{t+\tau_1}\sum_{t\in I} a_i(s)ds > 0,$$

$$\limsup_{t\to\infty}\Big\{\max_k \int_{r_k(t)}^{t}\sum_{k=1}^{n} a_k(s)ds\Big\} < \infty. \tag{5.3}$$

Moreover, assume that for all $\lambda > 0$ and some $T > 0$,

$$-\lambda + \inf_{t\geq T}\frac{\sum_{k=1}^{n} a_k(t)\exp\Big\{\lambda\int_{r_k(t)}^{t}\sum_{i=1}^{n} a_i(s)ds\Big\}}{\sum_{k=1}^{n} a_k(t)} > 0.$$

Then all solutions of (1.2) are oscillatory.

Corollary 5.2 *Suppose (5.3) hold and*

$$\liminf_{t\to\infty}\frac{\sum_{k=1}^{n} a_k(t)\int_{r_k(t)}^{t}\sum_{i=1}^{n} a_i(s)ds}{\sum_{k=1}^{n} a_k(t)} > \frac{1}{e}.$$

Then all solutions of (1.2) are oscillatory.

Statement 5.6 [17] Suppose there exist $\lambda > 0$ and a sufficiently large T such that

$$-\lambda + \sup_{t \geq T} \frac{\sum_{k=1}^{n} a_k(t) \exp\left\{\lambda \int_{r_k(t)}^{t} \sum_{i=1}^{n} a_i(s)ds\right\}}{\sum_{k=1}^{n} a_k(t)} \leq 0.$$

Then there exists a non-oscillatory solution of (1.2).

Statement 5.7 [18] Suppose that $t - r_i(t) = \tau_i > 0$ and for every $\lambda > 0$ and sufficiently large T,

$$-\lambda + \inf_{t \geq T} \min_{j=1,\dots,n} \sum_{k=1}^{n} p_{jk}(t) e^{\lambda \tau_k} > 0,$$

where $p_{jk}(t) = \frac{1}{\tau_j} \int_{t-\tau_j}^{t} a_k(s)ds$. Then all solutions of (1.2) are oscillatory.

Corollary 5.3 *If*

$$\liminf_{t \to \infty} \min_{j=1,\dots,n} \sum_{k=1}^{n} p_{jk}(t) > \frac{1}{e}$$

then all solutions of (1.2) are oscillatory.

Statement 5.8 [18] Suppose that $t - r_i(t) = \tau_i > 0$ and there exist $\lambda > 0$ and a sufficiently large T such that

$$-\lambda + \sup_{t \geq T} \max_{j=1,\dots,n} \sum_{k=1}^{n} p_{jk}(t) e^{\lambda \tau_k} \leq 0.$$

Then there exists a non-oscillatory solution of (1.2).

Denote $\tau_k(t) = t - r_k(t)$.

Statement 5.9 [6] Suppose (5.1) holds and for every $\lambda > 0$ and $i = 1, \dots, n$,

$$\liminf_{t \to \infty} \frac{1}{\lambda \tau_i(t)} \sum_{k=1}^{n} \int_{t}^{t+\tau_k(t)} a_k(s) e^{\lambda \tau_k(t)} ds > 1.$$

Then all solutions of (1.2) are oscillatory.

Corollary 5.4 *Suppose (5.1) holds and for every* $i = 1, \dots, n$,

$$\liminf_{t \to \infty} \frac{1}{\tau_i(t)} \sum_{k=1}^{n} \int_{t}^{t+\tau_k(t)} a_k(s) \tau_k(t) ds > \frac{1}{e}.$$

Then all solutions of (1.2) are oscillatory.

In the recent papers [14, 15] Li has generalized some results of Erbe and Kong [6]. An interesting oscillation criterion was obtained in [12]. Unfortunately, in this paper there is no example illustrating the strength of this result. Note also the original paper of Nadareishvili [16], where the author does not assume that $a_k(t)$ are nonnegative functions and $r_k(t) \leq t$.

Most of the results listed above generalized the well-known assertion on the oscillation of autonomous equations (see Corollary 3.3). All these results contained some restrictions on the parameters of the equations. For example, in [1, 18], the authors consider equations with constant delays, in [10] with bounded delays, and in [6, 8, 12] under condition (5.1).

These conditions are rather restrictive. Indeed, because of these restrictions one cannot get oscillation criteria for equations (3.36), (3.38) and (3.39). On the contrary, the results of this paper give an almost full description of the oscillatory properties of these equations. For example, if $a_1\mu^{a_2} > \frac{1}{e\ln\mu}$, then all solutions of (3.36) are oscillatory. If $a_1\mu^{a_2} < \frac{1}{e\ln\mu}$, then there exists a non-oscillatory solution of (3.36).

The oscillation results obtained in this paper are also not universal. To show it, consider the following equation

$$x'(t) + \frac{a_1}{t^{3/2}}x(\frac{t}{\mu}) + a_2x(t-\tau) = 0, \quad t \geq t_0 > 0, \tag{5.4}$$

where $a_1, a_2, \tau > 0$, $\mu > 1$. For this equation one cannot use the trick applied in Ex.3.3 for the similar (3.38), since the condition $B_{11} + B_{22} \neq 0$ in Theorem 3.2 does not hold for (5.4).

Apply now the Corollary of Statement 5.3. For $n = 2$ Condition (5.2) has the form

$$[a_1(t)a_2(t)]^{1/2}(\tau_1(t) + \tau_2(t)) > \frac{1}{e},$$

for $t \geq T$, where T is sufficiently large. Rewrite this condition for (5.4):

$$\Big[\frac{a_1a_2}{t^{3/2}}\Big]^{1/2}((1-1/\mu)t + \tau) > \frac{1}{e}. \tag{5.5}$$

It is obvious that (5.5) holds for $a_1 > 0, a_2 > 0$ and t sufficiently large. Hence the Corollary of Statement 5.3 implies that for every $a_1 > 0$ and $a_2 > 0$ all solutions of (5.4) are oscillatory.

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