Electronic Journal of Differential Equations, Vol. **2001**(2001), No. 35, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
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# Some observations on the first eigenvalue of the p-Laplacian and its connections with asymmetry *

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**Abstract**

In this work, we present a lower bound for the first eigenvalue of the $p$-Laplacian on bounded domains in $\mathbb{R}^2$. Let $\lambda_1$ be the first eigenvalue and $\lambda_1^*$ be the first eigenvalue for the ball of the same volume. Then we show that $\lambda_1 \geq \lambda_1^*(1 + C\alpha(\Omega)^3)$, for some constant $C$, where $\alpha$ is the asymmetry of the domain $\Omega$. This provides a lower bound sharper than the bound in Faber-Krahn inequality.

## 1 Introduction

Let $\Omega \subset \mathbb{R}^n$, be a bounded domain. For $1 < p < \infty$, let

$$\lambda_1 = \lambda_1(p, \Omega) = \inf \frac{\int_\Omega |Du|^p}{\int_\Omega |u|^p},$$

where the infimum is taken over all $u \in W_0^{1,p}(\Omega)$, $u \neq 0$. It is well known that $\lambda_1 = \lambda_1(\Omega, p) > 0$ and a non-zero minimizer, which we continue to call as $u = u(p, \Omega)$, exists and satisfies

$$\operatorname{div}(|Du|^{p-2}Du) + \lambda_1 |u|^{p-2}u = 0, \quad \text{in } \Omega, \tag{1.1}$$

where $u \in W_0^{1,p}(\Omega)$. The operator $\operatorname{div}(|Du|^{p-2}Du)$ is the $p$-Laplacian and this is the usual Laplacian when $p = 2$. For $p \neq 2$, this is a quasi-linear and a degenerate elliptic operator. The equation in (1.1) is to be interpreted in the weak sense, i. e.,

$$\int_\Omega |Du|^{p-2}Du \cdot D\psi = \lambda_1 \int_\Omega |u|^{p-2}u\psi, \ \ \forall\, \psi \in W_0^{1,p}(\Omega).$$

We refer to $\lambda_1$ as the first eigenvalue and $u$ as the first eigenfunction of the $p$-Laplacian on $\Omega$. It is well known that $\lambda_1$ is simple and $u$ has one sign [2, 10, 11]. The first eigenvalue is also known to be isolated [10]. Moreover, if $\Omega$ is a ball then

* *Mathematics Subject Classifications:* 35J60, 35P30.
*Key words:* Asymmetry, De Giorgi perimeter, p-Laplacian, first eigenvalue, Talenti's inequality.

Submitted September 3, 2000. Published May 16, 2001.

$u$ is radial, decreasing and has only one critical point. It will be useful to note that $u$ is $L^\infty(\Omega)$ [11]. It is also quite well known that $u$ is $C^{1,\alpha}_{loc}$ in $\Omega$ [5, 12, 18]. See [2] for a more detailed discussion of matters related to the regularity of $u$. It should also be pointed out that unlike the case of the Laplacian, i. e., $p = 2$, a complete characterization of the set of critical points of $u$, when $p \neq 2$, is still unknown. This fact or lack thereof becomes particularly important when working with level sets of $u$. The boundaries of such sets need not be smooth thus necessitating the use of the DeGiorgi perimeter. We discuss this further in section 2. Also see [1, 2, 16, 17].

Let $D^*$ denote the symmetrized domain for an open set $D$, i .e., $D^*$ is the ball, centered at the origin, with volume equal to that of $D$. Let $\lambda_1^* = \lambda_1(\Omega^*)$; then the Faber-Krahn inequality states that $\lambda_1 \geq \lambda_1^*$, where equality holds if and only if $\Omega$ is a ball [2]. Our attempt, in this work, will be to characterize this lower bound for $\lambda_1$, for $1 < p < \infty$, in terms of asymmetry. The notion of asymmetry, which was introduced in [9], is a measure of how close a set is to being a ball. More precisely, if $D$ is a compact set in $\mathbb{R}^n$, $n \geq 2$, then the asymmetry of $D$, denoted by $\alpha(D)$, is defined to be

$$\alpha(D) = \inf_x \frac{\operatorname{vol}(D \backslash B(x,R))}{\operatorname{vol}(D)}. \tag{1.2}$$

Here vol stands for volume, $B(x, R)$ is the ball centered at $x$, radius $R$, such that the volume of $B(x, R)$ is the same as that of $D$.

In [1, 7, 8, 9] lower bounds for capacities, for planar domains, were obtained in terms of asymmetry while in [3], an analogous upper bound for the Green's function was derived. The works in [9, 13] address the issue of the first eigenvalue of the Laplacian and present a sharper version of the Faber-krahn inequality in terms of deficiencies. This work generalizes the estimate in [9] to the case of the $p$-Laplacian on planar domains. We thank the referee whose comments have helped improved the exposition of this work. We also thank Juan Manfredi for his encouragement and interest in this work. We are also highly appreciative of Tom Salisbury of Department of Mathematics, York University, who kindly extended texing facilities to us.

## 2 Statement of the main result

For $D \subset \mathbb{R}^2$, let $|D|$ denote the area of a set $D$ and $\partial D$ denote its boundary. Let $L(\partial D)$ denote the length i.e. L is the Hausdorff 1-dimensional measure if $\partial D$ is smooth and the De Giorgi perimeter otherwise. From here on $\Omega$ will be a bounded domain in $\mathbb{R}^2$ with $\partial\Omega$ a finite union of rectifiable curves. Let $\alpha = \alpha(\Omega)$ denote the asymmetry of $\Omega$, $u = u(p, \Omega)$ be the first eigenfunction of the $p$-Laplacian, $1 < p < \infty$, and $\lambda_1$ be the first eigenvalue. We will take the first eigenfunction $u > 0$, we will also assume throughout that

$$\int_\Omega u^p = 1.$$

For $0 \le t \le \sup u$, set

$$\Omega_t = \{x \in \Omega : u(x) > t\}, \text{ and } \mu(t) = |\Omega_t|.$$

Note that $\mu(t)$ is decreasing and right continuous. It is easy to show that $\mu(t)$ is continuous if and only if $|\{u = t\}| = 0$. Clearly, $\mu(t)$ has at most countably many discontinuities. Since $u$ is only known to be $C^{1,\alpha}_{loc}$, it is not clear that $\mu(t)$ is continuous every where when $p \neq 2$ [1, 2, 16]. Let $u^*$ be the non-increasing rearrangement (Schwarz symmetrization) of $u$, defined as follows. First set $u^{\#}(a) = \inf\{t > 0; \mu(t) < a\}$. Let $(x, y)$ denote the coordinate of a point in $\Omega^*$. For such a point define $u^*(x, y) = u^{\#}(\pi(x^2+y^2)) = u^*(r)$, where $r = \sqrt{x^2 + y^2}$. By $\lambda_1^*$ we will mean $\lambda_1(\Omega^*)$, lastly set $M = \sup_\Omega u$.

We now state the main theorem of the paper.

**Theorem 2.1** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $\alpha = \alpha(\Omega)$ be its asymmetry, then there exists a constant $C > 0$, independent of $\Omega$, such that*

$$\lambda_1(\Omega) \ge \lambda_1(\Omega^*)(1 + C\alpha^3). \tag{2.1}$$

We adapt the method developed in [1, 3, 7] to achieve our goal, i. e., we characterize the propagation of asymmetry $\alpha$ via the level sets of $u$. This is expressed in terms of the isoperimetric inequality. See Lemmas 3.3 and 3.5 for a more precise statement. Our result relies on several lemmas proven in Section 3 and the proof of the Theorem appears in Section 4. We mention that we make considerable use of the co-area formula in our work. In this context we refer to [4, 6]. The reader may find some overlap between this work and [9] however we believe some aspects of our work may be of independent interest. Lastly, we are unable to determine whether or not the third power appearing in (2.2) is optimal. However, in the case of the Laplacian it has been conjectured that the above Theorem holds with the second power and if true, it would then be optimal [3, 13].

# 3 Preliminaries

In this section we present five lemmas which will lead to the proof of Theorem 2.1. For compactness of our presentation, we take $|\Omega| = 1$.

**Lemma 3.1** *Let $u(x)$ be a solution of (1.1); set $h(t) = \int_{\Omega_t} |Du|^p$. Then $h(t)$ is convex in $t$ and for $0 < t < M$,*

$$\lambda_1\Big(1 - t\int_\Omega u^{p-1}\Big) \le h(t) \le \lambda_1\,(1 - t/M)\int_{\Omega_t} u^p. \tag{3.1}$$

**Proof.** A proof of this lemma can be worked by using the co-area formula. However, we will provide a proof which uses appropriate test functions (also see [2]). Recall the weak formulation in Section 1, i. e.,

$$\int_\Omega |Du|^{p-2}Du \cdot D\psi = \lambda_1 \int_\Omega u^{p-1}\psi \tag{3.2}$$

where $\psi \in W_0^{1,p}(\Omega)$. Using the test function $(u-t)^+$ in (3.2), we find that

$$h(t) = \lambda_1 \int_{\Omega_t} u^{p-1}(u-t).$$

We now make some observations which will prove useful later.

For $\delta > 0$, $t \le u < t+\delta$ in $\Omega_t \backslash \Omega_{t+\delta}$. Then

$$\begin{aligned} h(t+\delta) - h(t) &= \lambda_1 \Big\{ \int_{\Omega_{t+\delta}} u^p - (t+\delta)u^{p-1} - \int_{\Omega_t} (u^p - tu^{p-1}) \Big\} \\ &= \lambda_1 \Big\{ -\int_{\Omega_t \backslash \Omega_{t+\delta}} u^p + t\int_{\Omega_t \backslash \Omega_{t+\delta}} u^{p-1} - \delta \int_{\Omega_{t+\delta}} u^{p-1} \Big\}. \\ &= \lambda_1 \Big\{ \int_{\Omega_t \backslash \Omega_{t+\delta}} u^{p-1}(t-u) - \delta \int_{\Omega_{t+\delta}} u^{p-1} \Big\}. \end{aligned}$$

It follows that

$$\frac{h(t+\delta) - h(t)}{\delta} \le -\lambda_1 \int_{\Omega_{t+\ delta}} u^{p-1}. \tag{3.3}$$

Now rearranging the above expression on the right hand side, we also see that

$$\begin{aligned} & h(t+\delta) - h(t) \qquad\qquad (3.4) \\ &= \lambda_1 \Big\{ -\int_{\Omega_t \backslash \Omega_{t+\delta}} u^p + t\int_{\Omega_t \backslash \Omega_{t+\delta}} u^{p-1} - \delta \int_{\Omega_t} u^{p-1} + \delta \int_{\Omega_t \backslash \Omega_{t+\delta}} u^{p-1} \Big\} \\ &= \lambda_1 \Big\{ \int_{\Omega_t \backslash \Omega_{t+\delta}} u^{p-1}(\delta + t - u) - \delta \int_{\Omega_t} u^{p-1} \Big\} \\ &\ge -\lambda_1 \delta \int_{\Omega_t} u^{p-1}. \end{aligned}$$

Clearly

$$\frac{h(t+\delta) - h(t)}{\delta} \ge -\lambda_1 \int_{\Omega_t} u^{p-1}. \tag{3.5}$$

A similar argument also yields

$$-\lambda_1 \int_{\Omega_{t-\delta}} u^{p-1} \le \frac{h(t) - h(t-\delta)}{\delta} \le -\lambda_1 \int_{\Omega_t} u^{p-1}.$$

Clearly, (3.3), (3.4) and the foregoing imply that

$$h'(t) = -\lambda_1 \int_{\Omega_t} u^{p-1} \ \text{ a. e. t.}$$

Equality will hold at every value of $t$ iff $|\{u = t\}| = 0$, i. e., iff $\mu(t)$ is continuous for all $t$. However, this is not known for $p \ne 2$. In this context also see [2, 16, 17]. But since $\mu(t)$ is decreasing, equality holds except on a countable ( possibly finite) set. Note that the right continuity of $\mu(t)$ does show that right hand derivative of $h$ exists at every $t$ and equals $-\lambda_1 \int_{\Omega_t} u^{p-1}$.

Now the inequalities (3.3) and (3.4) clearly imply that $h(t)$ is a convex function. If $\theta_1 < \theta < \theta_2$; then the following two inequalities hold, namely,

$$\frac{h(\theta) - h(\theta_1)}{\theta - \theta_1} \le -\lambda_1 \int_{\Omega_\theta} u^{p-1} \le \frac{h(\theta_2) - h(\theta)}{\theta_2 - \theta}. \tag{3.6}$$

Using the convexity of $h(t)$, we may now find lower and upper bounds for $h$. Noting that $h(0) = \lambda_1$ and $\mu(t)$ is right continuous, we see that (3.5) yields

$$h(t) \le h(0) - \lambda_1 t \int_{\Omega_t} u^{p-1} = \lambda_1 \left(1 - t \int_{\Omega_t} u^{p-1}\right),$$

and

$$h(t) \ge \lambda_1 \left(1 - t \int_{\Omega} u^{p-1}\right). \tag{3.7}$$

Clearly, convexity of $h(t)$ on $[0, M]$, its continuity at $t = 0$ and the facts $h(0) = \lambda_1$ and $h(M) = 0$ imply the easy inequality

$$h(t) \le \lambda_1(1 - t/M) \tag{3.8}$$

However, a simple argument provides us with a better upper bound for $h(t)$. Notice that

$$\frac{t}{M} \int_{\Omega_t} u^p \le t \int_{\Omega_t} u^{p-1} \frac{u}{M} \le t \int_{\Omega_t} u^{p-1}.$$

Thus

$$\int_{\Omega_t} u^{p-1}(u - t) = \int_{\Omega_t} u^p - t \int_{\Omega_t} u^{p-1} \le (1 - t/M) \int_{\Omega_t} u^p.$$

Clearly,

$$h(t) \le \lambda_1(1 - t/M) \int_{\Omega_t} u^p. \tag{3.9}$$

Putting together (3.6) and (3.8), we get the statement of the lemma. ♢

We now provide a simple upper bound for $u$.

**Lemma 3.2** *Let $u$ solve (1.1) and $\mu(t)$ be as defined before, then*

$$t \le \left(\frac{\lambda_1}{(4\pi)^{p/2}}\right)^{1/(p-1)} \left(1 - \mu(t)^{(p^2-2p+2)/(2p(p-1))}\right) \left(\frac{p^2 - 2p + 2}{2p(p-1)}\right). \tag{3.10}$$

*In particular,* $M(p) = \sup_\Omega u \le \left(\frac{\lambda_1}{(4\pi)^{p/2}}\right)^{1/(p-1)} \left(\frac{2p(p-1)}{p^2-2p+2}\right)$.

**Proof.** We will use Talenti's inequality [16]. Recall that $\int_\Omega u^p = 1$. Then for a. e. $t$,

$$\begin{aligned}(4\pi\mu(t))^{p/2(p-1)} &\leq -\mu(t)'\Big(-\frac{d}{dt}\int_{\Omega_t}|Du|^p\Big)^{1/(p-1)} \qquad (3.11)\\ &= -\mu(t)'\Big(-\lambda_1\frac{d}{dt}\int_{\Omega_t}u^{p-1}(u-t)\Big)^{1/(p-1)}\end{aligned}$$

Using Holder's inequality and the fact that $\int_\Omega u^p = 1$, (3.10) yields

$$\begin{aligned}\Big(\frac{(4\pi\mu(t))^{p/2}}{\lambda_1}\Big)^{1/(p-1)} &\leq -\mu(t)'\Big(\int_{\Omega_t}u^{p-1}\Big)^{1/(p-1)}\\ &\leq -\mu(t)'\Big(\int_{\Omega_t}u^p\Big)^{1/p}\mu(t)^{1/p(p-1)}\\ &\leq -\mu(t)'\mu(t)^{1/p(p-1)}\end{aligned}$$

Thus

$$\Big((4\pi)^{p/2}/\lambda_1\Big)^{1/(p-1)}\mu(t)^{(p^2-2)/(2p(p-1))} \leq -\mu(t)'.$$

Hence,

$$\left(\frac{(4\pi)^{p/2}}{\lambda_1}\right)^{1/(p-1)} \leq \Big(-\mu(t)^{(p^2-2p+2)/(2p(p-1))}\Big)'\left(\frac{2p(p-1)}{p^2-2p+2}\right).$$

But for all $p > 1$ it is clear that $p^2 - 2p + 2 > 0$ and $-\mu(t)^{(p^2-2p+2)/(2p(p-1))}$ is increasing and right continuous. Integrating from 0 to $t$, we find that

$$t \leq \left(\frac{2p(p-1)}{p^2-2p+2}\right)\left(\frac{\lambda_1}{(4\pi)^{p/2}}\right)^{1/(p-1)}\Big(1-\mu(t)^{(p^2-2p+2)/(2p(p-1))}\Big)$$

Thus we get the estimate in the statement of the lemma. ◇

**Remark.** Lemma 3.2 leads to an upper bound under the assumption $\lambda_1 \leq 2\lambda_1^*$.

The next three lemmas are crucial to proving the Theorem, they indicate how asymmetry enters into the calculations. In Lemmas 3.3 and 3.5, $k$ stands for a positive constant in $(0, 1/100)$, whose exact value will determined in Section 4. The basic approach to proving the Theorem is along the lines of [1, 3, 7] and this motivates the following lemma.

**Lemma 3.3** *Let $u$ solve (2.1), $\alpha$ be the asymmetry of $\Omega$. Assume that there exists a $T$ with $0 < T < M$ such that for a. e. $t \in [0, T]$, there exists a constant $k, 0 < k < 1/100$, with the property that*

$$L(\partial\Omega_t)^2 \geq 4\pi(1 + k\alpha^2)\mu(t).$$

*Then*

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*)\left(1 + \frac{Tp}{8M}k\alpha^2\right). \qquad (3.12)$$

**Proof.** Our starting point will be Lemma 2 in [16] and the outline of the proof is quite similar to the method used in Lemma 1 in [17]. From [16, Lemma 2] for a. e. $t$,

$$L(\partial\Omega_t)^{p/(p-1)} \le -\mu(t)' \Big( -\frac{d}{dt}\int_{\Omega_t} |Du|^p \Big)^{1/(p-1)},$$

where $L(\partial\Omega_t)$ is the De Giorgi perimeter. Employing the isoperimetric inequality,

$$(4\pi\mu(t))^{p/2(p-1)} \le -\mu(t)' \Big( -\frac{d}{dt}\int_{\Omega_t} |Du|^p \Big)^{1/(p-1)} \quad \text{a. e. t.}$$

Employing the hypothesis of the lemma, we get for a. e. $t$,

$$\big(4\pi(1+k\alpha^2)\mu(t)\big)^{p/2(p-1)} \le -\mu(t)' \Big( -\frac{d}{dt}\int_{\Omega_t} |Du|^p \Big)^{1/(p-1)}$$

Therefore,

$$-\frac{d}{dt}\int_{\Omega_t} |Du|^p \ge \frac{(4\pi(1+k\alpha^2)\mu(t))^{p/2}}{(-\mu(t)')^{p-1}}. \tag{3.13}$$

Recall that from Lemma 3.1, $\int_{\Omega_t}|Du|^p$ is convex and hence continuous on $[0, M]$. Since $-\int_{\Omega_t}|Du|^p$ is non-decreasing integrating (3.12) from 0 to $T$, we obtain

$$\int_{\Omega}|Du|^p - \int_{\Omega_T}|Du|^p \ge (1+k\alpha^2)^{p/2}\int_0^T \frac{4\pi\mu(s))^{p/2}}{(-\mu(s)')^{p-1}}ds$$

Rewriting and using that $p$-Dirichlet integrals diminish under symmetrization, we obtain

$$\lambda_1(\Omega) \quad \ge \quad \int_{\Omega_T^*}|Du^*|^p + (1+k\alpha^2)^{p/2}\int_0^T \frac{(4\pi\mu(s))^{p/2}}{(-\mu(s)')^{p-1}}ds \tag{3.14}$$

where $u^*$ is the Schwartz non-increasing radial rearrangement of $u$. Recall that $u$ is $C^{1,\alpha}_{loc}(\Omega)$, hence $u^*$ is locally Lipschitz continuous. Since $u^*(x) = u^*(|x|)$, we define $r(t) = \sqrt{\mu(t)/\pi}$. Clearly, $u^*(r(t)) = t$ where $r = |x|$. Thus the co-area formula yields

$$\int_{\Omega_t^*}|Du^*|^p = \int_t^\infty \Big( \int_{\partial\Omega_s^*}|Du^*|^{p-1}\Big) ds = \int_t^\infty |Du^*|^{p-1}(r(s))L(\partial\Omega_s^*)ds, \tag{3.15}$$

where $r(s) = \sqrt{\mu(s)/\pi}$. Thus for a. e. $t$,

$$\frac{d}{dt}\int_{\Omega_t^*}|Du^*|^p = -|Du^*|^{p-1}(r(t))L(\partial\Omega_t^*), \tag{3.16}$$

where $r = \sqrt{|\Omega_t^*|/\pi} = \sqrt{\mu(t)\pi}$. Note that the above also shows that $\int_{\Omega_t^*}|Du^*|^p$ is Lipschtiz continuous in $t$. However, using polar coordinates we may express

$$\int_{\Omega_t^*}|Du^*|^p = 2\pi\int_0^{\sqrt{\mu(t)/\pi}} |Du^*|^p r\,dr.$$

Differentiating the above and using (3.15)

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t^*}|Du^*|^p &= \sqrt{\pi}|Du^*|^p(t)r(t)\mu(t)'/\sqrt{\mu(t)}\\ &= |Du^*|^p\mu(t)' = -|Du^*|^{p-1}L(\partial\Omega_t^*).\end{aligned}$$

Simplifying we obtain for a. e. $t$ that

$$|Du^*| = -\frac{L(\Omega_t^*)}{\mu(t)'}. \tag{3.17}$$

Employing (3.16) in the co-area formula (3.14) results in

$$\int_{\Omega_t^*}|Du^*|^p = \int_t^\infty \frac{(4\pi\mu(s))^{p/2}}{(-\mu(s)')^{p-1}}ds. \tag{3.18}$$

Now since $k < 1/100$ and $\alpha \le 1$, clearly $(1+k\alpha^2)^{p/2} \ge 1 + kp\alpha^2/4$, for all $1 < p < \infty$. Using Lemmas 3.1, 3.2, (3.13), (3.17), we see that

$$\begin{aligned}\lambda_1(\Omega) &\ge \int_{\Omega_T^*}|Du^*|^p + (1+k\alpha^2)^{p/2}\int_{\Omega^*\setminus\Omega_T^*}|Du^*|^p\\ &\ge \int_{\Omega_T^*}|Du^*|^p + (1+kp\alpha^2/4)\int_{\Omega^*\setminus\Omega_T^*}|Du^*|^p\\ &= (1+kp\alpha^2/4)\int_{\Omega^*}|Du^*|^p - (kp\alpha^2/4)\int_{\Omega_T^*}|Du^*|^p\\ &\ge \lambda_1(\Omega^*)(1+kp\alpha^2/4) - (kp\alpha^2/4)\int_{\Omega_T}|Du|^p\\ &\ge \lambda_1(\Omega^*)(1+kp\alpha^2/4) - \lambda_1(\Omega)kp\alpha^2(1-T/M)/4. \end{aligned}\tag{3.19}$$

Simplifying (3.18), we get the statement of the lemma, namely,
$\lambda_1(\Omega) \ge \lambda_1(\Omega^*)\left(1 + kp\alpha^2 T/8M\right)$. ♢

Next we prove a relationship between $\lambda_1$ and the level set $\Omega_t$.

**Lemma 3.4** *With $u$ as before, $t \ge 0$ and $|\Omega| = 1$, we have*

$$\lambda_1(\Omega) \ge \frac{\lambda_1(\Omega^*)(1-t)^{p-1}}{|\Omega_t|^{(p-1)/2}}.$$

**Proof.** We start with the weak formulation for $u$ i.e.,

$$\int_\Omega |Du|^{p-2}Du\cdot D\phi = \lambda_1(\Omega)\int u^{p-1}\phi,\ \ \forall\ \ \phi\in W_0^{1,p}(\Omega). \tag{3.20}$$

Let $\Omega_t = \bigcup_{i=1}^\infty C_i$, where $C_i$'s are pairwise disjoint components of $\Omega_t$. In Remark 3.2 we will show that there can only be finitely many components of $\Omega_t$. Thus $\Omega_t = \bigcup_{i=1}^{n_t} C_i$. Setting $\phi_i = (u-t)_+$ in $C_i$ and zero elsewhere, (3.19) yields

$$\int_{C_i}|Du|^p = \lambda_1(\Omega)\int_{C_i}u^{p-1}(u-t)_+,\ \ \forall\ \ i=1,2, \tag{3.21}$$

Also, for each $i = 1, 2, \dots, \phi_i$ is a trial function for the minimum problem (i.e., the variational formulation of the eigenvalue problem) on $C_i$. Thus

$$\int_{C_i} |Du|^p \geq \lambda_1(C_i) \int_{C_i} (u-t)_+^p. \tag{3.22}$$

Employing Hölder's inequality, (3.20) together with (3.21) yields

$$\begin{aligned} \int_{C_i} |Du|^p &\leq \lambda_1(\Omega) \left( \int_{C_i} u^p \right)^{(p-1)/p} \left( \int_{C_i} (u-t)_+^p \right)^{1/p} \\ &\leq \lambda_1(\Omega) \left( \int_{C_i} u^p \right)^{(p-1)/p} \left( \frac{1}{\lambda_1(C_i)} \int_{C_i} |Du|^p \right)^{1/p}. \end{aligned}$$

Thus

$$\int_{C_i} |Du|^p \leq \frac{\lambda_1(\Omega)^{p/(p-1)}}{\lambda_1(C_i)^{1/(p-1)}} \int_{C_i} u^p.$$

Summing over $i$,

$$\int_{\Omega_t} |Du|^p \leq \lambda_1^{p/(p-1)}(\Omega) \sum_i \frac{1}{\lambda_1(C_i)^{1/(p-1)}} \int_{C_i} u^p. \tag{3.23}$$

Let $L = \inf\ \{\lambda_1(C_1), \lambda_1(C_2), \dots\}$. Let $C$ be an appropriate set in $\{C_i\}$ such that $\lambda_1(C) = L$. Then, (3.22) yields

$$\int_{\Omega_t} |Du|^p \leq \frac{\lambda_1^{p/(p-1)}(\Omega)}{\lambda_1(C)^{1/(p-1)}} \int_{\Omega_t} u^p.$$

Rearranging terms and employing Lemma 3.1,

$$\begin{aligned} \lambda_1(\Omega)^{p/(p-1)} &\geq \lambda_1(C)^{1/(p-1)} \left( \int_{\Omega_t} |Du|^p \right) \left( \int_{\Omega_t} u^p \right)^{-1} \\ &\geq \lambda_1(C)^{1/(p-1)} \lambda_1(\Omega) \left( 1 - t \int_\Omega u^{p-1} \right) \left( \int_{\Omega_t} u^p \right)^{-1}. \end{aligned} \tag{3.24}$$

Now observe that

$$\int_\Omega u^{p-1} \leq \left( \int_\Omega u^p \right)^{(p-1)/p} |\Omega|^{1/p} \leq \left( \int_\Omega u^p \right)^{(p-1)/p} = 1.$$

Clearly (3.23) yields,

$$\lambda_1(\Omega) \geq \lambda_1(C)(1-t)^{p-1} \left( \int_{\Omega_t} u^p \right)^{-(p-1)}. \tag{3.25}$$

Now $\lambda_1(C) \geq \lambda_1(C^*) = \lambda_1(\Omega^*)/|C|^{p/2}$. The latter follows from a scaling argument ( noting that $|\Omega^*| = 1$ and $\Omega \subset \mathbb{R}^2$); also observe that $|C| \leq |\Omega_t|$. Replacing $\lambda_1(C)$, in (3.24), by this lower bound and $|C|$ by $|\Omega_t|$ one is lead to

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*) \frac{\left[ (1-t) \left( \int_{\Omega_t} u^p \right)^{-1} \right]^{p-1}}{|\Omega_t|^{p/2}} \geq \lambda_1(\Omega^*) \frac{(1-t)^{p-1}}{|\Omega_t|^{p/2}}.$$

We will apply Lemma 3.4 in the case when $t << 1$. See Section 4. ♢

**Remark 3.2** For every $0 < t < M$, let $\Omega_t = \bigcup_i C_i$ where $C_i = C_i(t)$ is a component of $\Omega_t$. Then $\{C_i\}$ is a finite family.

**Proof.** Suppose that this family is infinite. Let $x_{M_i} \in C_i$ be such that $u(x_{M_i}) = M$. Let $x_i \in \partial C_i$ be such that $|x_i - x_{M_i}| = \text{dist}(x_{M_i}, \partial C_i)$. Clearly, $u(x_i) = t$. Since $\bar{\Omega}_t$ is compact, the local regularity results in [5, 12, 18] imply that $\sup_{\Omega_t}|Du| \le K(t) < \infty$. Thus

$$|u(x_i) - u(x_{M_i})| = |M - t| \le K(t)|x_i - x_{M_i}|.$$

Since $C_i$'s are infinite and $\sum_i |C_i| = |\Omega_t| < \infty$, $|C_i| \to 0$ as $i \to \infty$. This means that the right side of the above inequality goes to zero leading to a contradiction. $\diamondsuit$

We now study the situation when Lemma 3.3 fails to hold, namely, that for some $t$, $L(\partial\Omega_t)^2 < 4\pi(1 + k\alpha^2)|\Omega_t|$. Our effort in the next lemma is to estimate the size of such an $\Omega_t$ in terms of $\alpha$ and $k$.

**Lemma 3.5** *Let $0 < t < M$ be such that $L(\partial\Omega_t)^2 < 4\pi(1 + k\alpha^2)|\Omega_t|$, where $k$ is the constant in Lemma 3.3. Then*

$$|\Omega_t| \le (1 - \alpha)/(1 - 5\alpha\sqrt{k}/2). \tag{3.26}$$

**Proof.** Clearly, $\Omega_t$ is a set of finite perimeter. We first recall the definition of perimeter $L(\partial\Omega_t)$; we refer to [4]. We define

$$L(\partial\Omega_t) = \inf(\liminf_{i\to\infty} L(\partial S_i)),$$

where the infimum is taken over all sequences of polyhedra $S_i$'s ( polygonal regions in $\mathbb{R}^2$) with boundary $\partial S_i$ and satisfying

$$\lim_{i\to\infty} \int_Q |\chi_{S_i} - \chi_{\Omega_t}| = 0,$$

for every compact $Q \subset \mathbb{R}^2$. Here $\chi_D$ is the characteristic of a set $D$. Clearly then we may choose a sequence of polygonal regions $S_i$ such that $L(\partial S_i) \to L(\partial\Omega_t)$ with $|(S_i\backslash\Omega_t) \cup (\Omega_t\backslash S_i)| \to 0$ as $i \to \infty$. Moreover, the sequence may be so chosen that $L(\partial S_i) < 4\pi(1 + k\alpha^2)|S_i|$. We will continue to work with $\Omega_t$ but with the understanding that the estimates to follow hold for $S_i$ and the final statement for $\Omega_t$ comes from taking the limit. The proof is carried out in three steps.

(A) Let $\Omega_t = \bigcup_{i=1}^{\infty} C_i$, where $C_i$'s are disjoint components ( while, in our case, there can only be finitely many $C_i$'s, the proof we provide here applies to the infinite case as well). Let $H_j, j = 1, 2, \dots$ denote the holes in $\Omega_t$, i.e., the set $\Omega_t \cup (\bigcup_{j=1}^{\infty} H_j)$ consists of simply connected components, say $F_i, i = 1, 2, \dots$ Here $F_i$ denotes the simply connected component obtained by plugging the holes of $C_i$. We first prove an estimate for the total area of the holes $H_j$. Clearly, $L(\partial F_i) \le L(\partial C_i)$, and via the isoperimetric inequality,

$$\begin{aligned} 4\pi(\Sigma_i|F_i|) &\le \Sigma_i L(\partial F_i)^2 \le \Sigma_i L(\partial C_i)^2 \le (\Sigma_i L(\partial C_i))^2 = L(\Omega_t)^2 \\ &< 4\pi(1 + k\alpha^2)|\Omega_t| = 4\pi(1 + k\alpha^2)(\Sigma_i|C_i|). \end{aligned}$$

The latter follows via the assumption made in the statement of the lemma. While this leads to an estimate for the area of the holes, the argument we present next will gives us a much better estimate. Set $H = \bigcup_j H_j$, then the usual isoperimetric inequality implies

$$\begin{aligned} L(\Omega_t)^2 &= (\Sigma_i L(\partial C_i))^2 = (\Sigma L(\partial F_i) + \Sigma L(\partial H_j))^2 \\ &\geq 4\pi \left(\Sigma_i |F_i|^{1/2} + \Sigma_j |H_j|^{1/2}\right)^2 \geq 4\pi \left(|F|^{1/2} + |H|^{1/2}\right)^2, \end{aligned}$$

where $F = \bigcup_i F_i$. By our assumption on $\Omega_t$,

$$4\pi \left(|F|^{1/2} + |H|^{1/2}\right)^2 \leq L(\partial\Omega_t)^2 < 4\pi(1 + k\alpha^2)|\Omega_t|.$$

Recalling that $|F| = |\Omega_t| + |H|$, and expanding the left side, we have

$$\begin{aligned} 4\pi(1 + k\alpha^2)|\Omega_t| &> 4\pi\Big\{|F| + |H| + 2\{|F||H|\}^{1/2}\Big\} \\ &= 4\pi\Big\{|\Omega_t| + 2|H| + 2[(|\Omega_t| + |H|)|H|]^{1/2}\Big\}. \end{aligned}$$

Simplifying,

$$2\Big\{|H|(|\Omega_t| + |H|)\Big\}^{1/2} \leq k\alpha^2|\Omega_t|.$$

One then easily obtains

$$|H| \leq \frac{k^2\alpha^4}{4}|\Omega_t|. \tag{3.27}$$

(B) Our second step is to show that of the $C_i$'s all but one have small areas. In order to simplify our computations, we set $R_i = \sqrt{|C_i|/\pi}$, $i = 1, 2, \dots$. Label $R_i$'s such that $R_1 = \sup\{R_i, i = 1, \dots\}$. This supremum is attained since $\Sigma|C_i| = \pi\Sigma R_i^2 = |\Omega_t| < \infty$. Also note that $L(\partial C_i) \geq 2\pi R_i$, $\forall\ i$. Thus

$$4\pi^2(\Sigma R_i)^2 \leq (\Sigma L(\partial C_i))^2 = (L(\partial\Omega_t))^2 < 4\pi^2(1 + k\alpha^2)(\Sigma R_i^2).$$

Set $\varepsilon_i = R_i/R_1$, $i = 1, 2, \dots$, then

$$\Big(1 + \sum_{i>1}\varepsilon_i\Big)^2 \leq (1 + k\alpha^2)\Big(1 + \sum_{i>1}\varepsilon_i^2\Big).$$

Thus,

$$1 + 2\sum_{i>1}\varepsilon_i + \sum_{i>1}\varepsilon_i^2 \leq (1 + k\alpha^2)(1 + \sum_{i>1}\varepsilon_i^2),$$

hence,

$$2\sum_{i>1}\varepsilon_i \leq k\alpha^2(1 + \sum_{i>1}\varepsilon_i^2);$$

now, together with the fact $\sum_{i>1}\varepsilon_i^2 \leq \sum_{i>1}\varepsilon_i$, we get

$$\sum_{i>1}\varepsilon_i \leq \frac{k\alpha^2}{2 - k\alpha^2} \leq k\alpha^2.$$

Thus,

$$\sum_{i>1} \varepsilon_i^2 \leq (\sum_{i>1} \varepsilon_i)^2 \leq k^2\alpha^4,$$

implying that

$$\sum_{i>1} |C_i| \leq k^2\alpha^4 |C_1| \leq k^2\alpha^4 |\Omega_t|.$$

It n easy to see that

$$|C_1| \geq \frac{|\Omega_t|}{1+k^2\alpha^4} \geq |\Omega_t|(1-k^2\alpha^4). \tag{3.28}$$

(C) We now work with $F_1$; by hypothesis of the lemma and (3.27)

$$\begin{aligned} L(\partial F_1)^2 \leq L(\partial C_1)^2 \leq L(\partial \Omega_t)^2 &\leq 4\pi(1+k\alpha^2)(1+k^2\alpha^4)|C_1| \\ &\leq 4\pi\left(1+\frac{10}{9}k\alpha^2\right)|F_1|. \end{aligned} \tag{3.29}$$

The last inequality follows from noting that $k < 1/100$. Since $F_1$ is simply connected, we may calculate the inradius $I$ (see [14]) using (3.28),

$$\begin{aligned} I &\geq \frac{L(\partial F_1) - \sqrt{L(\partial F_1)^2 - 4\pi|F_1|}}{2\pi} \\ &\geq \frac{\sqrt{4\pi|F_1|} - \sqrt{4\pi(1+\frac{10}{9}k\alpha^2)|F_1| - 4\pi|F_1|}}{2\pi} \\ &\geq \sqrt{\frac{|F_1|}{\pi}}\left(1-\frac{11}{10}\sqrt{k}\alpha\right) \geq \sqrt{\frac{|\Omega_t|}{\pi}}\frac{\left(1-\frac{11}{10}\sqrt{k}\,\alpha\right)}{\sqrt{1+k^2\alpha^4}} \\ &\geq \sqrt{\frac{|\Omega_t|}{\pi}}\left(1-\frac{12}{10}\sqrt{k}\alpha\right) = R. \end{aligned}$$

Clearly the ball $B_R$ with an appropriate center lies in $F_1$, and so $B_R\backslash H$ lies in $C_1$. We now estimate $\Omega_t$ by using the properties of the in-ball, the definition of asymmetry $\alpha$ (see (1.2)) and (3.26),

$$\begin{aligned} \alpha|\Omega| &\leq |\Omega\backslash(B_R\backslash H)| = |\Omega| - |B_R\backslash H| \\ &\leq 1 - \left[\left(1-\frac{12}{10}\sqrt{k}\alpha\right)^2 - \frac{k^2\alpha^4}{4}\right]|\Omega_t| \\ &\leq 1 - \left(1-\frac{5}{2}\sqrt{k}\alpha\right)|\Omega_t| \end{aligned}$$

Thus,

$$|\Omega_t| \leq \frac{1-\alpha}{1-5\sqrt{k}\alpha/2}. \tag{3.30}$$

We now recall the discussion at the beginning of our proof. The inequality in (3.29) is derived for $S_i$ with $\alpha = \alpha_i = \alpha(S_i)$. As pointed out, taking the limit $i \to \infty$ provides justification for validity of (3.29) for $\Omega_t$. ◊

**Remark 3.3** If $t$ satisfies the conditions of Lemma 3.5, thenLemma 3.4 implies

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*) \left( \frac{1 - 5\sqrt{k}\alpha/2}{1-\alpha} \right)^{p/2} (1-t)^{p-1}. \tag{3.31}$$

# 4 Proof of the main result

We take $k = 1/625$; we recapitulate the above results as follows:
(a) If asymmetry propagates over a "t" interval $[0, T]$, i.e., $L(\partial\Omega_t)^2 \geq 4\pi(1 + k\alpha^2)|\Omega_t|$ a. e. $t \in [0, T]$, then

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*) \left( 1 + \frac{kT}{8M}\alpha^2 \right) . \tag{$*$}$$

This follows from Lemma 3.3, and Remark 3.1. Note that it is enough toassume that $\lambda_1 \leq 2\lambda_1^*$, for otherwise the theorem follows.
(b) If not, i.e., for some $t$ in $[0, T]$, we have $L(\partial\Omega_t)^2 < 4\pi(1 + k\alpha^2)|\Omega_t|$, then via Lemma 3.4, and Remark 3.3 with $k = 1/625$, we have

$$\begin{aligned} \lambda_1(\Omega) &\geq \lambda_1(\Omega^*)\Big( \frac{1 - 5\sqrt{k}\ \alpha/2}{1-\alpha} \Big)^{p/2} (1-t)^{p-1} \qquad (4.1) \\ &\geq \lambda_1(\Omega^*) \left( 1 + \frac{9\alpha}{10} \right)^{p/2} (1-t)^{p-1} \end{aligned}$$

We make the following simple observations keeping in mind that $0 \leq \alpha \leq 1$. Firstly

$$(1 + 9\alpha/10)^{p/2} \geq 1 + 9p\alpha/40 \text{ when } 1 < p. \tag{4.2}$$

Also

$$(1-t)^{p-1} \geq \begin{cases} 1 - (p-1)t & \text{if } p \geq 2 \text{ and } 0 < t \leq 1/p. \\ 1 - p(p-1)t & \text{if } 1 < p < 2 \text{ and} \\ & 0 < t \leq \frac{(p-1)}{p} < 1 - (1/p)^{1/(2-p)}. \end{cases} \tag{4.3}$$

To achieve the proof of the Theorem we will adapt a technique taken from [9].
**Case 1** Let $p \geq 2$. We start by observing that, using (4.32) and (4.33), the left side of (4.31) may be written as

$$\begin{aligned} \left( 1 + \frac{9\alpha}{10} \right)^{p/2} (1-t)^{p-1} &\geq \left( 1 + \frac{9p\alpha}{40} \right) (1 - (p-1)t) \\ &= 1 + p\left( \frac{9\alpha}{40} - \frac{(p-1)}{p}t - \frac{9(p-1)\alpha t}{40} \right). \qquad (4.4) \end{aligned}$$

We now reason as follows.
(i) Either asymmetry propagates over the " t" interval $[0, \alpha/10p]$ (in a. e. sense), in which case $(*)$ implies

$$\lambda_1(\Omega) \geq \lambda_1(\Omega^*) \left( 1 + k\frac{\alpha^3}{80pM} \right);$$

or (ii) it does not, i.e., for some $t \in [0, \alpha/10p]$, (b) holds. Employing (4.34) and noting that $\alpha \le 1,\ 2 \le p$ and $(p-1)/p \le 1$, we find that

$$\begin{aligned} \lambda_1(\Omega) &\ge \lambda_1(\Omega^*)\left[1 + p\left(\frac{9\alpha}{40} - \frac{(p-1)\alpha}{10p^2} - \frac{9(p-1)\alpha^2}{400p}\right)\right] \\ &= \lambda_1(\Omega^*)\left(1 + \frac{61p}{400}\alpha\right). \end{aligned}$$

**Case 2** Now take $1 < p < 2$. Thus (4.31), (4.32) and (4.33) for $0 < t < (p-1)/p$, give us

$$\begin{aligned} \left(1 + \frac{9\alpha}{10}\right)^{p/2}(1-t)^{p-1} &\ge \left(1 + \frac{9p\alpha}{40}\right)(1 - p(p-1)t) \\ &= 1 + p\left(\frac{9\alpha}{40} - (p-1)t - \frac{9p(p-1)\alpha t}{40}\right). \quad (4.5) \end{aligned}$$

Again, if (i) asymmetry propagates over the "$t$" interval $[0, (p-1)\alpha/10p]$, then (*) implies

$$\lambda_1(\Omega) \ge \lambda_1(\Omega^*)\left(1 + k\frac{(p-1)\alpha^3}{80pM}\right).$$

(ii) If not, then (4.35) together with the fact that $1 < p < 2$ implies that

$$\lambda_1(\Omega) \ge \lambda_1(\Omega^*)\left(1 + \frac{41p\alpha}{400}\right).$$

The statement of the Theorem follows from the conclusions in Cases 1 and 2.

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