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Oscillation of solutions to delay differential equations with positive and negative coefficients *

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Abstract

In this article we present infinite-integral conditions for the oscillation of all solutions of first-order delay differential equations with positive and negative coefficients.

1 Introduction

Consider the first-order delay differential equation

$$\dot{x}(t) + P(t)x(t-\sigma) - Q(t)x(t-\tau) = 0\,, \tag{1.1}$$

where $P(t)$ and $Q(t)$ are positive continuous real functions and σ, τ are positive constants. Equation (1.1) has the following general form

$$\dot{x}(t) + \sum_{i=1}^{n} P_i(t)x(t-\sigma_i) - \sum_{j=1}^{m} Q_j(t)x(t-\tau_j) = 0\,, \tag{1.2}$$

where $P_i(t), Q_j(t) \in C([t_0,\infty), R^+)$ and $\sigma_i, \tau_j \in [0,\infty)$, for $i = 1,\dots,n$ and $j = 1,\dots,m$. By a solution of (1.1) or (1.2), we mean a function $x(t) \in C([t_0-\rho), R)$ that for some t_0 satisfies (1.1) (or (1.2)) for all $t \geq t_0$, where $\rho = \max\{\sigma,\tau\}$ (or $\rho = \max\{\max_{1\leq i\leq n}\sigma_i, \max_{1\leq j\leq m}\tau_j\}$).

As usual a function $x(t)$ is called oscillatory if it has arbitrarily large zeros. Otherwise the solution is called non-oscillatory.

Qian and Ladas [1] obtained for (1.1) the well-known oscillation criterion

$$\liminf_{t\to\infty} \int_{t-\rho}^{t} [P(s) - Q(s+\tau-\sigma)]\,ds > \frac{1}{e}\,. \tag{1.3}$$

Elabbasy and Saker [32] obtained the oscillation criterion for the generalized equation,

$$\liminf_{t\to\infty} \int_{t-\rho}^{t} \sum_{i=1}^{p} [P_i(s) - \sum_{k\in J_i} Q_k(s+\tau_k-\sigma_i)]\,ds > \frac{1}{e}\,. \tag{1.4}$$

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It is easy to see that (1.3) is given by (1.4) when putting $n = m = 1$.

Many authors have considered the delay differential equation, with positive coefficient,

$$\dot{x}(t) + P(t)x(\tau(t)) = 0\,. \tag{1.5}$$

The first systematic study of oscillation for all solutions of (1.5) was made by Myshkis [3]. He proved that every solution of (1.5) oscillates if

$$\limsup_{t\to\infty}[t-\tau(t) < \infty, \quad \liminf_{t\to\infty}[t-\tau(t)] \liminf_{t\to\infty} P(t) > \frac{1}{e}\,. \tag{1.6}$$

In 1972, Ladas, Laksmikatham and Papadakis [4] proved that the same conclusion holds if

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} P(s)\,ds > 1\,. \tag{1.7}$$

In 1979, Ladas [5] and, in 1982, Kopltadaze and Canturija [2] replaced (1.7) by

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} P(s)\,ds > \frac{1}{e}\,. \tag{1.8}$$

Concerning the constant $1/e$ in (1.8), if the inequality

$$\int_{\tau(t)}^{t} P(s)\,ds \le \frac{1}{e} \tag{1.9}$$

holds eventually, then according to a result in [2], (1.5) has a non-oscillatory solution.

It is obvious that there is a gap between the conditions (1.7) and (1.8) when the limit

$$\lim_{t\to\infty} \int_{\tau(t)}^{t} P(s)\,ds \tag{1.10}$$

does not exist.

In 1995 Elbert and Stavrolakis [6] established infinite-integral conditions for oscillation (1.5) in the case where

$$\int_{\tau(t)}^{t} P(s)\,ds \ge \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int_{\tau(t)}^{t} P(s)\,ds = \frac{1}{e}\,. \tag{1.11}$$

They proved that if

$$\sum_{i=1}^{\infty} \Big[\int_{t_{i-1}}^{t_i} P(s) - \frac{1}{e} \Big]\,ds = \infty\,, \tag{1.12}$$

then every solution of (1.5) oscillates.

In 1996, Li [7] showed that if $\int_{\tau(t)}^{t} P(s)\,ds > 1/e$ for some $t_0 > 0$ and

$$\int_{t_0}^{\infty} P(t) \Big[\int_{\tau(t)}^{t} P(s)\,ds - \frac{1}{e} \Big]\,dt = \infty\,, \tag{1.13}$$

then every solution of (1.5) oscillates.

Domshlak and Stavrolakis [8] established sufficient conditions for the oscillation, in the critical case where

$$\lim_{t\to\infty} P(t) = \frac{1}{e\tau},$$

of the delay differential equation

$$\dot{x}(t) + P(t)x(t-\tau) = 0\,. \tag{1.14}$$

Recently Domshlak and Stavrolakis [9] and Jaros and Stavrolakis [10] considered the delay differential equation

$$\dot{x}(t) + a_1(t)x(t-\tau) + a_2(t)x(t-\sigma) = 0 \tag{1.15}$$

and established sufficient conditions for the oscillation of all solutions in the critical state that the corresponding limiting equation admits a non-oscillatory solution.

The oscillatory properties of various functional differential equations have been employed by many authors. For some contribution to the oscillation theory of delay differential equations we refer to the articles by Zhang and Goplsamy [11], Gyori and Ladas [12], Li [13], Arino, Ladas and Sficas [14], Ladas and Sficas [15], Ladas, Qian and Yan [16], Arino, Gyori and Jawhari [17], Hunt and Yorke [18], Gyori [19], Cheng [20], Kwang [21], Kulenovic, Ladas and Meimardou [22], Kulenovic and Ladas [23, 24, 25], Goplsamy, Kulenovic and Ladas [26], Ladas and Qian [27, 28], Elabbasy, Saker and Al-Shemas [29], Elabbasy and Saker [30] and Elabbasy, Saker and Saif [31], Elabbasy and Saker [32].

To a large extent, the study of functional differential equations is motivated by having many applications in Physics [33], Biology [34], Ecology [35], and the study of spread of infectious diseases [36].

Our aim in this paper is to give an infinite-integral conditions for oscillation of all solutions of (1.1) and (1.2) by using the generalized characteristic equation and the function of the form $\frac{x(t)}{x(t-\sigma_i)}$.

In section 2, we present an infinite-integral condition for oscillation of (1.1) which indicates that condition (1.3) is no longer necessary. In section 3, we extended the results in section 2 to establish infinite sufficient conditions for oscillation of (1.2) which indicates that condition (1.4) is no longer necessary. As far as we known, there are no other results for differential equations with positive and negative coefficients with more than one delay.

In the sequel, when we write a functional inequality we will assume that it holds for all sufficiently large values of t.

Lemma 1.1 ([12]) *Let $a \in (-\infty, 0)$, $\tau \in (0,\infty)$, $t_0 \in R$ and suppose that $x(t) \in C[[t_0,\infty), R]$ satisfies the inequality*

$$x(t) \le a + \max_{t-\tau\le s\le t} x(s) \quad \textit{for } t \ge t_0\,.$$

Then $x(t)$ cannot be a non-negative function.

Lemma 1.2 ([12]) *Assume that P_i and $\tau_i \in C[[t_0,\infty),R^+]$ for $i=1,\dots,n$. Then the differential inequality*

$$\dot{x}(t)+\sum_{i=1}^{n}P_i(t)x(t-\tau_i(t))\le 0,\ t\ge t_0 \tag{1.16}$$

has an eventually positive solution if and only if the equation

$$\dot{y}(t)+\sum_{i=1}^{n}P_i(t)y(t-\tau_i(t))=0,\ t\ge t_0 \tag{1.17}$$

has an eventually positive solution.

Lemma 1.3 ([13]) *Consider the delay differential equation*

$$\dot{x}(t)+\sum_{i=1}^{n}R_i(t)x(t-\tau_i)=0,\ t\ge t_0 \tag{1.18}$$

and assume that $\limsup_{t\to\infty}\int_t^{t+\tau i}R_i(s)\,ds>0$ for some i and $x(t)$ is an eventually positive solution of (1.18), then for the same i,

$$\liminf_{t\to\infty}\frac{x(t-\tau_i)}{x(t)}<\infty \tag{1.19}$$

Lemma 1.4 ([13]) *If (1.18) has an eventually positive solution, then*

$$\int_t^{t+\tau_i}R_i(s)\,ds<1\,,\ i=1,\dots,n \tag{1.20}$$

eventually.

2 Oscillation of solutions to (1.1)

Now we obtain an infinite-integral conditions for oscillation of all solutions of (1.1). We need the following Lemma.

Lemma 2.1 *Assume that:*

(h1) $P,Q\in C([t_0,\infty),R^+)$, $\sigma,\tau\in[0,\infty)$ *and* $\tau\le\sigma$

(h2) $P(t)\ge Q(t+\tau-\sigma)$, *for* $t\ge t_0+\sigma-\tau$

(h3) $\int_{t-\sigma}^{t-\tau}Q(s)\,ds\le 1$ *for* $t\ge t_0+\sigma$

Let $x(t)$ be an eventually positive solution of (1.1) and set

$$z(t)=x(t)-\int_{t-\sigma}^{t-\tau}Q(s+\tau)x(s)\,ds,\ t\ge t_0+\sigma-\tau\,. \tag{2.1}$$

Then $z(t)$ is a non-increasing positive function and satisfies the inequality

$$\dot{z}(t)+[P(t)-Q(t+\tau-\sigma)]\,z(t-\sigma)\le 0\,. \tag{2.2}$$

The proof of this lemma can be found as Lemma 2.6.1 in [12].

Theorem 2.2 *Assume that (h1), (h2) and (h3) from Lemma 2.1 are satisfied. Also assume that for $R(t) = P(t) - Q(t+\tau-\sigma)$,*

(h4) $\int_t^{t+\sigma} R(s)\,ds > 0$ *for* $t \ge t_0$ *for some* $t_0 > 0$.

(h5) $\int_{t_0}^{\infty} R(t)\ln\left[e\int_t^{t+\sigma} R(s)\,ds\right]\,dt = \infty$.

Then every solution of (1.1) oscillates.

Proof. On the contrary assume that 1.1) has an eventually positive solution $x(t)$. By Lemma 2.1 it follows that the function $z(t)$ is positive and satisfies (2.2). So Lemma 1.2 yields that the delay differential equation

$$\dot{y}(t) + [P(t) - Q(t+\tau-\sigma)]\,y(t-\sigma) = 0 \tag{2.3}$$

has an eventually positive solution. Let $\lambda(t) = -\dot{y}(t)/y(t)$. Then $\lambda(t)$ is non-negative and continuous, then there exists $t_1 \ge t_0$ such that $y(t_1) > 0$ and $y(t) = y(t_1)\exp(-\int_{t_1}^{t}\lambda(s)\,ds)$. Furthermore, if $\lambda(t)$ satisfies the generalized characteristic equation

$$\lambda(t) = R(t)\exp\Big(\int_{t-\sigma}^{t}\lambda(s)\,ds\Big)\,,$$

we can show that

$$e^{rx} \ge x + \frac{ln(er)}{r} \quad \text{for } r > 0\,. \tag{2.4}$$

Define $A(t) = \int_t^{t+\sigma} R(s)\,ds$. By using (2.4) we find that

$$\begin{aligned}
\lambda(t) &= R(t)\exp\Big(A(t)\frac{1}{A(t)}\int_{t-\sigma}^{t}\lambda(s)\,ds\Big) \\
&\ge R(t)\Big[\frac{1}{A(t)}\int_{t-\sigma}^{t}\lambda(s)\,ds + \frac{\ln(eA(t))}{A(t)}\Big]
\end{aligned}$$

or

$$\Big(\int_t^{t+\sigma} R(s)\,ds\Big)\lambda(t) - R(t)\int_{t-\sigma}^{t}\lambda(s)\,ds \ge R(t)\Big(\ln e\int_t^{t+\sigma} R(s)\,ds\Big) \tag{2.5}$$

Then for $N > T$,

$$\begin{aligned}
&\int_T^N \lambda(t)\Big(\int_t^{t+\sigma} R(s)\,ds\Big)\,dt - \int_T^N R(t)\int_{t-\sigma}^{t}\lambda(s)\,ds\,dt \\
&\quad \ge \int_T^N R(t)\Big(\ln e\int_t^{t+\sigma} R(s)\,ds\Big)\,dt\,.
\end{aligned} \tag{2.6}$$

By interchanging the order of integration, we find that

$$\int_T^N R(t)(\int_{t-\sigma}^t \lambda(s)\,ds)\,dt \geq \int_T^{N-\sigma} (\int_s^{s+\sigma} R(t)\lambda(s)dt)\,ds\,.$$

Hence

$$\int_T^N R(t)(\int_{t-\sigma}^t \lambda(s)\,ds)\,dt \geq \int_T^{N-\sigma} \lambda(s)(\int_s^{s+\sigma} R(t)dt)\,ds\,.$$

Then

$$\int_T^N R(t)(\int_{t-\sigma}^t \lambda(s)\,ds)\,dt \geq \int_T^{N-\sigma} \lambda(t)(\int_t^{t+\sigma} R(s)\,ds)\,dt$$

Hence

$$\begin{aligned} &\int_T^N \lambda(t)(\int_t^{t+\sigma} R(s)\,ds)\,dt - \int_T^{N-\sigma} \lambda(s)(\int_s^{s+\sigma} R(t)dt)\,ds \\ &\geq \int_T^N \lambda(t)(\int_t^{t+\sigma} R(s)\,ds)\,dt - \int_T^N R(t)\int_{t-\sigma}^t \lambda(s)\,ds\,dt\,. \end{aligned} \tag{2.7}$$

From (2.6) and (2.7), it follows that

$$\int_{N-\sigma}^N \lambda(t)(\int_t^{t+\sigma} R(s)\,ds)dt \geq \int_T^N (R(t))(\ln e\int_t^{t+\sigma} R(s)\,ds)dt\,. \tag{2.8}$$

On the other hand, by Lemma 1.4, we have

$$\int_t^{t+\sigma} R(s)\,ds < 1 \tag{2.9}$$

eventually. Then by (2.8) and (2.9), we find

$$\int_{N-\sigma}^N \lambda(t)dt \geq \int_T^N (R(t))\ln(e\int_t^{t+\sigma} R(s)\,ds)dt$$

or

$$\ln\frac{y(N-\sigma)}{y(N)} \geq \int_T^N R(t)\ln(e\int_t^{t+\sigma} R(s)\,ds)dt\,. \tag{2.10}$$

In view of (h5)

$$\lim_{t\to\infty}\frac{y(t-\sigma)}{y(t)} = \infty\,. \tag{2.11}$$

However, by Lemma 1.3,

$$\liminf_{t\to\infty}\frac{y(t-\sigma)}{y(t)} < \infty \tag{2.12}$$

which contradicts (2.11), and this completes the present proof. Therefore, every solution of (1.1) oscillates.

3 Oscillation of solutions to (1.2)

Our objective in this section is to establish infinite-integral conditions for oscillation of all solutions of (1.2). We need the following theorem for the proof of the main results in this section.

Theorem 3.1 *Assume that:*

(H1) $P_i, Q_j \in C([t_0,\infty), R^+)$, σ_i, $\tau_j \in [0,\infty)$ *for* $i = 1,\dots,n$ *and* $j = 1,\dots,m$

(H2) There exist a positive number $p \le n$ *and a partition of the set* $\{1,\dots,m\}$ *into* p *disjoint subsets* J_1, J_2, J_3, $\dots$,J_p, *such that* $j \in J_i$ *implies that* $\tau_{j\le}\sigma_i$

(H3) $P_i(t) \ge \sum_{k\in J_i} Q_k(t+\tau_k-\sigma_i)$ *for* $t \ge t_0+\sigma_i-\tau_k$, *and* $i = 1,\dots,p$,

(H4) $\sum_{i=1}^{p}\sum_{k\in J_j}\int_{t-\sigma_i}^{t-\tau_k} Q_k(s)\,ds \le 1$ *for* $t \ge t_0+\sigma_i$.

Let x(t) be an eventually positive solution of (1.2) and set

$$z(t) = x(t) - \sum_{i=1}^{p}\sum_{k\in J_j}\int_{t-\sigma_i}^{t-\tau_k} Q_k(s+\tau_k)x(s)\,ds, \quad t \ge t_0+\sigma_i-\tau_k\,. \tag{3.1}$$

Then $z(t)$ *is a non-increasing and positive function.*

Proof Assume that $t_1 \ge t_0+\rho$ is such that $x(t)$ is positive for $t \ge t_1-\rho$ $\rho = \max_{1\le i\le n}\{\sigma_i\}$. From (2.1) we have

$$\dot z(t) = \dot x(t) - \sum_{i=1}^{p}\sum_{k\in J_j} Q_k(t)x(t-\tau_k) + \sum_{i=1}^{p}\sum_{k\in J_j} Q_k(t+\tau_k-\sigma_i)x(t-\sigma_i)\,.$$

Hence

$$\dot z(t) = \dot x(t) - \sum_{j=1}^{m} Q_j(t)x(t-\tau_j) + \sum_{i=1}^{p}\sum_{k\in J_j} Q_k(t+\tau_k-\sigma_i)x(t-\sigma_i)\,.$$

From (1.2), we have

$$\dot z(t) = -\sum_{i=1}^{p} P_i(t)x(t-\sigma_i) + \sum_{i=1}^{p}\sum_{k\in J_j} Q_k(t+\tau_k-\sigma_i)x(t-\sigma_i) - \sum_{i=p+1}^{n} P_i(t)x(t-\sigma_i)\,.$$

As we know that

$$\sum_{i=p+1}^{n} P_i(t)x(t-\sigma_i) > 0\,,$$

we have

$$\dot z(t) \le -\left[\sum_{i=1}^{p}[P_i(t) - \sum_{k\in J_j} Q_k(t+\tau_k-\sigma_i)]x(t-\sigma_i)\right] \tag{3.2}$$

By using (H3) we have

$$\dot{z}(t) \leq 0 \quad for \ t \geq t_1 + \rho \,. \tag{3.3}$$

This implies that $z(t)$ is a non-increasing function. Now we prove that $z(t)$ is positive. For otherwise, there exists a $t_2 \geq t_1$ such that $z(t_2) \leq 0$. Since $\dot{z}(t) \leq 0$ for $t \geq t_1 + \rho$ and $\dot{z}(t) \neq 0$ on $[t_1 + \rho, \infty)$, there exists a $t_3 \geq t_2$ such that $z(t) \leq z(t_3)$ for $t \geq t_3$. Thus from (2.1) it follows that for $t \geq t_3$,

$$\begin{aligned} x(t) &= z(t) + \sum_{i=1}^{p} \sum_{k \in J_j} \int_{t-\sigma_i}^{t-\tau_k} Q_k(s+\tau_k) x(s)\, ds \\ &\leq z(t_3) + \sum_{i=1}^{p} \sum_{k \in J_j} \int_{t-\sigma_i}^{t-\tau_k} Q_k(s+\tau_k) x(s)\, ds \\ &\leq z(t_3) + \sum_{i=1}^{p} \sum_{k \in J_j} \int_{t-\sigma_i}^{t-\tau_k} Q_k(s+\tau_k)\, ds (\max_{t-\rho \leq s \leq t} x(s))\,. \end{aligned}$$

Hence

$$x(t) \leq z(t_3) + \sum_{i=1}^{p} \sum_{k \in J_j} \int_{t-\sigma_i}^{t-\tau_k} Q_k(s+\tau_k)\, ds (\max_{t-\rho \leq s \leq t} x(s))\,.$$

Hypothesis (H4) yields

$$x(t) \leq z(t_3) + \max_{t-\rho \leq s \leq t} x(s) \quad \text{for all } t \geq t_3\,,$$

where $z(t_3) \leq 0$. Lemma 1.1 implies that $x(t)$ cannot be non-negative function on $[t_3, \infty)$. Thus contradicting $x(t) > 0$. Therefore, $z(t)$ is a non-increasing and positive function.

Theorem 3.2 *Assume that (H1), (H2), (H3) and (H4) above are satisfied,* $\sigma_p = \max\{\sigma_1, \sigma_2, \sigma_3, \dots, \sigma_p\}$, $\sum_{i=1}^{p} \int_t^{t+\sigma_i} R_i(s)\, ds > 0$ *for* $t \geq t_0$ *for some* $t_0 > 0$. *Also assume that*

(H5) $\limsup_{t\to\infty} \int_t^{t+\sigma_p} R_p(s)\, ds > 0$

(H6) $\int_{t_0}^{\infty} (\sum_{i=1}^{p} R_i(t)) \ln \Big[e \sum_{i=1}^{p} \int_t^{t+\sigma_i} R_i(s)\, ds \Big]\, dt = \infty$ *where* $R_i(t) = P_i(t) - \sum_{k \in J_i} Q_k(t + \tau_k - \sigma_i)$.

Then every solution of (1.2) oscillates.

Proof. On the contrary assume that (1.2) has an eventually positive solution $x(t)$. By Theorem 2.1 it follows that the function $z(t)$ defined by (3.1) is an eventually positive function. Also by (3.2) we have

$$\dot{z}(t) + \sum_{i=1}^{p} [P_i(t) - \sum_{k \in J_j} Q_k(t + \tau_k - \sigma_i)] x(t - \sigma_i) \leq 0\,. \tag{3.4}$$

From the fact that eventually $0 < z(t) \le x(t)$, we see that $z(t)$ is a positive function and satisfies eventually

$$\dot{z}(t) + \sum_{i=1}^{p}[P_i(t) - \sum_{k\in J_j} Q_k(t+\tau_k-\sigma_i)]z(t-\sigma_i) \le 0\,. \tag{3.5}$$

Then by Lemma 1.2, we have that the delay differential equation

$$\dot{y}(t) + \sum_{i=1}^{p}[P_i(t) - \sum_{k\in J_j} Q_k(t+\tau_k-\sigma_i)]y(t-\sigma_i) = 0 \tag{3.6}$$

has an eventually positive solution. Let $\lambda(t) = -\dot{y}(t)/y(t)$. Then $\lambda(t)$ is a non-negative and continuous, and there exists $t_1 \ge t_0$ with $y(t_1) > 0$ such that $y(t) = y(t_1)\exp(-\int_{t_1}^{t}\lambda(s)\,ds)$. Furthermore, $\lambda(t)$ satisfies the generalized characteristic equation

$$\lambda(t) = \sum_{i=1}^{p} R_i(t)\exp(\int_{t-\sigma_i}^{t}\lambda(s)\,ds)$$

with $R_i(t) = P_i(t) - \sum_{k\in J_i}^{p} Q_k(t+\tau_k-\sigma_i)$

Let $B(t) = \sum_{i=1}^{p}\int_{t}^{t+\sigma_i} R_i(s)\,ds$. By using (2.4) we find that

$$\begin{aligned}
\lambda(t) &= \sum_{i=1}^{p} R_i(t)\exp\big(B(t)\frac{1}{B(t)}\int_{t-\sigma_i}^{t}\lambda(s)\,ds\big)\\
&\ge \sum_{i=1}^{p} R_i(t)\big[\frac{1}{B(t)}\int_{t-\sigma_i}^{t}\lambda(s)\,ds + \frac{\ln(eB(t))}{B(t)}\big]
\end{aligned}$$

or

$$\sum_{i=1}^{p}\int_{t}^{t+\sigma_i} R_i(s)\,ds\lambda(t) - \sum_{i=1}^{p} R_i(t)\int_{t-\sigma_i}^{t}\lambda(s)\,ds \ge \sum_{i=1}^{p} R_i(t)(\ln e\int_{t}^{t+\sigma_i} R_i(s)\,ds)$$

Then for $N > T$,

$$\begin{aligned}
&\int_{T}^{N}\lambda(t)(\sum_{i=1}^{p}\int_{t}^{t+\sigma_i} R_i(s)\,ds)dt - \int_{T}^{N}\sum_{i=1}^{p} R_i(t)\int_{t-\sigma_i}^{t}\lambda(s)\,ds\,dt\\
&\qquad \ge \int_{T}^{N}\sum_{i=1}^{p} R_i(t)(\ln e\int_{t}^{t+\sigma_i} R_i(s)\,ds)\,dt\,.
\end{aligned} \tag{3.7}$$

Interchanging the order of integration, we find that

$$\int_{T}^{N}\sum_{i=1}^{p} R_i(t)\int_{t-\sigma_i}^{t}\lambda(s)\,ds\,dt \ge \int_{T}^{N-\sigma_i}(\int_{s}^{s+\sigma_i}\sum_{i=1}^{p} R_i(t)\lambda(s)dt)\,ds\,.$$

Hence

$$\int_T^N (\sum_{i=1}^p R_i(t)) \int_{t-\sigma_i}^t \lambda(s)\,ds\,dt \geq \int_T^{N-\sigma_i} \lambda(s)(\int_s^{s+\sigma_i} \sum_{i=1}^p R_i(t)dt)\,ds\,.$$

Then

$$\int_T^N (\sum_{i=1}^p R_i(t)) \int_{t-\sigma_i}^t \lambda(s)\,ds\,dt \geq \sum_{i=1}^p \int_T^{N-\sigma_i} \lambda(t)(\int_t^{t+\sigma_i} R_i(s)\,ds)\,dt\,. \tag{3.8}$$

From (3.7) and (3.8), it follows that

$$\begin{aligned} &\int_T^N \lambda(t)(\sum_{i=1}^p \int_t^{t+\sigma_i} R_i(s)\,ds)dt - \int_T^{N-\sigma_i} \lambda(t) \int_t^{t+\sigma_i} \sum_{i=1}^p R_i(s)\,ds\,dt \\ &\qquad \geq \int_T^N \sum_{i=1}^p R_i(t)(\ln e \sum_{i=1}^p \int_t^{t+\sigma_i} R_i(s)\,ds)\,dt\,. \end{aligned} \tag{3.9}$$

Hence

$$\begin{aligned} &\sum_{i=1}^p \int_{N-\sigma_i}^N \lambda(t)(\int_t^{t+\sigma_i} R_i(s)\,ds)dt \\ &\geq \int_T^N (\sum_{i=1}^p R_i(t))(\ln e \int_t^{t+\sigma_i} \sum_{i=1}^p R_i(t)\,ds)\,dt\,. \end{aligned} \tag{3.10}$$

On the other hand, by Lemma 1.4, we have

$$\int_t^{t+\sigma_i} R_i(s)\,ds < 1, \quad i = 1, \dots, p \tag{3.11}$$

eventually. Then by (3.10) and (3.11), we find

$$\sum_{i=1}^p \int_{N-\sigma_i}^N \lambda(t)dt \geq \int_T^N (\sum_{i=1}^p R_i(t))\ln(e \int_t^{t+\sigma_i} \sum_{i=1}^p R_i(t)\,ds)\,dt$$

or

$$\sum_{i=1}^p \ln \frac{y(N-\sigma_i)}{y(N)} \geq \int_T^N (\sum_{i=1}^p R_i(t))\ln(e \int_t^{t+\sigma_i} \sum_{i=1}^p R_i(t)\,ds)\,dt\,. \tag{3.12}$$

In view of (H6) we have

$$\lim_{t\to\infty} \prod_{i=1}^p \frac{y(t-\sigma_i)}{y(t)} = \infty\,. \tag{3.13}$$

This implies that

$$\lim_{t\to\infty} \frac{y(t-\sigma_p)}{y(t)} = \infty\,. \tag{3.14}$$

However by Lemma 1.3, we have

$$\liminf_{t\to\infty} \frac{y(t-\sigma_p)}{y(t)} < \infty$$

This contradicts (3.14) and completes the present proof. Therefore, every solution of (1.2) oscillates.

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