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ON OSCILLATION AND ASYMPTOTIC BEHAVIOUR OF A NEUTRAL DIFFERENTIAL EQUATION OF FIRST ORDER WITH POSITIVE AND NEGATIVE COEFFICIENTS

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Abstract. In this paper sufficient conditions are obtained so that every solution of

$$(y(t) - p(t)y(t - \tau))' + Q(t)G(y(t - \sigma)) - U(t)G(y(t - \alpha)) = f(t)$$

tends to zero or to $\pm\infty$ as t tends to ∞, where τ, σ, α are positive real numbers, $p, f \in C([0,\infty), R), Q, U \in C([0,\infty), [0,\infty))$, and $G \in C(R, R)$, G is non decreasing with $xG(x) > 0$ for $x \neq 0$. The two primary assumptions in this paper are $\int_{t_0}^{\infty} Q(t) = \infty$ and $\int_{t_0}^{\infty} U(t) < \infty$. The results hold when G is linear, super linear,or sublinear and also hold when $f(t) \equiv 0$. This paper generalizes and improves some of the recent results in [5, 7, 8, 10].

1. Introduction

The study of neutral delay differential equation (NDDE)has been the centre of attraction of many researchers all over the world for the last several years. The authors in a recent paper [10] which improved [7] substantially considered the first order forced nonlinear neutral delay differential equation

$$(y(t) - p(t)y(t - \tau))' + Q(t)G(y(t - \sigma)) - U(t)G(y(t - \alpha)) = f(t), \tag{1.1}$$

where τ, σ, α are positive numbers, $p, f \in C([0,\infty), R)$, $Q, U \in C([0,\infty), [0,\infty))$. The authors in [10] proved that every non-oscillatory solution of (1.1) tends to zero as t tends to ∞, using the following hypothesis.

- (H0) $G \in C(R, R)$ with G non-decreasing and $xG(x) > 0$ for $x \neq 0$.
- (H1) $\int_{t_0}^{\infty} Q(t)\,dt = \infty$.
- (H2) $\int_{t_0}^{\infty} U(t)\,dt < \infty$.
- (H3) There exists a bounded function $F \in C'([0,\infty), R)$ such that $F'(t) = f(t)$ and $\lim_{t\to\infty} F(t) = 0$.
- (H4) $\liminf_{|u|\to\infty} \frac{G(U)}{U} \leq \beta$ where $\beta > 0$.
- (H5) $Q(t) > U(t - \sigma + \alpha)$.
- (H6) $\sigma > \alpha$ or $\sigma < \alpha$.

Further the following ranges of $p(t)$ were considered in [10].

- (A1) $0 \leq p(t) \leq p < 1$,

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(A2) $-1 < -p \le p(t) \le 0$,
(A3) $-p_2 \le p(t) \le -p_1 < -1$,
(A4) $1 < p_1 \le p(t) \le p_2$,
(A5) $-p_2 \le p(t) \le 0$,
(A6) $0 \le p(t) \le p_2$,

where p, p_1, p_2 are positive real numbers.

We strongly feel (1.1) is not yet studied systematically. In this paper we find sufficient conditions so that all non-oscillatory solutions of (1.1) tend to zero or $\pm\infty$, as t tends to ∞ and these conditions improve [10]. Almost all authors [4, 7, 10, 11, 12] studied (1.1) with assumption (H6). But in this paper we could do away with the conditions (H6). Further we succeeded to relax (H4) and (H5) as suggested by the authors in their comments in [10] for further research. We have given two more theorems, one with p(t) as in (A6) and another when $p(t)$ is oscillatory and noted some errors in the literature. In this work the assumption (H2) permits to take $U(t) \equiv 0$. Thus this paper extends and generalizes the work [5, 8]. At the last but not the least an example is given to illustrate the significance of our work.

By a solution of (1.1), we mean a real valued continuous function y on $[t_y - \rho, \infty)$ such that $(y(t) - p(t)y(t-\tau))$ is once continuously differentiable for $t \ge t_y$ and (1.1) is satisfied identically for $t \ge t_y$, where $\rho = \max(\tau, \sigma)$. A solution of (1.1) is said to be oscillatory if and only if it has arbitrarily large zeros. Otherwise it is said to be non-oscillatory.So far as existence and uniqueness of solutions of (1.1) are concerned one may refer [6], but in this work we assume the existence of solutions of (1.1) and study their qualitative behaviour. In the sequel, unless otherwise specified, when we write a functional inequality, it will be assumed to hold for all sufficiently large values of t.

2. Main Results

Theorem 2.1. *Suppose that $p(t)$ satisfies (A1) or (A2). Let (H0)-(H3) hold. Then every solution of (1.1) oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be any non-oscillatory positive solution of (1.1) on $[t_y, \infty)$. For $t \ge t_0 = t_y + \rho$, we set

$$z(t) = y(t) - p(t)y(t-\tau), \tag{2.1}$$

$$K(t) = \int_t^\infty U(s)G(y(s-\alpha))\,ds, \tag{2.2}$$

$$w(t) = z(t) + K(t) - F(t). \tag{2.3}$$

Thus for $t \ge t_0$, we obtain

$$w'(t) = -G(y(t-\sigma))Q(t) \le 0. \tag{2.4}$$

Hence $w(t) \le 0$ or $w(t) \ge 0$ for $t > t_1 > t_0$ and $\lim_{t\to\infty} w(t) = l$ where $-\infty \le l < \infty$. We claim $y(t)$ is bounded. Otherwise there exists a sequence $< T_n >$ such that $n \to \infty$ implies

$$T_n \to \infty, y(T_n) \to \infty \,\text{and}\, y(T_n) = \max(y(s) : t_1 \le s \le T_n). \tag{2.5}$$

We may choose n sufficiently large such that $T_n - \rho > t_1$. Suppose that $p(t)$ satisfies (A1). Then using (H3), for any $t > t_2 > t_1$ we obtain

$$w(T_n) \ge y(T_n) - p(T_n)y(T_n - \tau) + K(T_n) - F(T_n) \ge (1-p)y(T_n) - \epsilon. \tag{2.6}$$

As $n \to \infty$,we see that $w(t_n) \to \infty$,a contradiction. Hence $y(t)$ is bounded. Similarly it can be shown if $p(t)$ satisfies (A2) then also $y(t)$ is bounded. Consequently $z(t)$ and $w(t)$ are bounded. Hence it follows from (H0), (H2) and (2.2) that $K(t)$ is convergent and

$$K(t) \to 0 \quad \text{as } t \to \infty. \tag{2.7}$$

Then

$$\lim_{t\to\infty} w(t) = \lim_{t\to\infty} z(t) = l. \tag{2.8}$$

Next we claim $\liminf_{t\to\infty} y(t) = 0$. Otherwise for large $t \geq t_2$, $y(t) > m > 0$ and since G is nondecreasing,

$$\int_{t_2}^{\infty} G(y(s-\sigma))Q(s)\,ds > G(\beta)\int_{t_2}^{\infty} Q(s)\,ds = \infty, \tag{2.9}$$

by (H1). However, integrating (2.4) between t_2 to ∞ we obtain

$$\int_{t_2}^{\infty} G(y(s-\sigma))Q(s)\,ds < \infty, \tag{2.10}$$

a contradiction. Hence our claim holds. If $p(t)$ satisfies (A1) then $\liminf_{t\to\infty} z(t) < \liminf_{t\to\infty} y(t) = 0$. Now two distinct cases arise. Consider the first one $l \geq 0$. Then $z(t) \geq 0$ for large t. Then it follows that $\lim_{t\to\infty} z(t) = 0$. Then

$$\begin{aligned}
0 = \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty}(y(t) - p(t)y(t-\tau)) \\
&\geq \limsup_{t\to\infty} y(t) + \liminf_{t\to\infty}(-py(t-\tau)) \\
&= (1-p)\limsup_{t\to\infty} y(t);
\end{aligned}$$

which implies $\limsup_{t\to\infty} y(t) = 0$. Hence $\lim_{t\to\infty} y(t) = 0$. Next consider the second case $l \leq 0$. Then $z(t) \leq 0$. We claim $\limsup_{t\to\infty} y(t) = 0$. Otherwise suppose $\limsup_{t\to\infty} y(t) = \mu > 0$. Then we can find a sequence$\langle t_n\rangle$ such that $y(t_n) \to \mu$ as $n \to \infty$. As $y(t_n - \tau)$ is bounded, hence there can be a subsequence $< t_{n_k} >$ such that $y(t_{n_k} - \tau) \to \lambda$ where $\lambda \leq \mu$. Then

$$\begin{aligned}
z(t_{n_k}) &= y(t_{n_k}) - p(t_{n_k})y(t_{n_k} - \tau) \\
&\geq y(t_{n_k}) - py(t_{n_k} - \tau) \\
&\to \mu - p\lambda \geq (1-p)\mu > 0.
\end{aligned}$$

Hence we have a contradiction because $z(t) \leq 0$. Thus our claim holds and consequently $\lim_{t\to\infty} y(t) = 0$. If the function $p(t)$ satisfies (A2), then $z(t) \geq 0$. Since $\liminf_{t\to\infty} y(t) = 0$, we get a infinite sequence $\langle t_n\rangle$ such that $n \to \infty$ implies $t_n \to \infty$ and consequently $y(t_n) \to 0$. Then $z(t_n) \leq y(t_n) + py(t_n - \tau)$. Taking limit $n \to \infty$ we get $l \leq p\mu$, where $\limsup_{t\to\infty} y(t) = \mu$. As $z(t) \geq y(t)$, because of (A2) it is found that $\limsup_{t\to\infty} z(t) \geq \mu$. This implies $l \geq \mu$. Thus $\mu(p-1) \geq 0$. Then μ must be zero because $p < 1$. Hence $\limsup_{t\to\infty} y(t) = 0$. Thus $\lim_{t\to\infty} y(t) = 0$. The proof for the case when $y(t) < 0$ is similar. Thus the theorem is completely proved. □

Next we state a Lemma found in [6, page19].

Lemma 2.2. *Let $u, v, p : [0,\infty) \to R$ be such that $u(t) = v(t) - p(t)v(t-c)$, $t \geq c$, where $c \geq 0$. Suppose that $p(t)$ is in one of the ranges (A2), (A3) or(A6). If $v(t) > 0$ for $t \geq 0$ and $\liminf_{t\to\infty} v(t) = 0$ and $\lim_{t\to\infty} u(t) = L$ exists then $L = 0$.*

Theorem 2.3. *Suppose (H0)-(H3) hold.*

(i) *If $p(t)$ lies in the ranges (A3) then every solution of* (1.1) *oscillates or tends to zero as $t \to \infty$.*
(ii) *If $p(t)$ satisfies (A4) then every bounded solution of* (1.1) *oscillates or tends to zero as $t \to \infty$.*

Proof. Consider first the proof for (i) and suppose that $p(t)$ satisfies (A3). Let $y(t)$ be any nonoscillatory positive solution of (1.1) on $[t_y, \infty)$. For $t \geq t_0 = t_y + \rho$, we set $z(t), K(t)$ and $w(t)$ as in (2.1), (2.2) and (2.3) respectively and obtain (2.4). As in the Theorem 2.1 we prove that $y(t)$ is bounded. Then it follows that (2.7) and (2.8) hold. Next we use (2.9) and (2.10) to prove $\liminf_{t\to\infty} y(t) = 0$ as in Theorem 2.1. If $p(t)$ satisfies (A3), then we apply Lemma 2.2 and obtain $\lim_{t\to\infty} z(t) = 0$. Then since $y(t) \leq z(t)$, $\limsup_{t\to\infty} y(t) \leq 0$. Consequently, $\lim_{t\to\infty} y(t) = 0$.

The proof for (ii) follows similarly and we obtain $\lim_{t\to\infty} z(t) = 0$. Then we note that $0 \leq \lim_{t\to\infty} z(t) \leq (1-p_1) \limsup_{t\to\infty} y(t)$. Hence $\lim_{t\to\infty} y(t) = 0$ since $p_1 > 1$. The proof for the case $y(t) < 0$ for $t \geq t_y$ is similar. Thus the theorem is proved. □

Remark 2.4. Theorems 2.1 and 2.3 hold when G is linear, sublinear or super linear. These two theorems improve [10, Theorems 2.2, 2.4, 2.7] because the conditions (H4),(H5) and (H6) are not used in our result. These conditions are used in previous papers [4, 10, 12].

Theorem 2.5. *Suppose that $p(t)$ satisfies (A5). Let $(H_0), (H_2)$ and (H3) hold. Then suppose that*

(H7) $\int_\rho^\infty Q^*(t)\,dt = \infty$ *where* $Q^*(t) = min\,[Q(t), Q(t-\tau)]$,
(H8) $G(-u) = -G(u)$,
(H9) *for* $u > 0, v > 0$, $G(u)G(v) \geq G(u)G(v)$ *and* $G(u) + G(v) \geq \delta G(u+v)$, *where* $\delta > 0$ *is a constant.*

Then every solution of (1.1) *oscillates or tends to zero as $t \to \infty$.*

Proof. Let $y(t)$ be an eventually positive solution of (1.1) for $t > t_y$. Then we set $z(t), K(t)$ and $w(t)$ as in (2.1) ,(2.2) and (2.3) respectively and obtain (2.4). Then $w'(t) \leq 0$. Hence $w(t)$ is monotonic and single sign. Consequently $\lim_{t\to\infty} w(t) = l$ where$-\infty \leq l < \infty$. We claim $y(t)$ is bounded, otherwise $y(t)$ is unbounded implies $z(t)$ is unbounded. Hence there exists an increasing sequence $\langle t_n \rangle$ such that $t_n \to \infty, z(t_n) \to \infty$ as $n \to \infty$ and $z(t_n) = \max(z(t) : t_1 \leq t \leq t_n)$. Then $n \to \infty$ implies $w(t_n) = z(t_n) + K(t_n) - F(t_n) \to \infty$. Thus we get a contradiction.Hence $y(t)$ is bounded which implies $\lim_{t\to\infty} z(t) = l$. Further $l < 0$ is not possible since $z(t) \geq 0$ for large t. Thus $0 \leq l < \infty$. Again bounded ness of $y(t)$ and (H2) yield (2.7). If l=0 then $\lim_{t\to\infty} y(t) = 0$ and if $l > 0$ then for $t \geq t_2 > t_1, z(t) > \lambda > 0$. Using definition of $Q^*(t)$ and (H9) one may obtain

$$
\begin{aligned}
0 &= w'(t) + Q(t)G(y(t-\sigma)) + G(-p(t-\sigma))w'(t-\tau) \\
&\quad + G(-p(t-\sigma))G(y(t-\sigma-\tau))Q(t-\tau) \\
&\geq w'(t) + G(p_2)w'(t-\tau) + Q^*(t)[G(y(t-\sigma)) + G(-p(t-\sigma))G(y(t-\sigma-\tau))] \\
&\geq w'(t) + G(p_2)w'(t-\tau) + \delta Q^*(t)G(z(t-\sigma)) \\
&\geq w'(t) + G(p_2)w'(t-\tau) + \delta G(\lambda)Q^*(t).
\end{aligned}
$$

Integrating the above inequality from t_2 to ∞ and using (H7) we arrive at the contradiction that $w(t) + G(p_2)w(t-\tau) \to -\infty$ as $t \to \infty$. The proof for the case $y(t) < 0$ is similar and it may be noted that (H8) is required in this case. Thus the theorem is proved. $\square$

Remark 2.6. The prototype of function satisfying (H8) and (H9) is $G(u) = (\beta + |u|^{\lambda})|u|^{\mu} \operatorname{sgn} u$ with $\lambda > 0$, $\mu > 0$, $\beta \geq 1$.

Remark 2.7. *The above Theorem substantially improves* [10, Theorems 2.11] *because the authors there have used the following three additional conditions for their work.*

(i) *$U(t)$ is monotonic increasing.*
(ii) $\alpha > \sigma$.
(iii) $Q^*(t) \geq U(t - \sigma + \alpha - \tau)$.

Remark 2.8. Condition (H7) implies (H1), but the converse is not true.

Theorem 2.9. *Suppose $p(t)$ is oscillating and tends to zero as $t \to \infty$ with $-p_2 \leq p(t) \leq p_1 < 1$,where p_1 and p_2 are positive real numbers. If (H0)-(H3) hold then every solution of* (1.1) *oscillates or tend to* 0 *as $t \to \infty$.*

Proof. Suppose $y(t)$ does not oscillate. Then $y(t) > 0$ or $y(t) < 0$ for $t \geq t_0$. Let $y(t) > 0$ for $t \geq t_1 > t_0$. The proof for the case $y(t) < 0$ is similar. Set $z(t), w(t)$, and $k(t)$ as in (2.1), (2.2) and (2.3) respectively and obtain (2.4). Hence $w(t)$ is monotonic and single sign. $w(t) > 0$ or $w(t) < 0$ for $t > t_2 > t_1$. We claim $y(t)$ is bounded. Otherwise there exists a sequence$< T_n >$such that $T_n \to \infty, y(T_n) \to \infty$ as $n \to \infty$ and $y(T_n) = \max\ (y(s) : t_1 \leq s \leq T_n)$. We may choose n sufficiently large such that $T_n - \rho > t_2$. Then using (H3) for any $t > t_3 > t_2$ we obtain

$$w(T_n) \geq y(T_n) - p(T_n) + K(T_n) - F(T_n) \geq y(T_n)(1 - p_1) - \epsilon.$$

As $n \to \infty$, we see that $w(t_n) \to \infty$, a contradiction. Hence $y(t)$ is bounded. Consequently $z(t)$ and $w(t)$ are bounded. Use of (H0) and (H2) yields (2.7). Next we prove $\liminf_{t\to\infty} y(t) = 0$ as in Theorem 2.1. Since $p(t) \to 0$ as $t \to \infty$ and $y(t)$ is bounded therefore we have $\lim_{t\to\infty} p(t)y(t-\tau) = 0$. From the facts that $\lim_{t\to\infty} w(t) = l$ exists, $\lim_{t\to\infty} k(t) = 0$ and $\lim_{t\to\infty} F(t) = 0$ it follows that $\lim_{t\to\infty} y(t)$ exists and must be equal to zero. Thus the theorem is proved. $\square$

Remark 2.10. For the results with $p(t)$ oscillating, we may refer Theorem 6(ii) of [3] where the proof is wrong because they have used Lemma2.2 in their proof which is not permissible since $p(t)$ does not satisfy the conditions of the lemma. Again we have another result with $p(t)$ oscillating is [9, Theorem 2.4] where $p(t)$ is periodic and $-1 < -p_4 \leq p(t) \leq p_5 < 1$ with $p_5 + p_4 < 1$.

Theorem 2.11. *Let $p(t)$ be in range (A6). Suppose that (H0), (H2) and (H3) hold. Then (i) every unbounded solution of* (1.1) *oscillates or tends to $\pm\infty$ as $t \to \infty$ and (ii) every bounded solution of* (1.1) *oscillates or tends to zero as $t \to \infty$ if the following condition holds:*

(H10) *suppose that, for every sequence $\langle \sigma_i \rangle \subset (0, \infty), \sigma_i \to \infty$ as $i \to \infty$ and for every $\beta > 0$ such that the intervals $(\sigma_i - \beta, \sigma_i + \beta)$, $i = 1, 2, \dots,$ are non overlapping,*
$\Sigma_{i=1}^{\infty} \int_{\sigma_i - \beta}^{\sigma_i + \beta} Q(t)\, dt = \infty.$

Proof. First let us prove (i) and suppose $y(t)$ is an unbounded positive solution of (1.1) for $t \geq T_0$. Then set $z(t), K(t)$ and $w(t)$ as in (2.1), (2.2) and (2.3) respectively and obtain (2.4). Hence $w'(t) \leq 0$, for $t \geq T_0 + \sigma$. Then $w(t) > 0$ or $w(t) < 0$ for $t > T_1 > T_0+\sigma$. In either case $\lim_{t\to\infty} w(t) = l$ where $-\infty \leq l < \infty$.If $l \neq -\infty$ then $\lim_{t\to\infty} w(t)$ exists. Integrating (2.4) from T_1 to t and then taking limit as $t \to \infty$, one obtains (2.10). Since $y(t)$ is unbounded, there exists a sequence $\langle t_n \rangle \subset [T_1, \infty)$ such that $t_n \to \infty$ and $y(t_n) \to \infty$ as $n \to \infty$. Hence, for every $M > 0$, there exists $N_1 > 0$ such that $y(t_n) > M$ for $n \geq N_1$. Since $y(t)$ is continuous, there exists $\delta_n > 0$ such that $y(t) > M$ for $t \in (t_n - \delta_n, t_n + \delta_n)$ and $n \geq N_1$ and $\liminf_{n\to\infty} \delta_n > 0$. Hence $\delta_n > \delta > 0$ for $n \geq N_2$. Choose $N > \max\,(N_1, N_2)$ such that $t_N > T_1$. Hence

$$\begin{aligned}
\int_{t_N+\delta_N+\sigma}^{\infty} Q(t)G(y(t-\sigma))\,dt &\geq \sum_{i=N+1}^{\infty} \int_{t_i-\delta_i+\sigma}^{t_i+\delta_i+\sigma} Q(t)G(y(t-\sigma))\,dt \\
&\geq G(M) \sum_{i=N+1}^{\infty} \int_{t_i-\delta_i+\sigma}^{t_i+\delta_i+\sigma} Q(t)\,dt \\
&\geq G(M) \sum_{i=N+1}^{\infty} \int_{t_i+\sigma-\delta}^{t_i+\sigma+\delta} Q(t)\,dt.
\end{aligned}$$

Then from (H10) it follows that $\int_{t_N+\delta_N+\sigma}^{\infty} Q(t)G(y(t-\sigma))\,dt = \infty$, a contradiction to (2.10). If $l = -\infty$ then from (2.3) it follows that $w(t) + F(t) \geq z(t)$. Hence using (H3) one may obtain $\lim_{t\to\infty} z(t) = -\infty$. As $z(t) > -p(t)y(t-\tau) \geq -p_2 y(t-\tau)$ for $t \geq T_1$, then $\lim_{t\to\infty} y(t) = \infty$. If $y(t) < 0$, and unbounded for large t then we proceed similarly to obtain $\lim_{t\to\infty} y(t) = -\infty$. Next let us prove (ii) and assume $y(t)$ to be an eventually positive and bounded solution of (1.1) for large t. Then we proceed as in the first case and obtain (2.10) because $l \neq -\infty$. We claim $\limsup_{t\to\infty} y(t) = 0$. Otherwise let $\limsup_{t\to\infty} y(t) = \mu > 0$. Then there exists a sequence $\langle t_n \rangle$ such that $y(t_n) > M > 0$ for large n. Since $y(t)$ is continuous there exists $\delta_n > 0$ such that $y(t) > M$ for $t \in (t_n - \delta_n, t_n + \delta_n)$ and $n \geq N_1$ and $\liminf_{n\to\infty} \delta_n > 0$. Then proceeding as in the y(t) unbounded case, we get

$$\int_{t_N+\delta_N+\sigma}^{\infty} Q(t)G(y(t-\sigma))\,dt = \infty,$$

by (H10) which contradicts (2.10). Hence $\lim_{t\to\infty} y(t) = 0$. The proof for the case when $y(t) < 0$ is similar. Thus the proof is complete. □

Remark 2.12. Condition (H10) implies (H1) but not conversely.

Remark 2.13. In [10] the authors in their comments for further research suggested to develop a theorem for (1.1) when $p(t)$ is in the range (A6).

Remark 2.14. In all the results above we do not have any restriction on the sign of coefficient function $f(t)$. It may be positive, negative, zero or oscillating.

Example. Consider the NDDE

$$(y(t) - e^{-1}y(t-1))' + 2y^3(t-3) - t^{-2}y^3(t-2) = 2e^{9-3t} - t^{-2}e^{6-3t}, \quad t > 0. \tag{2.11}$$

This equation satisfies all the conditions of Theorem 2.1 of this paper for $p(t)$ in the range (A1). Hence all solutions of (2.11) either oscillate or tend to zero as $t \to \infty$. As such $y(t) = e^{-t}$ is a solution which tends to 0 as $t \to \infty$. Here $G(u) = u^3$ is super linear. Since $\alpha = 2 < 3 = \sigma$, the results in [4, 7, 11, 12] cannot be applied to this NDDE. Even [10, Theorem 2.2] (where $\sigma > \alpha$) cannot be applied to (2.11) because it does not satisfy the sub linear condition (H4).

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