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# LIGHT RAYS IN STATIC SPACETIMES WITH CRITICAL ASYMPTOTIC BEHAVIOR: A VARIATIONAL APPROACH

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Abstract. Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a Lorentzian manifold equipped with the static metric $\langle \cdot, \cdot \rangle_z = \langle \cdot, \cdot \rangle - \beta(x) dt^2$. The aim of this paper is investigating the existence of lightlike geodesics joining a point $z_0 = (x_0, t_0)$ to a line $\gamma = \{x_1\} \times \mathbb{R}$ when coefficient $\beta$ has a quadratic asymptotic behavior by means of a variational approach.

## 1. Introduction and main result

The aim of this paper is investigating the existence of lightlike geodesics in suitable semi-Riemannian manifolds by using variational tools and topological methods. First of all, we recall the main definitions.

A *Lorentzian manifold* is a couple $(\mathcal{M}, \langle \cdot, \cdot \rangle_z)$ where $\mathcal{M}$ is a smooth connected finite-dimensional manifold and $\langle \cdot, \cdot \rangle_z$ is a Lorentzian metric, that is a smooth symmetric $(0, 2)$ tensor field which induces on the tangent space of each point of $\mathcal{M}$ a bilinear form of index 1.

The importance of the study of these manifolds comes from General Relativity, since some 4-dimensional Lorentzian manifolds are solutions of Einstein's equations. Differently from a Riemannian metric, a Lorentzian one does not induce a positive definite bilinear form on its tangent space, thus each one of its tangent vectors $v \neq 0$ can be *timelike*, *lightlike* or *spacelike* if the scalar product $\langle v, v \rangle_z$ is negative, null or positive, respectively, while $v = 0$ is always spacelike.

In order to have informations on the geometry of a Lorentzian manifold, it is important to look for its geodesics and, similarly to the definition given for a geodesic in a Riemannian manifold, the following definition can be stated.

**Definition 1.1.** Let $\mathcal{M}$ be a smooth finite dimensional manifold equipped with a Lorentzian metric $\langle \cdot, \cdot \rangle_z$. A geodesic in $\mathcal{M}$ is a smooth curve $z : I \to \mathcal{M}$ which solves the equation
$$D_s \dot{z}(s) = 0 \quad \text{for all } s \in I,$$
where $D_s$ denotes the covariant derivative along $z$ induced by the Levi-Civita connection of $\langle \cdot, \cdot \rangle_z$ and $I$ is a real interval.

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Let $z = z(s)$ be a geodesic on $\mathcal{M}$. It is easy to check that there exists a constant $E(z) \in \mathbb{R}$ such that

$$\langle \dot{z}(s), \dot{z}(s) \rangle_z \equiv E(z) \quad \text{for all } s \in I.$$

So, all the tangent vectors $\dot{z}(s)$, $s \in I$, have the same causal character and a geodesic $z = z(s)$ is *timelike*, *lightlike* or *spacelike* if $E(z)$ is negative, null or positive, respectively.

From a physical point of view the most significant geodesics are the timelike and the lightlike ones, named *causal geodesics*. In particular, in General Relativity gravitational fields can be descrived by means of suitable Lorentzian manifolds in which lightlike geodesics allow one to represent light rays.

In general, the study of geodesics in Lorentzian manifolds is not possible up to consider special models. Here, we deal with (standard) static manifolds defined as follows.

**Definition 1.2.** A Lorentzian manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle_z)$ is called (standard) static if there exists a finite dimensional Riemannian manifold $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$ such that $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ and $\langle \cdot, \cdot \rangle_z$ is given by

$$\langle \zeta, \zeta \rangle_z = \langle \xi, \xi \rangle - \beta(x)\tau^2 \tag{1.1}$$

for any $z = (x, t) \in \mathcal{M}_0 \times \mathbb{R}$ and $\zeta = (\xi, \tau) \in T_z\mathcal{M} \equiv T_x\mathcal{M}_0 \times \mathbb{R}$, where $\beta : \mathcal{M}_0 \to \mathbb{R}$ is a smooth and strictly positive scalar field.

Physically interesting examples of static spacetimes are anti-de Sitter, Schwarzschild or Reissner-Nordström ones; in the two-dimensional case, static spacetimes are essentially equivalent to Generalized Robertson-Walker ones (for more details, see [3]).

Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a Lorentzian manifold endowed with the static metric defined in (1.1). Our main result is stated as under the following hypotheses:

- (H1) $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$ is a complete $\mathcal{C}^3$ $n$-dimensional Riemannian manifold;
- (H2) $\beta$ has an asymptotic quadratic behaviour, that is there exist $\lambda \geq 0$, $\mu_1, \mu_2 \in \mathbb{R}$ and a point $\bar{x} \in \mathcal{M}_0$ such that

$$\beta(x) \leq \lambda d^2(x, \bar{x}) + \mu_1 d^p(x, \bar{x}) + \mu_2 \tag{1.2}$$

  where $d(\cdot, \cdot)$ is the distance induced on $\mathcal{M}_0$ by its Riemannian metric $\langle \cdot, \cdot \rangle$ and $0 \leq p < 2$.

**Theorem 1.3.** *Suppose that* (H1), (H2) *are satisfied. Then, there exist at least two non trivial lightlike geodesics joining the point $z_0 = (x_0, t_0)$ to the vertical line $\gamma = \{x_1\} \times \mathbb{R}$ if $x_0 \neq x_1$. Furthermore, if $\mathcal{M}_0$ is non contractible in itself, then there exist two sequences of such lightlike geodesics $z_n^+ = (x_n^+, t_n^+)$, $z_n^- = (x_n^-, t_n^-)$ such that $t_n^+(1) \nearrow +\infty$ and $t_n^-(1) \searrow -\infty$ as $n \nearrow +\infty$.*

In the previous years, the existence of causal geodesics in a static spacetime has been widely exploited. A first result follows from Avez-Seifert Theorem (see [3, Theorem 2.14]), in which it is proved that in a globally hyperbolic spacetime two causally related points can be joined by a causal geodesic. In fact, if $\mathcal{M}_0$ is a complete Riemannian manifold and $\beta$ has an (at most) quadratic growth the corresponding static spacetime is globally hyperbolic and Avez-Seifert Theorem can be applied (see [10, Corollary 3.4]).

Furthermore, the existence of lightlike geodesics joining a point $z_0 = (x_0, t_0)$ to a line $\gamma = \{x_1\} \times \mathbb{R}$ has been obtained by means of geometrical tools in assumptions (H1) and (H2) (see Remark 3.4).

On the other hand, by using variational methods, the existence and the multiplicity of such lightlike geodesics have been proved when $\beta$ is bounded (see [8, Subsection 6.3]) and then when $\beta$ has a subquadratic growth (see [5]). But in [2] the geodesic connectedness has been guaranteed if $\beta$ grows quadratically and this condition is optimal as showed by a family of explicit counterexamples (for more details, see [2, Section 7]).

Here, we want to improve both the variational results, by using hypothesis (H2), and the geometric existence one, obtaining a multiplicity theorem.

## 2. Variational tools

Let $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ be a static Lorentzian manifold equipped with metric (1.1). Furthermore, fix $z_0 = (x_0, t_0) \in \mathcal{M}$ and $\gamma = \{x_1\} \times \mathbb{R} \subset \mathcal{M}$.

If $(\mathcal{M}_0, \langle \cdot, \cdot \rangle)$ is a $\mathcal{C}^3$ $n$-dimensional Riemannian manifold, by the Nash Embedding Theorem we can assume that $\mathcal{M}_0$ is a submanifold of $\mathbb{R}^N$ and $\langle \cdot, \cdot \rangle$ is the restriction to $\mathcal{M}_0$ of its Euclidean metric, still denoted by $\langle \cdot, \cdot \rangle$.

On the other hand, without loss of generality, we can take $I = [0, 1]$, as geodesics are independent by affine reparametrizations, and $t_0 = 0$, since the coefficient $\beta$ of metric (1.1) does not depend on the coordinate $t$ (in fact, if $z(s) = (x(s), t(s))$ is a geodesic and $T > 0$ is a real number, the curve $z_T(s) = (x(s), t(s) + T)$ is still a geodesic).

Hence, our problem is reduced to look for solutions of

$$\begin{gathered} D_s \dot{z}(s) = 0 \quad \forall\, s \in I, \\ \langle \dot{z}(s), \dot{z}(s) \rangle_z = 0 \quad \forall\, s \in I, \\ x(0) = x_0, \quad x(1) = x_1, \quad t(0) = 0. \end{gathered} \tag{2.1}$$

Let $H^1(I, \mathbb{R}^N)$ be the Sobolev space of the absolutely continuous curves $x = x(s)$ whose derivative is square summable. Such a space can be endowed with the norm

$$\|x\|^2 = \int_0^1 \langle \dot{x}, \dot{x} \rangle ds + \int_0^1 \langle x, x \rangle ds.$$

If $\Omega^1(x_0, x_1)$ is the set of $H^1$-curves in $\mathcal{M}_0$ joining $x_0$ to $x_1$ and defined in $I$, then it is

$$\Omega^1(x_0, x_1) \equiv \{x \in H^1(I, \mathbb{R}^N) : x(I) \subset \mathcal{M}_0,\ x(0) = x_0,\ x(1) = x_1\}.$$

If $\mathcal{M}_0$ is complete, $\Omega^1(x_0, x_1)$ is a complete Riemannian manifold (see [9]) and its tangent space is

$$T_x \Omega^1(x_0, x_1) = \{\xi \in H^1(I, T\mathcal{M}_0) : \xi(s) \in T_{x(s)} \mathcal{M}_0 \ \forall s \in I, \xi(0) = \xi(1) = 0\}.$$

Furthermore, we can define $H^1(0)$, the subspace of $H^1(I, \mathbb{R})$ of those curves $t = t(s)$ such that $t(0) = 0$.

It is well known that problem (2.1) has a variational structure; so it is quite standard to prove that $\bar{z} = \bar{z}(s)$ is a lightlike geodesic joining $z_0 = (x_0, 0)$ to $\gamma$ if and only if it is a critical point of the $\mathcal{C}^1$ functional

$$f(z) = \frac{1}{2} \int_0^1 \langle \dot{z}, \dot{z} \rangle_z ds, \quad \text{in } Z = \Omega^1(x_0, x_1) \times H^1(0), \tag{2.2}$$

with critical level $f(\overline{z}) = 0$. But a direct investigation of the zero critical level of $f$ is not easy as the functional in (2.2) is unbounded both from below and from above. In order to overcome this difficulty, Fortunato, Giannoni and Masiello in [7] stated a new variational principle similar to the Fermat one so to introduce a new functional, arrival time $T = T(x)$, which is bounded from below on Riemannian manifold $\Omega^1(x_0, x_1)$.

**Theorem 2.1** (Fermat principle). *Let $\overline{z} : I \to \mathcal{M}$, $\overline{z} = \overline{z}(s)$, be a smooth curve such that $\overline{z} = (\overline{x}, \overline{t})$. Then, the following statements are equivalent:*

(a) *$\overline{z}$ is a solution of problem (2.1) with arrival time $\overline{t}(1) = T > 0$;*
(b) *$\overline{x}$ is a critical point of functional*

$$F(x) = \sqrt{\int_0^1 \langle \dot{x}, \dot{x} \rangle ds \cdot \int_0^1 \frac{1}{\beta(x)} ds} \quad \text{in } \Omega^1(x_0, x_1),$$

*with critical level $T = F(\overline{x}) > 0$ and*

$$\overline{t}(s) = T\Big( \int_0^1 \frac{1}{\beta(\overline{x})} ds \Big)^{-1} \int_0^s \frac{1}{\beta(\overline{x})} d\sigma \quad \text{for all } s \in I. \tag{2.3}$$

*Proof.* The proof can be found, for example, in [8]. Anyway, here, for completeness, we outline its main arguments. Fixed $T \in \mathbb{R}$, let us define

$$W_T = \{t \in H^1(0) : t(1) = T\}.$$

It is easy to see that $W_T$ is an affine submanifold of $H^1(I, \mathbb{R})$ whose tangent space is given by $H_0^1(I, \mathbb{R}) = \{\tau \in H^1(I, \mathbb{R}) \mid \tau(0) = \tau(1) = 0\}$. Thus, the space of curves joining $z_0$ to $(x_1, T)$ is

$$Z_T = \Omega^1(x_0, x_1) \times W_T$$

with tangent space given by

$$T_z Z_T \equiv T_x \Omega^1(x_0, x_1) \times H_0^1(I, \mathbb{R}) \quad \text{for any } z = (x, t) \in Z_T.$$

Let $f_T$ be the restriction of functional $f$ to $Z_T$, so, by (1.1), for each $z = (x, t) \in Z_T$ it is

$$f_T(z) = \frac{1}{2} \int_0^1 \langle \dot{x}, \dot{x} \rangle ds - \frac{1}{2} \int_0^1 \beta(x) \dot{t}^2 ds \tag{2.4}$$

whose differential is given by

$$f'_T(z)[\zeta] = \int_0^1 \langle \dot{x}, \dot{\xi} \rangle ds - \frac{1}{2} \int_0^1 \beta'(x)[\xi] \dot{t}^2 ds - \int_0^1 \beta(x) \dot{t} \dot{\tau} ds \tag{2.5}$$

for all $\zeta = (\xi, \tau) \in T_x \Omega^1(x_0, x_1) \times H_0^1(I, \mathbb{R})$. If, for simplicity, we assume

$$\begin{aligned} \frac{\partial f_T}{\partial x}(z)[\xi] &= f'_T(z)[(\xi, 0)] \quad \forall\, \xi \in T_x \Omega^1(x_0, x_1), \\ \frac{\partial f_T}{\partial t}(z)[\tau] &= f'_T(z)[(0, \tau)] \quad \forall\, \tau \in H_0^1(I, \mathbb{R}), \end{aligned}$$

then, $z$ is a critical point of functional $f_T$ in $Z_T$ if and only if $z \in N_T$ and $\frac{\partial f_T}{\partial x}(z)[\xi] = 0$ for all $\xi \in T_x \Omega^1(x_0, x_1)$, where $N_T = \{z \in Z_T \mid \frac{\partial f_T}{\partial t}(z) \equiv 0\}$ is the kernel of $\frac{\partial f_T}{\partial t}$ in $Z_T$.

Let us remark that, by (2.5) and simple calculations it follows that $N_T$ is the graph of the $\mathcal{C}^1$ map $\Phi_T : \Omega^1(x_0,x_1) \to W_T$ such that

$$\Phi_T(x)(s) = T\Big(\int_0^1 \frac{1}{\beta(x(\sigma))}d\sigma\Big)^{-1}\int_0^s \frac{1}{\beta(x(\sigma))}d\sigma \quad \text{for all } s \in I. \tag{2.6}$$

So, considered the restriction of $f_T$ to $N_T$, we can define a new $\mathcal{C}^1$ functional

$$J_T(x) = f_T(x,\Phi_T(x)), \quad x \in \Omega^1(x_0,x_1), \tag{2.7}$$

which can be explicitely written as

$$J_T(x) = \frac{1}{2}\int_0^1 \langle \dot{x},\dot{x}\rangle ds - \frac{1}{2}T^2\Big(\int_0^1 \frac{1}{\beta(x)}ds\Big)^{-1}, \quad x \in \Omega^1(x_0,x_1).$$

Now, let us point out that $\overline{z} = (\overline{x},\overline{t})$ solves (2.1) with $\overline{t}(1) = T$ if and only if $\overline{z} \in Z_T$ is such that $f_T'(\overline{z}) = 0$ and $f_T(\overline{z}) = 0$; hence, if and only if $\overline{x} \in \Omega^1(x_0,x_1)$ is such that $J_T'(\overline{x}) = 0$, $J_T(\overline{x}) = 0$ and $\overline{t} = \Phi_T(\overline{x})$.

Therefore, our problem (2.1) is reduced to search for $T > 0$ and $x \in \Omega^1(x_0,x_1)$ such that

$$\begin{gathered} J_T'(x) = 0, \\ J_T(x) = 0, \end{gathered} \tag{2.8}$$

i.e., to search for a couple $(x,T) \in \Omega^1(x_0,x_1) \times \mathbb{R}_+^*$ solution of the problem

$$\begin{gathered} \frac{\partial H}{\partial x}(x,T) = 0, \\ H(x,T) = 0, \end{gathered} \tag{2.9}$$

with $H(x,T) = 2J_T(x)$. By solving the equation $H(x,T) = 0$ we obtain

$$T^2 = \int_0^1 \langle \dot{x},\dot{x}\rangle ds \cdot \int_0^1 \frac{1}{\beta(x)}ds;$$

thus, in order to have a positive $T$, we can consider

$$F(x) = \sqrt{\int_0^1 \langle \dot{x},\dot{x}\rangle ds \cdot \int_0^1 \frac{1}{\beta(x)}ds}. \tag{2.10}$$

We can easily see that $\mathcal{G} = \{(x,t) \in \Omega^1(x_0,x_1) \times \mathbb{R}_+ \mid H(x,T) = 0\}$ is the graph of $F$. So, by applying the abstract theorem in [7, Theorem 2.3] it follows that, if $(\overline{x},T)$ is a solution of problem (2.9) with $T > 0$, then $\overline{x}$ is a critical point of $F$ such that $T = F(\overline{x}) > 0$ and vice versa. $\square$

**Remark 2.2.** By differentiating the map $x \in \Omega^1(x_0,x_1) \longmapsto H(x,F(x)) \in \mathbb{R}$ we have

$$\frac{\partial H}{\partial x}(x,F(x)) + \frac{\partial H}{\partial T}(x,F(x))F'(x) = 0 \quad \text{for all } x \in \Omega^1(x_0,x_1). \tag{2.11}$$

**Remark 2.3.** Functional $F$ is continuous in all $\Omega^1(x_0,x_1)$, but eventually it is not differentiable only at the zero level, i.e. on constant curves. Thus, if $x_0 \neq x_1$ the manifold $\Omega^1(x_0,x_1)$ does not contain any constant curve, so $F$ is a smooth functional on all $\Omega^1(x_0,x_1)$.

**Remark 2.4.** By reasoning as outlined in the previous proof it is also possible to prove that:

(a) $\overline{z} = (\overline{x},\overline{t})$ is a solution of problem (2.1) with arrival time $\overline{t}(1) = T < 0$;

if and only if

(b) $\overline{x}$ is a critical point of functional
$$F_-(x) = -\sqrt{\int_0^1 \langle \dot{x}, \dot{x} \rangle ds \cdot \int_0^1 \frac{1}{\beta(x)} ds} = -F(x) \quad on\ \Omega^1(x_0, x_1),$$
with critical level $T = F_-(\overline{x}) < 0$, and $\overline{t} = \overline{t}(s)$ is given by (2.3).

Now, our aim is reduced to look for critical points of $F$ in $\Omega^1(x_0, x_1)$. Thus, we need the Ljusternik-Schnirelmann Theory (for more details, see, e.g., [13]).

**Definition 2.5.** Let $X$ be a topological space and $A \subseteq X$. The Ljusternik-Schnirelman category of $A$ in $X$ ($\operatorname{cat}_X A$) is the least number of closed and contractible subsets of $X$ covering $A$. If this is not possible we say that $\operatorname{cat}_X A = +\infty$. We denote $\operatorname{cat} X = \operatorname{cat}_X X$.

**Definition 2.6.** Let $\Omega$ be a Riemannian manifold and $f$ a $\mathcal{C}^1$ functional on $\Omega$. $f$ is said to satisfy the Palais-Smale condition if any $(x_n)_n \subset \Omega$ such that
$$(f(x_n))_n \text{ is bounded and } \lim_{n\to+\infty} f'(x_n) = 0$$
converges in $\Omega$ up to subsequences.

**Theorem 2.7** (Ljusternik-Schnirelmann). *Let $\Omega$ be a complete Riemannian manifold and $f$ a $\mathcal{C}^1$ functional on $\Omega$. If $f$ satisfies the Palais-Smale condition and is bounded from below, then $f$ attains its infimum and has at least cat$\Omega$ critical points. Furthermore, if $\sup_\Omega F = +\infty$ and cat$\Omega = +\infty$ there exists a sequence of critical points $(x_n)_n \subset \Omega$ such that $F(x_n) \nearrow +\infty$.*

**Remark 2.8.** If $1 \le k \le cat\Omega$, then each critical level $c_k$ of the previous theorem is given by
$$c_k = \inf_{A\in\Gamma_k} \sup_{x\in A} f(x)$$
where $\Gamma_k = \{A \subseteq X : \operatorname{cat}_\Omega A \ge k\}$.

At last, in order to estimate the Ljusternik-Schnirelmann category of $\Omega^1(x_0, x_1)$, we need the following result (see [6]).

**Proposition 2.9** (Fadell-Husseini). *If $\mathcal{M}_0$ is a manifold not contractible in itself, then for all $x_0, x_1 \in \mathcal{M}_0$ the manifold of curves $\Omega^1(x_0, x_1)$ has infinite category and possesses compact subsets of arbitrarily high category.*

## 3. Proof of the main theorem

Obviously, functional $F$ is bounded from below as it is
$$F(x) \ge 0 \quad \text{for all } x \in \Omega^1(x_0, x_1).$$
Anyway, in order to prove the Palais-Smale condition, a stronger property is needed. For this aim, we recall a technical lemma (for more details on the proof, see [2, Proposition 4.1] or also [1, Lemma 2.4]).

**Lemma 3.1.** *Let $\beta$ satisfy assumption* (H2). *If $(x_k)_k \subset \Omega^1(x_0, x_1)$ is such that $\|\dot{x}_k\| \to +\infty$, then*
$$\int_0^1 \frac{\|\dot{x}_k\|^2}{\beta(x_k)} ds \to +\infty \quad as\ k \to +\infty,$$
*where $\|\dot{x}_k\|^2 = \int_0^1 \langle \dot{x}_k, \dot{x}_k \rangle ds$.*

An easy consequence of Lemma 3.1 is the following result.

**Lemma 3.2.** *Functional $F$ is coercive in $\Omega^1(x_0, x_1)$, i.e.*

$$F(x) \to +\infty \quad \text{if } \|\dot{x}\| \to +\infty.$$

**Proposition 3.3.** *Assume that $\mathcal{M}_0$ is complete and $\beta$ has a quadratic growth as in (1.2). Then functional $F$ satisfies the Palais-Smale condition in $\Omega^1(x_0, x_1)$.*

*Proof.* Let $(x_k)_k \subset \Omega^1(x_0, x_1)$ be such that

$$(F(x_k))_k \text{ is bounded and } \lim_{k\to+\infty} F'(x_k) = 0. \tag{3.1}$$

From Lemma 3.2 and (3.1) we have that $(\|\dot{x}_k\|)_k$ is bounded. Furthermore, it is also easy to see that

$$\sup\{d(x_k(s), x_0) \mid s \in I, k \in \mathbb{N}\} < +\infty.$$

Thus, there exists $R > 0$ such that the family $\{x_k(s) : s \in I, k \in \mathbb{R}\}$ is contained in $B_R(0) = \{x \in \mathcal{M}_0 : d(x, x_0) \le R\}$ which is compact, so there exist $M, \nu > 0$ such that

$$\nu \le \beta(x_k(s)) \le M \quad \text{for all } s \in I, k \in \mathbb{N}. \tag{3.2}$$

Furthermore, $(x_k)_k$ is bounded in $H^1(I, \mathbb{R}^N)$. Hence, there exists $x \in H^1(I, \mathbb{R}^N)$ such that, up to subsequences, it is $x_k \rightharpoonup x$ weakly in $H^1(I, \mathbb{R}^N)$ and $x_k \to x$ uniformly in $I$. Since $\mathcal{M}_0$ is complete, $x \in \Omega^1(x_0, x_1)$. What remains to prove is that this convergence is also strong in $\Omega^1(x_0, x_1)$. For simplicity, consider $z_k = (x_k, t_k)$ and $T_k = F(x_k)$ with $t_k = \Phi_{T_k}(x_k)$. Hence, by (2.6) and (3.2) sequence $(t_k)_k$ is bounded in $H^1(I, \mathbb{R})$.

On the other hand, by [4, Lemma 2.1] there exist two sequences $(\xi_k)_k, (\nu_k)_k \subset H^1(I, \mathbb{R}^N)$ such that

$$\begin{gathered} \xi_k \in T_{x_k}\Omega^1(x_0, x_1), \quad x_k - x = \xi_k + \nu_k \quad \text{for all } k \in \mathbb{R}, \\ \xi_k \rightharpoonup 0 \text{ weakly in } H^1(I, \mathbb{R}^N) \text{ and } \nu_k \to 0 \text{ strongly in } H^1(I, \mathbb{R}^N). \end{gathered} \tag{3.3}$$

By (3.1) it is

$$F'(x_k)[\xi_k] = o(1), \tag{3.4}$$

while by Remark 2.2 it follows

$$\frac{\partial H}{\partial x}(x_k, T_k) + \frac{\partial H}{\partial T}(x_k, T_k) F'(x_k) = 0,$$

where $(\frac{\partial H}{\partial T}(x_k, T_k))_k$ is bounded. Evaluating this operator on $\xi_k \in T_{x_k}\Omega^1(x_0, x_1)$, we obtain

$$\frac{\partial H}{\partial x}(x_k, T_k)[\xi_k] + \frac{\partial H}{\partial T}(x_k, T_k) F'(x_k)[\xi_k] = 0,$$

so, by (3.4) it is

$$\frac{\partial H}{\partial x}(x_k, T_k)[\xi_k] = o(1).$$

Hence, being

$$\frac{\partial H}{\partial x}(x_k, T_k)[\xi_k] = 2J'_{T_k}(x_k)[\xi_k],$$

it follows

$$J'_{T_k}(x_k)[\xi_k] = o(1).$$

So, reasoning as in [8, Lemma 3.4.1], since

$$J'_{T_k}(x_k)[\xi_k] = f'_{T_k}(x_k, t_k)[(\xi_k, 0)],$$

we have that

$$o(1) = f'_{T_k}(x_k, t_k)[(\xi_k, 0)] = \int_0^1 \langle \dot{x}_k, \dot{\xi}_k \rangle ds - \frac{1}{2} \int_0^1 \beta'(x_k)[\xi_k] \dot{t}_k^2 ds.$$

Whence, being the sequence $(\|\dot{t}_k\|)_k$ bounded, (3.3) implies

$$\int_0^1 \beta'(x_k)[\xi_k] \dot{t}_k^2 ds = o(1),$$

so it is

$$\int_0^1 \langle \dot{x}_k, \dot{\xi}_k \rangle ds = o(1),$$

and, by applying again (3.3), we obtain

$$\int_0^1 \langle \dot{\xi}_k, \dot{\xi}_k \rangle ds = o(1),$$

so $\xi_k \to 0$ strongly in $H^1(I, \mathbb{R}^N)$ . Hence, sequence $(x_k)_k$ converges strongly to $x$. $\square$

*Proof of Theorem 1.3.* By Lemma 3.2 and Proposition 3.3 we can apply Theorem 2.7 to functional $F$ in the complete Riemannian manifold $\Omega^1(x_0, x_1)$ obtaining that $F$ has at least a critical point. Moreover, if $\mathcal{M}_0$ is non contractible in itself, then for Proposition 2.9, functional $F$ has infinitely many critical points $(x_k)_k \subset \Omega^1(x_0, x_1)$ such that

$$\lim_{k \to +\infty} F(x_k) = +\infty;$$

whence, by Theorem 2.1 there exists a sequence of geodesics $(z_n = (x_n, t_n))_n$ such that $t_n(1) \nearrow +\infty$ as $n \nearrow +\infty$. Furthermore, by reasoning in the same way, Remark 2.4 allows us to complete the proof by finding geodesics with negative arrival time. $\square$

**Remark 3.4.** The existence of lightlike geodesics joining a point to a line, in a static spacetime satisfying assumptions (H1)and (H2), has already been proved by means of geometrical tools. In fact, by using a correspondence between the trajectories of particles under a potential and the geodesics on a static spacetime, in [11, Section 4], Sánchez proved that a point and a line can be joined by a lightlike geodesic if and only if the conformal Riemannian metric $g^* = \beta^{-1}\langle \cdot, \cdot \rangle$ is geodesically connected. On the other hand, by Hopf-Rinow Theorem, $g^*$ is geodesically connected if and only if it is complete. But, by [12, Theorem 3.1], the metric $g^*$ is complete when $\langle \cdot, \cdot \rangle$ is complete and $\beta$ grows at most quadratically, as in our hypotheses.

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