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Multiple positive solutions for equations involving critical Sobolev exponent in $\mathbb{R}^N$ *

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Abstract

This article concerns with the problem

$$-\text{div}(|\nabla u|^{m-2}\nabla u) = \lambda h u^q + u^{m^*-1}, \quad \text{in} \quad \mathbb{R}^N .$$

Using the Ekeland Variational Principle and the Mountain Pass Theorem, we show the existence of $\lambda^* > 0$ such that there are at least two non-negative solutions for each λ in $(0, \lambda^*)$.

1 Introduction

In this work, we study the existence of solutions for the problem

$$(P) \qquad \begin{cases} -\Delta_m u = \lambda h u^q + u^{m^*-1}, \ \mathbb{R}^N \\ u \geq 0, \ u \neq 0, \ u \in D^{1,m}(\mathbb{R}^N) \end{cases}$$

where $\Delta_m u = \text{div}\,(|\nabla u|^{m-2}\,\nabla u)$, $\lambda > 0$, $N > m \geq 2$, $m^* = Nm/(N-m)$, $0 < q < m-1$, h is a nonnegative function with $L^\Theta(\mathbb{R}^N)$ with $\Theta = \frac{Nm}{Nm-(q+1)(N-m)}$, and

$$D^{1,m}(\mathbb{R}^N) = \{u \in L^{m^*}(\mathbb{R}^N) \mid \frac{\partial u}{\partial x_i} \in L^m(\mathbb{R}^N)\}$$

endowed with the norm $\|u\| = \left(\int |\nabla u|^m\right)^{1/m}$.

The case $q = 0$, $m = 2$ was studied by Tarantello [20], and a more general case with $m \geq 2$ by Cao, LI & Zhou [5]. In these two references, [5] and [20], it is proved that (P) has multiple solutions. In the case $m = 2$, $h \in L^p(\mathbb{R}^N)$ with $p_1 \leq p \leq p_2$ and $1 < q < 2^* - 1$, Pan [18] proved the existence of a positive solution for (P). In the more general case, $m \geq 2$, $h \in L^\Theta(\mathbb{R}^N)$, Gonçalves & Alves [10] proved the existence of a positive solution for (P).

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By a solution to (P), we mean a function $u \in D^{1,m}(\mathbb{R}^N)$, $u \geq 0$ and $u \neq 0$ satisfying

$$\int |\nabla u|^{m-2} \nabla u \nabla \Phi = \lambda \int h u^q \Phi + \int u^{m^*-1}\Phi, \quad \forall \Phi \in D^{1,m}(\mathbb{R}^N).$$

Hereafter, $\int$, $D^{1,m}$, L^p and $|.|_p$ stand for $\int_{\mathbb{R}^N}$, $D^{1,m}(\mathbb{R}^N)$, $L^p(\mathbb{R}^N)$ and $|.|_{L^p}$ respectively.

In the search of solutions we apply minimizing arguments to the energy functional

$$I(u) = \frac{1}{m}\int |\nabla u|^m - \frac{\lambda}{q+1}\int h\left(u^+\right)^{q+1} - \frac{1}{m^*}\int \left(u^+\right)^{m^*} \tag{1}$$

associated to (P), where $u^+(x) = \max\{u(x), 0\}$. Note that the condition $h \in L^\Theta$ implies that $I \in C^1\left(D^{1,m}, \mathbb{R}\right)$.

To show the existence of at least two critical points of the energy functional, we shall use the Ekeland Variational Principle [8], and the Mountain Pass Theorem of Ambrosetti & Rabinowitz [2] without the Palais-Smale condition. Using the Ekeland Variational Principle, we obtain a solution u_1 with $I(u_1) < 0$, and by the Mountain Pass Theorem we prove the existence of a second solution u_2 with $I(u_2) > 0$. Techniques for finding the solutions u_1 and u_2 are borrowed from Cao, Li & Zhou [5]. Then we combine these techniques with arguments developed by Chabrowski [6], Noussair, Swanson & Jianfu [17], Jianfu & Xiping [12], Azorero & Alonzo [9], Gonçalves & Alves [10] and Alves, Gonçalves & Miyagaki [1] to obtain the following result

Theorem 1 *There exists a constant $\lambda^* > 0$, such that (P) has at least two solutions, u_1 and u_2, satisfying*

$$I(u_1) < 0 < I(u_2) \quad \forall \lambda \in (0, \lambda^*)\,.$$

2 Preliminary Results

In this section we establish some results needed for the proof of Theorem 1.

Definition. A sequence $\{u_n\} \subset D^{1,m}$ is called a $(PS)_c$ sequence, if $I(u_n) \to c$ and $I'(u_n) \to 0$.

Lemma 1 *If $\{u_n\}$ is a $(PS)_c$ sequence, then $\{u_n\}$ is bounded, and $\{u_n^+\}$ is a $(PS)_c$ sequence.*

Proof. Using the hypothesis that $\{u_n\}$ is a $(PS)_c$ sequence, there exist n_o and $M > 0$ such that

$$I(u_n) - \frac{1}{m^*} I'(u_n)u_n \leq M + \|u_n\| \quad \forall n \geq n_o\,. \tag{2}$$

Now, using (1) and the Hölder's inequality, we have

$$I(u_n) - \frac{1}{m^*} I'(u_n)u_n \geq \frac{1}{N} \|u_n\|^m + c_1 \|u_n\|^{q+1} \tag{3}$$

where c_1 is a constant depending of $N, m, q, \|h\|_\Theta$ and Θ. It follows from (2) and (3) that $\{u_n\}$ is bounded. Now, we shall show that $\{u_n^+\}$ is a also $(PS)_c$ sequence. Since $\{u_n\}$ is bounded, the sequence $u_n^- = u_n - u_n^+$ is also bounded. Then

$$I'(u_n)u_n^- \to 0$$

and we conclude that

$$\left\|u_n^-\right\| \to 0. \tag{4}$$

From (4) we achieve that

$$\|u_n\| = \left\|u_n^+\right\| + o_n(1). \tag{5}$$

Therefore, by (4) and (5)

$$I(u_n) = I(u_n^+) + o_n(1)$$

and

$$I'(u_n) = I'(u_n^+) + o_n(1),$$

which consequently implies that $\{u_n^+\}$ is a $(PS)_c$ sequence. $\square$

From Lemma 1, it follows that any $(PS)_c$ sequence can be considered as a sequence of nonnegative functions.

Lemma 2 *If $\{u_n\}$ is a $(PS)_c$ sequence with $u_n \rightharpoonup u$ in $D^{1,m}$, then $I'(u) = 0$, and there exists a constant $M > 0$ depending of $N, m, q, |h|_\Theta$ and Θ, such that*

$$I(u) \geq -M\lambda^\Theta$$

Proof. If $\{u_n\}$ is a $(PS)_c$ sequence with $u_n \rightharpoonup u$, using arguments similar to those found in [10], [12] and [17], we can obtain a subsequence, still denoted by u_n, satisfying

$$u_n(x) \to u(x) \quad \text{a.e. in} \quad \mathbb{R}^N \tag{6}$$

$$\nabla u_n(x) \to \nabla u(x) \quad \text{a.e. in} \quad \mathbb{R}^N \tag{7}$$

$$u(x) \geq 0 \quad \text{a.e. in} \quad \mathbb{R}^N. \tag{8}$$

From (6), (7) and using the hypothesis that $\{u_n\}$ is bounded in $D^{1,m}$, we get

$$I'(u) = 0\,, \tag{9}$$

which in implies $I'(u)u = 0$, and

$$\|u\|^m = \lambda \int h u^{q+1} + \int u^{m^*}\,.$$

Consequently

$$I(u) = \lambda \left(\frac{1}{m} - \frac{1}{q+1}\right) \int h u^{q+1} + \frac{1}{N} \int u^{m^*}.$$

Using Hölder and Young Inequalities, we obtain

$$I(u) \geq -\frac{1}{N}\,|u|_{m^*}^{m^*} - M\lambda^{\Theta} + \frac{1}{N}\,|u|_{m^*}^{m^*} = -M\lambda^{\Theta}$$

where $M = M(N, m, q, \Theta, \|h\|_{\Theta})$. □

For the remaining of this article, we will denote by S the best Sobolev constant for the imbedding $D^{1,m} \hookrightarrow L^{m^*}$.

Lemma 3 *Let $\{u_n\} \subset D^{1,m}$ be a $(PS)_c$ sequence with*

$$c < \frac{1}{N} S^{N/m} - M\lambda^{\Theta}\,,$$

where $M > 0$ is the constant given in Lemma 2. Then, there exists a subsequence $\{u_{n_j}\}$ that converges strongly in $D^{1,m}$.

Proof By Lemmas 1 and 2, there is a subsequence, still denoted by $\{u_n\}$ and a function $u \in D^{1,m}$ such that $u_n \rightharpoonup u$. Let $w_n = u_n - u$. Then by a lemma in Brezis & Lieb [3], we have

$$\|w_n\|^m = \|u_n\|^m - \|u\|^m + o_n(1) \tag{10}$$

$$\|w_n\|_{m^*}^{m^*} = |u_n|_{m^*}^{m^*} - |u|_{m^*}^{m^*} + o_n(1)\,. \tag{11}$$

Using the Lebesgue theorem (see Kavian [13]), it follows that

$$\int h u_n^{q+1} \longrightarrow \int h u^{q+1}. \tag{12}$$

From (10), (11) and (12), we obtain

$$\|w_n\|^m = |w_n|_{m^*}^{m^*} + o_n(1) \tag{13}$$

and

$$\frac{1}{m}\,\|w_n\|^m - \frac{1}{m^*}\,|w_n|_{m^*}^{m^*} = c - I(u) + o_n(1). \tag{14}$$

Using the hypothesis that $\{w_n\}$ is bounded in $D^{1,m}$, there exists $l \geq 0$ such that

$$\|w_n\|^m \to l \geq 0. \tag{15}$$

From (13) and (15), we have

$$|w_n|_{m^*}^{m^*} \to l, \tag{16}$$

and using the best Sobolev constant S and recalling that

$$\|w_n\|^m \geq S\left(\int |w_n|^{m^*}\right)^{m/m^*}, \tag{17}$$

we deduce from (15), (16) and (17) that

$$l \geq S l^{m/m^*}. \tag{18}$$

Now, we claim that $l = 0$. Indeed, if $l > 0$, from (18)

$$l \geq S^{N/m}. \tag{19}$$

By (14), (15) and (16), we have

$$\frac{1}{N} l = c - I(u). \tag{20}$$

From (19), (20) and Lemma 2 we get

$$c \geq \frac{1}{N} S^{N/m} - M\lambda^{\Theta},$$

but this result contradicts the hypothesis. Thus, $l = 0$ and we conclude that

$$u_n \to u \quad \text{in} \quad D^{1,m}.$$

3 Existence of a first solution (Local Minimization)

Theorem 2 *There exists a constant $\lambda_1^* > 0$ such that for $0 < \lambda < \lambda_1^*$ Problem (P) has a weak solution u_1 with $I(u_1) < 0$.*

Proof. Using arguments similar to those developed in [5], we have

$$I(u) \geq \left(\frac{1}{m} - \epsilon\right) \|u\|^m + o\left(\|u\|^m\right) - C(\epsilon)\lambda^{m/(m-(q+1))},$$

where $C(\epsilon)$ is a constant depending on $\epsilon > 0$. The last inequality implies that for small ϵ, there exist constants γ, ρ and $\lambda_1^* > 0$ such that

$$I(u) \geq \gamma > 0, \quad \|u\| = \rho, \quad \text{and} \quad 0 < \lambda < \lambda_1^*.$$

Using the Ekeland Variational Principle, for the complete metric space $\overline{B}_\rho(0)$ with $d(u,v)=\|u-v\|$, we can prove that there exists a $(PS)_{\gamma_o}$ sequence $\{u_n\}\subset \overline{B}_\rho(0)$ with

$$\gamma_o=\inf\{I(u)\mid u\in\overline{B}_\rho(0)\}.$$

Choosing a nonnegative function $\Phi\in D^{1,m}\backslash\{0\}$, we have that $I(t\Phi)<0$ for small $t>0$ and consequently $\gamma_o<0$.

Taking $\lambda_1^*>0$, such that

$$0<\frac{1}{N}S^{N/m}-M\lambda^{\Theta}\quad \forall\lambda\in(0,\lambda_1^*)$$

from Lemma 3, we obtain a subsequence $\{u_{n_j}\}\subset\{u_n\}$ and $u_1\in D^{1,m}$, such that

$$u_{n_j}\to u \quad\text{in}\quad D^{1,m}\,.$$

Therefore,

$$I'(u_1)=0\quad\text{and}\quad I(u_1)=\gamma_o<0\,,$$

which completes this proof. $\square$

4 Existence of a second solution (Mountain Pass)

In this section, we shall use arguments similar to those explored by Cao, Li & Zhou [5], Chabrowski [6], Noussair, Swanson & Jianfu [17], Jianfu & Xiping [12] and Gonçalves & Alves [10] to obtain the following

Theorem 3 *There exists a constant $\lambda_2^*>0$ such that for $0<\lambda<\lambda_2^*$ Problem (P) has a weak solution u_2 with $I(u_2)>0$.*

Proof. By arguments found in [5] and [10], we can prove that there exists $\delta_1>0$ such that for all $\lambda\in(0,\delta_1)$, the functional I has the Mountain Pass Geometry, that is:

(i) There exist positive constants r,ρ with $I(u)\ge r>0$ for $\|u\|=\rho$

(ii) There exists $e\in D^{1,m}$ with $I(e)<0$ and $\|e\|>\rho$.

Then by [16], there exists a $(PS)_{\gamma_1}$ sequence $\{v_n\}$ with

$$\gamma_1=\inf_{g\in\Gamma}\max_{t\in[0,1]}I(g(t))$$

where

$$\Gamma=\{g\in C([0,1],D^{1,m})\mid g(0)=0\quad\text{and}\quad g(1)=e\}\,.$$

Using the next claim, which is a variant of a result found in [5], we can complete the proof of this theorem.

Claim. There exists $\lambda_2^* > 0$ such that for the constant M given by Lemma 2,

$$0 < \gamma_1 < \frac{1}{N} S^{N/m} - M\lambda^{\Theta} \quad \forall \lambda \in (0, \lambda_2^*)\,.$$

Assuming this claim, by Lemma 3 there exists a subsequence $\{v_{n_j}\} \subset \{v_n\}$ and a function $u_2 \in D^{1,m}$ such that $v_{n_j} \to u_2$. Therefore,

$$I'(u_2) = 0 \quad \text{and} \quad I(u_2) = \gamma_1 > 0\,.$$

Which concludes the present proof. □

Verification of the above claim. For $x \in \mathbb{R}^N$, let

$$\Psi(x) = \frac{\left[N\left(\frac{N-m}{m-1}\right)^{m-1}\right]^{(N-m)/m^2}}{\left[1 + |x|^{m/(m-1)}\right]^{\frac{N-m}{m}}}\,.$$

Then it is well known that (see [7] or [19])

$$\|\Psi\|^m = |\Psi|_{m^*}^{m^*} = S^{N/m}\,. \tag{21}$$

Let $\delta_2 > 0$ such that

$$\frac{1}{N} S^{N/m} - M\lambda^{\Theta} > 0 \quad \forall \lambda \in (0, \delta_2)\,.$$

Then from (1) and (21), we have

$$I(t\Psi) \le \frac{t^m}{m} S^{N/m}\,,$$

and there exists $t_o \in (0,1)$ with

$$\sup_{0 \le t \le t_o} I(t\Psi) < \frac{1}{N} S^{N/m} - M\lambda^{\Theta} \quad \forall \lambda \in (0, \delta_2)\,.$$

Moreover, from (1) and (21), we have

$$I(t\Psi) = \left(\frac{t^m}{m} - \frac{t^{m^*}}{m^*}\right) S^{N/m} - \frac{\lambda t^{q+1}}{q+1} \int h\Psi^{q+1}\,,$$

and remarking that

$$\left(\frac{t^m}{m} - \frac{t^{m^*}}{m^*}\right) \le \frac{1}{N} \quad \forall t \ge 0,$$

we obtain

$$I(t\Psi) \le \frac{1}{N} S^{N/m} - \frac{\lambda t^{q+1}}{q+1} \int h\Psi^{q+1}\,;$$

therefore,

$$\sup_{t\geq t_o} I(t\Psi) \leq \frac{1}{N}S^{N/m} - \frac{\lambda t_0^{q+1}}{q+1}\int h\Psi^{q+1}.$$

Now, taking $\lambda > 0$ such that

$$-\frac{\lambda t_0^{q+1}}{q+1}\int h\Psi^{q+1} < -M\lambda^{\Theta}$$

that is,

$$0 < \lambda < \left(\frac{t_0^{q+1}\int h\Psi^{q+1}}{M(q+1)}\right)^{1/(\Theta-1)} = \delta_3$$

we deduce that

$$\sup_{t\geq t_o} I(t\Psi) < \frac{1}{N}S^{N/m} - M\lambda^{\Theta} \quad \forall\lambda \in (0,\delta_3)\,.$$

Choosing $\lambda_2^* = \min\{\delta_1,\delta_2,\delta_3\}$, we have

$$\sup_{t\geq 0} I(t\Psi) < \frac{1}{N}S^{N/m} - M\lambda^{\Theta} \quad \forall\lambda \in (0,\lambda_2^*)\,.$$

and consequently

$$0 < \gamma_1 < \frac{1}{N}S^{N/m} - M\lambda^{\Theta} \quad \forall\lambda \in (0,\lambda_2^*)$$

which proves the claim.

Proof of Theorem 1. Theorem 1 is an immediate consequence of Theorems 2 and 3.

Remark. Using Lemma 3 and the same arguments explored by Azorero & Alonzo, in the case $0 < q < p$ [9], we can easily show that for small λ the following problem has infinitely many solutions with negative energy levels.

$$(P)_* \qquad \begin{aligned} -\Delta_m u &= \lambda h\,|u|^{q-1}\,u + |u|^{m^*-2}\,u, \quad \text{in} \quad \mathbb{R}^N \\ u &\in D^{1,m} \end{aligned}$$

This result is obtained using the concept and properties of genus, and working with a truncation of the energy functional associated with $(P)_*$.

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