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Pressure conditions for the local regularity of solutions of the Navier–Stokes equations *

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Abstract

We obtain a relationship between the integrability of the pressure gradient and the the integrability of the velocity for local solutions of the Navier–Stokes equations with finite energy. In particular, we show that if the pressure gradient is sufficiently integrable, then the corresponding velocity is locally bounded and smooth in the spatial variables. The result is proven by using De Giorgi type estimates in L_p^{weak} spaces.

1 Introduction and statement of results

One of the most important unresolved questions in the theory of the Navier–Stokes equations is the local behavior of solutions in three spatial dimensions, in particular, questions of their local regularity. This problem was studied by Serrin [9], who showed that if the velocity is sufficiently integrable, then it is locally bounded and smooth in the spatial variables. Indeed, let $\Omega \subset \mathbb{R}^N$ be an open set, let $T > 0$, and set $\Omega_T = \Omega \times (0,T)$. Suppose that $\mathbf{v} \in V^2_{\text{loc}}(\Omega_T) = L_{2,\text{loc}}(0,T; W^1_{2,\text{loc}}(\Omega)) \cap L_{\infty,\text{loc}}(0,T; L_{2,\text{loc}}(\Omega))$ is a weak solution of the Navier–Stokes system

$$\begin{aligned} \mathbf{v}_t - \Delta \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = 0\,, \\ \operatorname{div}\mathbf{v} = 0\,. \end{aligned} \tag{1}$$

Serrin showed that if $\mathbf{v} \in L_{q,r}(\Omega_T) = L_r(0,T; L_q(\Omega))$ for some q, r satisfying

$$\frac{N}{q} + \frac{2}{r} < 1 \tag{2}$$

then $\mathbf{v}$ is locally bounded and smooth in the spatial variables. Kahane [3] later showed that this also implies that the velocity is locally analytic in the spatial variables. The case where the inequality in (2) is replaced by an equality was studied by Takahashi, [10].

In a sequence of papers including [5, 6, 7, 8], Scheffer adopted a different approach. Rather than finding conditions which guaranteed regularity, he

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constructed solutions to initial and initial boundary value problems that were partially regular; that is the Hausdorff dimension in space and time of the set of possible singularities could be estimated from above. Caffarelli, Kohn and Nirenberg in [2] extended these techniques to show that if $\Omega_T \subset \mathbb{R}^3 \times \mathbb{R}$ and $\mathbf{v}$ is a local suitable weak solution of (1), then the one–dimensional Hausdorff measure in space and time of the set of possible singularities is zero. A suitable weak solution is a pair of functions $\mathbf{v} \in V^2_{\mathrm{loc}}(\Omega_T)$ and $p \in L_{5/4,\mathrm{loc}}(\Omega_T)$ that satisfy (1) and a generalized energy inequality [2, Equation 2.5].

Since this partial regularity result requires some regularity of the pressure, a natural question is the relationship between the regularity of the pressure and the regularity of the velocity. The purpose of this paper is to examine this relationship and to prove the following results.

Theorem 1 *Let $(\mathbf{v}, p)$ be a local weak solution of the Navier–Stokes equations in a domain $\Omega_T \subset \mathbb{R}^3 \times \mathbb{R}$, and suppose that $\mathbf{v} \in V^2_{\mathrm{loc}}(\Omega_T)$ and $\nabla p \in L_{\mu,\mathrm{loc}}(\Omega_T)$. Then $\mathbf{v} \in L_{q,\mathrm{loc}}(\Omega_T)$ for any q that satisfies*

$$q < \frac{5}{5/\mu - 2}, \tag{3}$$

where $1 < \mu \leq \frac{5}{3}$. Furthermore, if $\mu > \frac{5}{3}$, then $\mathbf{v}$ is locally bounded and smooth in the spatial variables.

This can be generalized to domains in an arbitrary number of spatial dimensions in the following fashion.

Theorem 2 *Let $(\mathbf{v}, p)$ be a local weak solution of the Navier–Stokes equations in a domain $\Omega_T \subset \mathbb{R}^N \times \mathbb{R}$, and suppose that $\mathbf{v} \in V^2_{\mathrm{loc}}(\Omega_T)$ and $\nabla p \in L_{\mu,\mathrm{loc}}(\Omega_T)$. Then $\mathbf{v} \in L_{q,\mathrm{loc}}(\Omega_T)$ for any q that satisfies*

$$q < \frac{N+2}{(N+2)/\mu - 2}, \tag{4}$$

where $1 < \mu \leq \frac{N+2}{3}$. Furthermore, if $\mu > \frac{N+2}{3}$, then $\mathbf{v}$ is locally bounded and smooth in the spatial variables.

A consequence of these results is the fact that singularities of the velocity $\mathbf{v}$ are only possible at singularities of the pressure gradient ∇p.

We point out that the norm $\|\nabla p\|_{L_{(N+2)/3}(\Omega_T)}$ is dimensionless in the same sense that the norms $\|\mathbf{v}\|_{L_{q,r}(\Omega_T)}$ are dimensionless when $N/q+2/r = 1$. Indeed, if $\mathbf{v}(x,t)$ and $p(x,t)$ satisfy the Navier-Stokes equations (1) in a domain $\Omega \times (0,T)$, then the functions $\mathbf{v}_\lambda(x,t) = \lambda \mathbf{v}(\lambda x, \lambda^2 t)$ and $p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$ also satisfy (1) in the dilated domains $\Omega/\lambda \times (0, T/\lambda^2)$ for each $\lambda > 0$; however, the given norms of $\mathbf{v}_\lambda$ and p_λ are independent of λ. Indeed,

$$\begin{aligned}\|\mathbf{v}_\lambda\|_{L_{q,r}(\Omega/\lambda\times(0,T/\lambda^2))} &= \left\{\int_0^{T/\lambda^2}\left(\int_{\Omega/\lambda}|\lambda\mathbf{v}(\lambda x,\lambda^2 t)|^q\,dx\right)^{r/q} dt\right\}^{1/r} \\ &= \left\{\int_0^T\left(\int_\Omega \lambda^q|\mathbf{v}(x,t)|^q\frac{dx}{\lambda^N}\right)^{r/q}\frac{dt}{\lambda^2}\right\}^{1/r} \\ &= \lambda^{1-(N/q+2/r)}\|\mathbf{v}\|_{L_{q,r}(\Omega_T)} = \|\mathbf{v}\|_{L_{q,r}(\Omega_T)}\end{aligned} \tag{5}$$

and

$$\begin{aligned}&\|\nabla p_\lambda\|_{L_{(N+2)/3}(\Omega/\lambda\times(0,T/\lambda^2))} \\ &\quad= \left\{\int_0^{T/\lambda^2}\int_{\Omega/\lambda}|\lambda^2\cdot\lambda\nabla p(\lambda x,\lambda^2 t)|^{(N+2)/3}dx\,dt\right\}^{3/(N+2)} \\ &\quad= \left\{\int_0^T\int_\Omega \lambda^{N+2}|\nabla p|^{(N+2)/3}\frac{dx}{\lambda^N}\frac{dt}{\lambda^2}\right\}^{3/(N+2)} \\ &\quad= \|\nabla p\|_{L_{(N+2)/3}(\Omega_T)}.\end{aligned} \tag{6}$$

In this sense, Theorems 1 and 2 are analogues of the aforementioned result of Serrin [9].

The basic idea of the proof is to use as a test function a smoothed and cutoff variant of $(v^i \mp k)_\pm^\epsilon$. Provided ϵ is sufficiently small, the nonlinear term is integrable and we can remove the smoothing to obtain a local energy estimate. From this we can obtain an estimate of $\operatorname{meas}[|v^i|>k]$ and consequently of v^i in $L_{p,\mathrm{loc}}^{\mathrm{weak}}(\Omega_T)$.

Recall the definitions of the spaces $L_q^{\mathrm{weak}}(\mathcal{U})$; a measurable function f is an element of $L_q^{\mathrm{weak}}(\mathcal{U})$ if and only if

$$|f|_{L_q^{\mathrm{weak}}(\mathcal{U})} \equiv \sup_{k>0} k\big(\operatorname{meas}\{x\in\mathcal{U} : |f(x)|>k\}\big)^{1/q} < \infty\,. \tag{7}$$

The quantity $|f|_{L_q^{\mathrm{weak}}(\mathcal{U})}$ is not a norm, but it is a quasi-norm. The inequality

$$|f|_{L_q^{\mathrm{weak}}(\mathcal{U})} \le \|f\|_{L_q(\mathcal{U})} \tag{8}$$

follows immediately from

$$k^q\operatorname{meas}[|f|>k] \le \int_{\mathcal{U}}|f|^q\,\chi[|f|>k]\,dx \le \int_{\mathcal{U}}|f|^q\,dx \tag{9}$$

so that $L_q(\mathcal{U})\subset L_q^{\mathrm{weak}}(\mathcal{U})$. However $L_q^{\mathrm{weak}}(\mathcal{U})\neq L_q(\mathcal{U})$, as the function $f(x) = 1/x$ satisfies $f\in L_1^{\mathrm{weak}}(0,1)$, but $f\notin L_1(0,1)$. Finally, if $q'<q$ and $\mathcal{U}$ is

bounded, then $L_q^{\text{weak}}(\mathcal{U}) \subset L_{q'}(\mathcal{U})$; indeed

$$\begin{aligned}\|f\|^{q'}_{L_{q'}(\mathcal{U})} &= q' \int_0^\infty k^{q'-1}\,\text{meas}[|f|>k]\,dk \\ &\le q'\,\text{meas}\,\mathcal{U} + q'|f|^q_{L_q^{\text{weak}}(\mathcal{U})} \int_1^\infty k^{q'-q-1}\,dk < \infty.\end{aligned} \tag{10}$$

For further details about the spaces $L_q^{\text{weak}}(\mathcal{U})$ see [1, Chp. 1] or [4, IX.4].

Once we have the the energy inequality, we can show that $\mathbf{v} \in L_{\beta,\text{loc}}(\Omega_T)$ implies $v^i \in L^{\text{weak}}_{\alpha(\beta),\text{loc}}$ for some $\alpha(\beta)$ and each i; consequently $\mathbf{v} \in L_{q,\text{loc}}(\Omega_T)$ for all $q < \alpha(\beta)$. Carefully iterating the process yields the result.

Remark: The techniques of this paper exploit the structure of the nonlinear term and the fact that $\mathbf{v}$ is solenoidal; in particular these techniques fail for non-solenoidal solutions of the more general system

$$\mathbf{u}_t - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = 0\,. \tag{11}$$

2 Proof of Theorem 1

For simplicity, suppose that $\Omega_T \subset \mathbb{R}^3 \times \mathbb{R}$, and let

$$Q_R = Q_R(x_o,t_o) = B_R(x_o) \times (t_o - R^2, t_o) \subset \Omega_T.$$

Let $t_o - R^2 < \tau < t_o$ and define $Q_R^\tau = B_R(x_o) \times (t_o - R^2, \tau)$. Choose $0 < \sigma < 1$, and let $\zeta \in C^\infty(\Omega_T)$ be a cutoff function so that $\zeta(x,t) = 1$ if $(x,t) \in Q_{\sigma R}$, so that $\zeta(x,t) = 0$ if either $|x - x_o| = R$ or $t = t_o - R^2$, and so that $|\zeta_t| + |\nabla\zeta|^2 \le C_{\sigma,R}$. Further, let $\{J_\eta(x,t)\}_{\eta>0}$ be a family of symmetric mollifying kernels in space and time; given a function $f(x,t)$, we shall denote the mollification $(J_\eta * f)$ by f_η

Let $k > 0$, $\omega > 0$, and choose $0 < \epsilon < \frac{2}{3}$ so small that $\frac{10}{10-3\epsilon} \le \mu$. Suppose that $\mathbf{v} \in V^2_{\text{loc}}(\Omega_T)$; recall the Sobolev embedding $V^2_{\text{loc}}(\Omega_T) \hookrightarrow L_{10/3,\text{loc}}(\Omega_T)$ when $\Omega_T \subseteq \mathbb{R}^3 \times \mathbb{R}$. Fix $i \in \{1,2,3\}$, and consider the function

$$\phi^i(x,t) = \left\{[(v^i_\eta(x,t) - k)_+ + \omega]^\epsilon \zeta^2(x,t)\right\}_\eta. \tag{12}$$

If η is sufficiently small, ϕ^i is a valid test function; multiplying the i^{th} component of the first of (1) and integrating over Ω_T we obtain

$$\begin{aligned} 0 = &\iint_{Q_R^\tau} \left(\frac{\partial}{\partial t} v_\eta^i\right) [(v_\eta^i - k)_+ + \omega]^\epsilon \zeta^2 \,dx\,dt \\ &+ \sum_{j=1}^{3} \iint_{Q_R^\tau} \left(\frac{\partial}{\partial x_j} v_\eta^i\right) \frac{\partial}{\partial x_j} \{[(v_\eta^i - k)_+ + \omega]^\epsilon \zeta^2\} dx\,dt \\ &+ \sum_{j=1}^{3} \iint_{Q_R^\tau} \left[v^j \frac{\partial v^i}{\partial x_j}\right]_\eta [(v_\eta^i - k)_+ + \omega]^\epsilon \zeta^2 \,dx\,dt \\ &+ \iint_{Q_R^\tau} \frac{\partial p_\eta}{\partial x_i} [(v_\eta^i - k)_+ + \omega]^\epsilon \zeta^2 \,dx\,dt \\ = &\, I_1 + I_2 + I_3 + I_4 \end{aligned}$$

for each fixed i.

To estimate I_1, first let $\omega \downarrow 0$ so that

$$\begin{aligned} \lim_{\omega\downarrow 0} I_1 &= \iint_{Q_R^\tau} \left\{\frac{\partial}{\partial t}(v_\eta^i - k)_+\right\} (v_\eta^i - k)_+^\epsilon \zeta^2 \,dx\,dt \\ &= \frac{1}{\epsilon+1} \iint_{Q_R^\tau} \left\{\frac{\partial}{\partial t}\left[(v_\eta^i - k)_+^{\epsilon+1}\right]\right\} \zeta^2 \,dx\,dt \\ &= \frac{1}{\epsilon+1} \int_{B_R} (v_\eta^i - k)_+ \zeta^2 \Big|_{t=\tau} dx - \frac{2}{\epsilon+1} \iint_{Q_R^\tau} (v_\eta^i - k)_+^{\epsilon+1} \zeta\zeta_t \,dx\,dt. \end{aligned} \tag{13}$$

Then, sending $\eta \downarrow 0$ we obtain for almost every τ

$$\begin{aligned} \lim_{\eta\downarrow 0} \lim_{\omega\downarrow 0} I_1 = &\frac{1}{\epsilon+1} \int_{B_R(x_o)} (v^i - k)_+^{\epsilon+1} \zeta^2 \Big|_{t=\tau} dx \\ &- \frac{2}{\epsilon+1} \iint_{Q_R^\tau} (v^i - k)_+^{\epsilon+1} \zeta\zeta_t \,dx\,dt. \end{aligned} \tag{14}$$

To estimate the I_2 term, first rewrite the integral as

$$\begin{aligned} I_2 = &\,\epsilon \sum_{j=1}^{3} \iint_{Q_R^\tau} \left\{\frac{\partial}{\partial x_j}[(v_\eta^i - k)_+ + \omega]\right\}^2 [(v_\eta^i - k)_+ + \omega]^{\epsilon-1} \zeta^2 \,dx\,dt \\ &+ 2 \sum_{j=1}^{3} \iint_{Q_R^\tau} \frac{\partial v_\eta^i}{\partial x_j} [(v_\eta^i - k)_+ + \omega]^\epsilon \zeta\zeta_{x_j} \,dx\,dt \\ = &\,\frac{4\epsilon}{(\epsilon+1)^2} \sum_{j=1}^{3} \iint_{Q_R^\tau} \left\{\frac{\partial}{\partial x_i}[(v_\eta^i - k)_+ + \omega]^{\frac{\epsilon+1}{2}}\right\}^2 \zeta^2 \,dx\,dt \\ &+ 2 \sum_{j=1}^{3} \iint_{Q_R^\tau} \frac{\partial v_i}{\partial x_j} [(v_\eta^i - k)_+ + \omega]^\epsilon \zeta\zeta_{x_j} \,dx\,dt. \end{aligned} \tag{15}$$

Using Fatou's lemma in the first term and the fact that $\epsilon \le 1$ in the second, we send $\omega \downarrow 0$ to obtain

$$\begin{aligned}\liminf_{\omega\downarrow 0} I_2 \ge C_\epsilon \iint_{Q_R^\tau} \left|\nabla\left\{(v_\eta^i - k)_+^{\frac{\epsilon+1}{2}}\right\}\right|^2 \zeta^2\, dx\, dt \\ - 2\iint_{Q_R^\tau} |\nabla(v_\eta^i - k)_+|\,(v_\eta^i - k)_+^\epsilon \zeta\,|\nabla\zeta|\, dx\, dt. \quad (16)\end{aligned}$$

Using Young's inequality and Fatou's lemma once more, we obtain

$$\begin{aligned}\liminf_{\eta\downarrow 0}\liminf_{\omega\downarrow 0} I_2 \ge C_\epsilon \iint_{Q_R^\tau} \left|\nabla\left\{(v^i - k)_+^{\frac{\epsilon+1}{2}}\right\}\right|^2 \zeta^2\, dx\, dt \\ - C_\epsilon \iint_{Q_R^\tau} (v^i - k)_+^{\epsilon+1}|\nabla\zeta|^2\, dx\, dt. \quad (17)\end{aligned}$$

To estimate I_3, note that because $\epsilon \le 2/3$ the integral is uniformly bounded and we can pass to the limit as $\omega \downarrow 0$ then as $\eta \downarrow 0$ to obtain

$$\begin{aligned}\lim_{\eta\downarrow 0}\lim_{\omega\downarrow 0} I_3 &= \sum_{j=1}^{3} \iint_{Q_R^\tau} v^j \frac{\partial v^i}{\partial x_j}(v^i - k)_+^\epsilon \zeta^2\, dx\, dt \\ &= \frac{1}{\epsilon+1}\sum_{j=1}^{3} \iint_{Q_R^\tau} v^j \left\{\frac{\partial}{\partial x_j}(v^i - k)_+^{\epsilon+1}\right\}\zeta^2\, dx\, dt.\end{aligned} \quad (18)$$

Then, because $\mathbf{v}$ is solenoidal we can integrate by parts to obtain

$$\lim_{\eta\downarrow 0}\lim_{\omega\downarrow 0} I_3 = \frac{-2}{\epsilon+1}\sum_{j=1}^{3}\iint_{Q_R^\tau} v^j (v^i - k)_+^{\epsilon+1}\zeta\zeta_{x_j}\, dx\, dt. \quad (19)$$

As for I_4, we have

$$I_4 = \iint_{Q_R^\tau} \frac{\partial p_\eta}{\partial x_i}\left[(v_\eta^i - k)_+ + \omega\right]^\epsilon \zeta^2\, dx\, dt \quad (20)$$

so since $\nabla p \in L_{\mu,\mathrm{loc}}(\Omega_T)$, $|\mathbf{v}| \in L_{10/3,\mathrm{loc}}(\Omega_T)$ and $\frac{10}{10-3\epsilon} \le \mu$, we can pass to the limit, obtaining

$$\lim_{\eta\downarrow 0}\lim_{\omega\downarrow 0} I_4 = \iint_{Q_R^\tau} |\nabla p|(v^i - k)_+^\epsilon \zeta^2\, dx\, dt. \quad (21)$$

Combine these results and use the arbitrariness of τ to obtain

$$\begin{aligned}
\|(v^i-k)_+^{(\epsilon+1)/2}\zeta\|^2_{V^2(Q_R)} &= \operatorname*{ess\,sup}_{t_o-R^2<\tau<t_o} \int_{B_R} \left|(v^i-k)_+^{(\epsilon+1)/2}\zeta\right|^2 \Big|_\tau dx \\
&\quad + \iint_{Q_R} \left|\nabla\left\{(v^i-k)_+^{(\epsilon+1)/2}\zeta\right\}\right|^2 dx\,dt \\
&\le C_{\epsilon,\sigma,R} \iint_{Q_R} (v^i-k)_+^{\epsilon+1}\,dx\,dt \qquad (22)\\
&\quad + C_{\epsilon,\sigma,R} \iint_{Q_R} |\mathbf{v}|(v^i-k)_+^{\epsilon+1}\,dx\,dt \\
&\quad + C_\epsilon \iint_{Q_R} |\nabla p|(v^i-k)_+^{\epsilon}\,dx\,dt
\end{aligned}$$

which is our energy estimate.

The restriction $\epsilon \le \frac{2}{3}$, needed to pass to the limit in I_3 implies that $\frac{\epsilon+1}{2} < 1$ so we can not use this inequality to directly prove strong L_q estimates. We can, however, use this technique to prove a weak-L_q estimate. Indeed, estimate the left side as follows.

$$\begin{aligned}
\|(v^i-k)_+^{(\epsilon+1)/2}\zeta\|^2_{V^2(Q_R)} &\ge C\|(v^i-k)_+^{(\epsilon+1)/2}\|^2_{L_{10/3}(Q_{\sigma R})} \\
&\ge C\left\{\iint_{Q_{\sigma R}} (v^i-k)_+^{5(\epsilon+1)/3}\,dx\,dt\right\}^{3/5} \\
&\ge Ck^{\epsilon+1}\left[\operatorname{meas}\{(x,t)\in Q_{\sigma R} : v^i(x,t) > 2k\}\right]^{3/5}. \qquad (23)
\end{aligned}$$

Suppose that $\mathbf{v} \in L_{\beta,\mathrm{loc}}(\Omega_T)$ for some $\beta \ge 10/3$. We can then estimate the first term on the right side of (22) as

$$\begin{aligned}
\iint_{Q_R} (v^i-k)_+^{\epsilon+1}\,dx\,dt &\le \left(\iint_{Q_R} (v^i-k)_+^\beta\,dx\,dt\right)^{(\epsilon+1)/\beta} \left(\operatorname{meas}[v^i>k]\right)^{1-(\epsilon+1)/\beta} \\
&\le \|\mathbf{v}\|^{\epsilon+1}_{L_\beta(Q_R)} \left\{|\mathbf{v}|^\beta_{L^{\text{weak}}_\beta(Q_R)} \frac{1}{k^\beta}\right\}^{1-(\epsilon+1)/\beta} \\
&\le \|\mathbf{v}\|^\beta_{L_\beta(Q_R)} \left(\frac{1}{k}\right)^{\beta-\epsilon-1}. \qquad (24)
\end{aligned}$$

Similarly,

$$\iint_{Q_R} |\mathbf{v}|(v^i-k)_+^{\epsilon+1}\,dx\,dt \le \|\mathbf{v}\|^\beta_{L_\beta(Q_R)} \left(\frac{1}{k}\right)^{\beta-\epsilon-2}. \qquad (25)$$

Lastly

$$\begin{aligned}\iint_{Q_R} |\nabla p|(v^i-k)_+^\epsilon \, dx\, dt \leq \left(\iint_{Q_R} |\nabla p|^\mu \, dx\, dt\right)^{1/\mu} \times \\ \left(\iint_{Q_R} |\mathbf{v}|^\beta \, dx\, dt\right)^{\epsilon/\beta} \left(\operatorname{meas}[v^i>k]\right)^{1-1/\mu-\epsilon/\beta} \\ \leq \|\nabla p\|_{L_\mu(Q_R)} \|\mathbf{v}\|^\epsilon_{L_\beta(Q_R)} \left(|\mathbf{v}|^\beta_{L^{\text{weak}}_\beta(Q_R)} \frac{1}{k^\beta}\right)^{1-1/\mu-\epsilon/\beta} \\ \leq \|\nabla p\|_{L_\mu(Q_R)} \|\mathbf{v}\|^{\beta(1-1/\mu)}_{L_\beta(Q_R)} \left(\frac{1}{k}\right)^{\beta(1-1/\mu)-\epsilon} .\end{aligned} \tag{26}$$

Combining these estimates, we find

$$\begin{aligned}\operatorname{meas}\{(x,t)\in Q_{\sigma R} : v^i(x,t)>2k\} \leq C\|\mathbf{v}\|^{5\beta/3}_{L_\beta(Q_R)} \left(\frac{1}{k}\right)^{5\beta/3} \\ +C\|\mathbf{v}\|^{5\beta/3}_{L_\beta(Q_R)} \left(\frac{1}{k}\right)^{5(\beta-1)/3} + C\|\nabla p\|^{5/3}_{L_\mu(Q_R)} \|\mathbf{v}\|^{5\beta(1-1/\mu)/3}_{L_\beta(Q_R)} \left(\frac{1}{k}\right)^{5[\beta(1-1/\mu)+1]/3} .\end{aligned} \tag{27}$$

Repeating this process with test functions $(v^i+k)_-$, we obtain an estimate of the form

$$|v^i|_{L^{\text{weak}}_{\alpha(\beta)}(Q_{\sigma R})} \leq C(\epsilon,\beta,R,\sigma,\mu,\|\nabla p\|_{L_\mu(Q_R)},\|\mathbf{v}\|_{L_\beta(Q_R)}) \tag{28}$$

for each i, where

$$\alpha(\beta) = \min\left\{\frac{5}{3}(\beta-1), \frac{5}{3}\left[\beta\left(\frac{\mu-1}{\mu}\right)+1\right]\right\}. \tag{29}$$

Thus $\mathbf{v}\in L_{\beta,\text{loc}}(\Omega_T)$ implies $\mathbf{v}\in L_{q,\text{loc}}(\Omega_T)$ for every $q<\alpha(\beta)$.

Our result is proven by iteration. Since $V^2_{\text{loc}}(\Omega_T)\hookrightarrow L_{10/3,\text{loc}}(\Omega_T)$, set $\beta_o=\frac{10}{3}$, and inductively define

$$\beta_{n+1} = \alpha(\beta_n) = \min\left\{\frac{5}{3}(\beta_n-1), \frac{5}{3}\left[\beta_n\left(\frac{\mu-1}{\mu}\right)+1\right]\right\}. \tag{30}$$

Now (28) implies that $\mathbf{v}\in L_{q,\text{loc}}(\Omega_T)$ for every $q<\beta_n$, for every n. Set

$$\begin{aligned}\gamma(\beta) &= \tfrac{5}{3}(\beta-1), \\ \delta(\beta) &= \tfrac{5}{3}\left[\beta\left(\frac{\mu-1}{\mu}\right)+1\right].\end{aligned}$$

If $\beta\geq\frac{10}{3}$, then $\gamma(\beta)\geq\frac{7}{6}\beta$. On the other hand, $\delta(\beta)\geq\beta$ if and only if

$$\beta \leq \frac{5}{5/\mu-2} \tag{31}$$

so that the sequence $\beta, \delta(\beta), \delta(\delta(\beta)), \ldots$ converges to $\frac{5}{5/\mu-2}$ independently of the choice of β. Thus $\mathbf{v} \in L_{q,\mathrm{loc}}(\Omega_T)$ for all $q < \frac{5}{5/\mu-2}$. If $\mu > \frac{5}{3}$, we can choose $q > 5$ and apply the regularity result of Serrin [9] which guarantees the local boundedness and smoothness of $\mathbf{v}$ in the spatial variables.

Remark. Theorem 2, the general result in N spatial dimensions, is proven in the same fashion as Theorem 1.

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