THE COSMOLOGICAL CONSTANT PROBLEM

THESIS

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For Soniya C. Nandlal, Phulmattee Sookoo and Meera M. Nandlal.
Thanks for the prayers, the blessings and all the love.

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CHAPTER I

INTRODUCTION

1.1 OBJECTIVES

In 1917 Albert Einstein applied his newly developed theory of general relativity to the universe. In order to provide a model for a static universe he added a cosmological constant Λ to his equation. This constant was equivalent to a pressure, which balanced the attractive force of gravity. When Edwin Hubble discovered that the universe was expanding, Einstein retracted the cosmological constant and considered it his greatest theoretical mistake.

Quantum field theory predicts a large zero-point electromagnetic energy density for the vacuum. This vacuum energy has properties similar to a cosmological constant. The zero-point energy of the vacuum has also been experimentally verified by the Casimir effect. Since the vacuum possesses a zero-point energy, this energy should produce gravitational effects. However, no such effects have ever been observed. The discrepancy between theory and observation is 120 orders of magnitude and serve as the basis for the cosmological constant problem.

In this thesis an equation for the zero-point energy density of the electromagnetic vacuum will be derived and its magnitude will be determined. Also, assuming that the zero-point energy of the vacuum exists, the equation for the Casimir effect will be derived.

This will be followed by a brief description of the experiment, which verifies the Casimir effect and thus confirms the existence of the zero-point energy density of the vacuum.

The gravitational energy density of the solar system will be calculated and compared with the vacuum energy density to demonstrate the discrepancy of 120 orders of magnitude. The cosmological constant problem will then be discussed along with the experiments which are presently being carried out to measure its value.

1.2 NATURAL UNITS

In particle physics and quantum field theory natural units are used to simplify computations.[1] In this system of units action and velocity are the fundamental dimensions. Planck's constant $\hbar$ is the unit of action and the velocity of light c is the unit of velocity. We then have $\hbar = c = 1$. Also, in natural units (since $E = mc^2$) many quantities have the dimension of mass and this is usually expressed in units of Mev. Another advantage of using this system is that $\varepsilon_o = 1$ which implies that $\mu_o = 1$. This allows Maxwell's equations to be written in their simplest form.

1.3 VECTORS

Vectors will be represented either by bold-faced upper or lower case letters and the complex conjugate of a vector will be represented by a letter with an asterisk. So if **A** represents a vector then **A*** represents its complex conjugate and Å represents the derivative of **A** with respect to time.

1.4 VACUUM

The vacuum refers to the ground state or the state of lowest energy of a quantum system, which is void of normal material.

1.5 HARMONIC OSCILLATOR

In this section a comparison will be made between the harmonic oscillator and the electromagnetic radiation energy density. It will be shown that both equations have the same form and must therefore have the same solutions. This result will then be used to calculate the energy density of the electromagnetic vacuum.

The Hamiltonian for a one-dimensional harmonic oscillator is given by[2,3]

$$H = \{ m(dx/dt)^2 + m(\omega x)^2 \} / 2 = p^2 / 2m + m(\omega x)^2 / 2 , \quad (1.1)$$

and has an energy spectrum of

$$E = (n + 1/2)\ \omega, \quad (1.2)$$

with a zero-point energy given by

$$E = \omega / 2. \quad (1.3)$$

For the two-dimensional isotropic harmonic oscillator with coordinate x and y, we have

$$H = [m\{(dx/dt)^2 + (dy/dt)^2\} + m\omega^2\{x^2 + y^2\}] / 2 . \qquad (1.4)$$

Introducing the complex number z = x + iy gives

$$H = \{m(dz/dt)(dz^*/dt) + m\omega^2 zz^*\} / 2$$

$$= P_z P_z^* / 2m + m\omega^2 zz^* / 2. \qquad (1.5)$$

The zero-point energy of the two-dimensional harmonic oscillator is the sum of the zero-point energies of two individual one-dimensional harmonic oscillators. Thus

$$E = \omega / 2 + \omega / 2 = \omega. \qquad (1.6)$$

1.6 ELECTROMAGNETIC RADIATION ENERGY DENSITY

The time average energy density of the electromagnetic field is given by[4]

$$\rho = \{\mathbf{E}^2 + \mathbf{B}^2\} / 2, \qquad (1.7)$$

where

$$\mathbf{E} = \mathbf{E}_\circ \,\mathrm{Sin}\,(\,\mathbf{k}\cdot\mathbf{x} - \omega t + \delta\,)\,,\ \ \mathbf{B} = \mathbf{B}_\circ \,\mathrm{Sin}\,(\,\mathbf{k}\cdot\mathbf{x} - \omega t + \delta\,). \qquad (1.8)$$

Both **E** and **B** are plane waves with wave vector **k**, frequency ω and phase δ. These waves represent photons and therefore have two polarization states. In order to develop an analogy between the harmonic oscillator and the radiation field, the Coulomb or radiation gauge will be used with a zero scalar potential.[5] It is also important to note that the Coulomb and Lorentz gauges are equal for a zero scalar potential.

The **E** and **B** fields are given in terms of the vector potential by

$$\mathbf{E} = -\dot{\mathbf{A}}\,, \quad \mathbf{B} = \nabla \times \mathbf{A}\,, \nabla \cdot \mathbf{A} = 0\,. \tag{1.9}$$

Let

$$\mathbf{A} = \mathbf{A}_{\circ}\, e^{\,i\{-\omega t + \mathbf{k} \cdot \mathbf{x}\}}\,. \tag{1.10}$$

Notice that **A** has the same form as the de Broglie wave for a free particle with energy ω, and momentum **k** . Now from (1.7)

$$\rho = \{\mathbf{E}^2 + \mathbf{B}^2\}/\,2 = \{\mathbf{E} \cdot \mathbf{E}^* + \mathbf{B} \cdot \mathbf{B}^*\}/\,2 \tag{1.11}$$

and since from (1.9)

$$\mathbf{E} = -\dot{\mathbf{A}}\,, \tag{1.12}$$

then

$$\mathbf{E}^{*} = -\mathring{\mathbf{A}}^{*}, \tag{1.13}$$

and so

$$\mathbf{E} \cdot \mathbf{E}^{*} = \mathring{\mathbf{A}}^{*} \cdot \mathring{\mathbf{A}}\,. \tag{1.14}$$

It can also be shown (see Appendix A) that

$$\mathbf{B} \cdot \mathbf{B}^{*} \;=\; \mathbf{k}^{2}\,\mathbf{A} \cdot \mathbf{A}^{*}\,. \tag{1.15}$$

Now substituting Eq. (1.14) and Eq. (1.15) into Eq. (1.11), we obtain

$$\rho \;=\{\, \mathring{\mathbf{A}}^{*} \cdot \mathring{\mathbf{A}} + \mathbf{k}^{2}\,\mathbf{A} \cdot \mathbf{A}^{*}\}/2\;\;. \tag{1.16}$$

Comparing Eq. (1.5) with Eq. (1.16) , we find that for a given time and for m = 1, the complex particle displacement becomes the **A** field.

We can also infer from the comparison of Eq. (1.5) with Eq. (1.16) that the solution to Eq. (1.16) gives the energy spectrum of the harmonic oscillator, where n is the number of photons with wave vector **k** and frequency ω.

Also, each mode will have a zero-point energy given by $\omega / 2$, corresponding to no photons present.

1.7 ELECTROMAGNETIC ENERGY DENSITY OF THE VACUUM

The method of periodic boundary conditions[6] will be used to count the modes of the electromagnetic field. Consider standing waves in a cubical box[7] of volume $V = L^3$ where the complex vector potential $\mathbf{A}$ vanishes along the edges of the box, then

$$(k_x, k_y, k_z) = 2\pi / L \ (n_x, n_y, n_z), \tag{1.17}$$

where n_x, n_y, n_z are integers. The wave vectors k_x, k_y, k_z will then have discrete values. However, as L becomes large and approximates the size of the universe, the discrete values of $\mathbf{k}$ approach a continuum.

The number of modes is given by

$$N = (L / 2\pi)^3 k^3 = (L / 2\pi)^3 p^3. \tag{1.18}$$

We can consider p^3 as a volume V_k in momentum space. Then the volume density of each mode is given by

$$n = N / L^3 = V_k / (2\pi)^3. \tag{1.19}$$

This result implies that the number of modes dn between ω and $\omega + d\omega$ is equal to the volume in momentum space in a thin spherical shell divided by $(2\pi)^3$, or

$$dn = 4\pi\omega^2 d\omega / (2\pi)^3 . \tag{1.20}$$

Since there are two polarization states and the energy of each mode is given by $\omega / 2$ and depends only on **k**, the total energy density is given by

$$\rho_{vac} = \int \omega dn = \int\{4\pi\omega^3 d\omega / (2\pi)^3\} = (\omega_{max})^4 / 8\pi^2, \tag{1.21}$$

where the limits of integration are from 0 to ω_{max}.

This result gives the energy density for the electromagnetic vacuum and reveals that $\rho \rightarrow \infty$ as $\omega \rightarrow \infty$. We must therefore choose a cut off frequency in order to obtain a finite value for the vacuum energy density. The Planck time (see Appendix C) occurs 10^{-44} seconds after the big bang and at any time earlier than the Planck time, all of our physical theories break down. The Planck energy (see Appendix C) is the energy of the photons at the Planck time. This energy will be chosen as the maximum since at any time later than the Planck time, the temperature of the universe decreases and thus the energy of the photons decreases. It is also believed that at the Planck energy, the gravitational interaction is as strong as the electromagnetic and strong interactions.[8] Using the planck energy of 10^{19} Gev, we have from Eq. (1.21) that the electromagnetic energy density of the vacuum (see Appendix C) in fermi (f) is

$$\rho_{vac} = 8.4 \text{ x } 10^{76} \text{ f}^{-4} . \tag{1.22}$$

In chapter 3 this result will be compared to the gravitational energy density.

CHAPTER II

CASIMIR ANALYSIS

2.1 CASIMIR EFFECT

In 1948, H. B. G. Casimir showed that two uncharged conducting plates placed in a vacuum will attract each other. This phenomenon has since been known as the Casimir effect. The Casimir effect can be understood by considering the change in the zero-point energy of the vacuum in the vicinity of the plates. When the plates are sufficiently close together there are fewer allowed modes between the plates than outside the plates. This energy difference does work on the system and draws the plates together. The Casimir effect, therefore, provides evidence for the existence of the zero-point electromagnetic energy density of the vacuum.

2.2 CASIMIR FORCE

Consider two parallel-uncharged conducting rectangular plates of length $L_x = L_y = L$, separated by a distance d. In order for the tangential components of the electric field to vanish[9] on the walls of the conducting plates, we require that[10, 11]

$$k_x = j\pi/L\,, \quad k_y = m\pi/L\,, \quad k_z = n\pi/d\,, \tag{2.1}$$

where j, m and n are positive integers including zero.

The allowed frequencies are given by

$$\omega_{jmn} = k_{jmn} = \pi \, [\{j^2 + m^2\}/L^2 + n^2/d^2\} \,]^{1/2}. \qquad (2.2)$$

The zero-point energy of the electromagnetic field between the plates for two polarization states is given by

$$\Sigma_{j,m,n} (2)\,(1/2)\,\omega_{jmn} = \Sigma_{j,m,n} \, \pi[\{j^2 + m^2\}/L^2 + n^2/d^2\} \,]^{1/2}. \quad (2.3)$$

The factor of two arises because there are two independent polarizations if the integers j, m, n, $\neq 0$. If one of the integers is zero then there is only one polarization state and a factor of 1/ 2 will be required in Eq. (2.3). If L is large compared to d, then the values of j and m forms a continuum and the sums over j and m are replaced by integrals. Now

$$(k_x, k_y, k_z) = (j\pi/L\,,\, m\pi/L\,,\, n\pi/d\,)\,, \qquad (2.4)$$

and for continuous j and m we have

$$(dj,\, dm\,) = (L/\pi\, dk_x\,,\, L/\pi\, dk_y). \qquad (2.5)$$

Also for

$$(dk_x , dk_y) = (dx , dy), \tag{2.6}$$

the zero-point energy within the plates is given by,

$$E_o(d) = L^2 / \pi \sum_n \int dx \int dy \{ x^2 + y^2 + \pi^2 n^2 /d^2\}^{1/2} , \tag{2.7}$$

where the summation and the limits of integration are from zero to infinity.

Eq. (2.7) implies that the zero-point vacuum energy is infinite in any finite volume. On the other hand if d is arbitrarily large, then the sum over n can be replaced by an integral, so Eq. (2.7) becomes

$$E_o(\infty) = L^2 d / \pi^3 \int dx \int dy \int dz\{ x^2 + y^2 + z^2\}^{1/2} . \tag{2.8}$$

The limits for Eq. (2.8) are also from zero to infinity so this equation also gives an infinite quantity. The potential energy of the system when the plates are separated by the distance d is given by

$$U(d) = E_o(d) - E_o(\infty), \tag{2.9}$$

or

$$U(d) = L^2 / \pi^2 \Sigma_n \int dx \int dy \{ x^2 + y^2 + \pi^2 n^2 / d^2 \}^{1/2} -$$

$$d / \pi \int dx \int dy \int dz \{ x^2 + y^2 + z^2 \}^{1/2}. \qquad (2.10)$$

Introducing polar coordinates r,θ in the x, y plane, we have that

$$dx\, dy = r dr d\theta \qquad (2.11)$$

and

$$r^2 = x^2 + y^2 . \qquad (2.12)$$

Since θ ranges from 0 to π/2 for k_x, $k_y > 0$, Eq. (2.10) now becomes

$$U(d) = L^2 / \pi^2 [\Sigma_n \int d\theta \int r dr \{ r^2 + \pi^2 n^2 / d^2 \}^{1/2} -$$

$$d / \pi \int d\theta \int dz \int r dr \{ r^2 + z^2 \}^{1/2}] \qquad (2.13)$$

where the limits on r are from zero to infinity. Changing variables gives

$$U(d) = L^2\pi^2/4d^3 \;[\Sigma_n \int dp\{p + n^2\}^{1/2} -$$

$$\int dz \int dp\{p + z^2\}^{1/2}]. \qquad (2.14)$$

Eq. (2.14) gives the difference between two infinite quantities and can be rewritten as

$$U(d) = L^2\pi^2/4d^3 \;[1/2\,F(0) + \Sigma_{n=1} F(n) - \int dz\, F(z)], \qquad (2.15)$$

where the limits of the sum are from one to infinity and the limits on the integral are from zero to infinity and for u = n or z,

$$F(u) \equiv \int dp\,(p + u^2)^{1/2}. \qquad (2.16)$$

The Euler-Maclaurin summation formula,[12]

$$\Sigma_{n=1} F(n) - \int dz\, F(z) = -1/2\,F(0) - 1/12\,F'(0) + 1/720\,F'''(0)\ldots \qquad (2.17)$$

where primes denote derivatives, will be used to solve Eq. (2.15), with the assumption that $F(\infty) \to 0$. This assumption is valid since a real conductor will become transparent to electromagnetic waves with frequencies above the plasma frequency of the conductor; the plates will remain unaffected for frequencies above a cutoff frequency.[13] We are in essence assuming that the Casimir force is a low frequency effect.

Integrating Eq.(2.16) gives

$$F(u) = \int dp\,(p + u^2)^{1/2} = 2/3\,(p + u^2)^{3/2}$$

$$= F(\infty) - F(0) = 0 - (2/3)\,u^3 = (-2/3)\,u^3\,, \qquad (2.18)$$

where the limits of integration is from 0 to ∞. From Eq. (2.18) we have that $F'(0) = 0$, $F'''(0) = -4$, and all higher order derivatives are equal to zero. Substituting these results into Eq. (2.15) we obtain

$$U(d) = L^2\pi^2 / 4d^2\,(-4 / 720) = L^2\pi^2 / 720\,d^3 \qquad (2.19)$$

which gives a force (in fermi(f^{-2})) of

$$F(d) = -\nabla\,U(d) = -\pi^2\,L^2 / (240\,d^4)\,, \qquad (2.20)$$

or

$$F(d) = -\pi^2\,L^2\hbar c / (240\,d^4)\,, \qquad (2.21)$$

in dynes.

This result is the Casimir force.

2.3 LAMOREAUX EXPERIMENT

The most precise experimental results, to date, for detecting the Casimir effect were obtained by Lamoreaux.[14] The experiment is extremely difficult to perform with parallel plates. Lamoreaux used one conducting plate and a spherical lens, both of which were made of quartz and coated in copper and gold. Since the Casimir force is geometry dependent another force equation had to be derived. This equation does not have an exact solution; however, if the radius of curvature of the lens R is much greater than the separation of the plates A then the attractive force (in dynes) between the lens and the plate is,

$$F(u) = 2\pi R\hbar c \{ \pi^2/ (720 A^3)\} . \tag{2.22}$$

In the experiment a plate was attached to a sensitive torsion balance. If one plate moved towards the other the pendulum would twist. A laser was used to measure the twisting of the pendulum to an accuracy of 0.01 microns. When a current was applied to a system of piezoelectric components one of the plates moved. At the same time an electronic feedback system was used to maintain the position of the torsion pendulum. The zero-point energy effect is related to the amount of current needed to maintain the pendulum's position. Lamoreaux found that the measured force fell within 5 percent of the value predicted by Eq. (2.22).

CHAPTER III

GRAVITATION

3.1 GRAVITATIONAL ENERGY DENSITY

Newton's theory of gravitation will be used to determine the energy density of the vacuum and the result will be compared to the zero-point energy density, which was obtained in the previous chapter.

In order to accomplish this goal let us assume that the universe possesses an energy density; then since matter and energy are equivalent there should be some gravitational effects. The cosmological principle[15] assures us that the universe is homogenous and isotropic; therefore, the energy density of the universe must be uniformly distributed. Although general relativity is necessary for analysis on a large scale, it is possible, due to a homogenous and isotropic universe, to choose a volume large enough to apply Newton's laws and small enough to remove any general relativistic effects. On the other hand Newton's laws have stood the test of time in our solar system; this implies that the energy density is not sufficiently large to affect planetary motion. Now according to Newton's laws, the solar force F_s on a planet of mass M a distance r from the sun M_s is given by

$$F_s = G\,M\,M_s / r^2. \tag{3.1}$$

Now an energy density in the vacuum ρ_g would produce a force F_v on the planets. To determine F_v, consider a sphere of radius r which contains a uniform matter density given by M (r).

Then

$$F_v = G\,M(r)\,M/\,r^2, \tag{3.2}$$

and

$$M(r) = \int \rho_g\,dv = 4/3\;\pi\,r^3\rho_g \tag{3.3}$$

which gives

$$F_v = \{4\pi\rho_g\,G\,M\,r\}/\,3. \tag{3.4}$$

It can also be shown (see Appendix B) that the uniform energy density outside the sphere will not produce any force on the planet. The consistency of Newton's Laws assures us that F_v is much smaller than F_s. Comparing Eq. (3.1) to Eq. (3.4) gives

$$\rho_g < M_s / [\,(4\pi/3)\, r^3]\,. \tag{3.5}$$

Using $M_s = 1.99 \times 10^{30}$ kg and the orbital radius of Pluto, $r = 5.9 \times 10^{12}$ m, then (see Appendix C)

$$\rho_g < 2.3 \times 10^{-9}\ \mathrm{kg/m^3} = 6.5 \times 10^{-27}\ f^{-4}. \tag{3.6}$$

This result clearly indicates that the gravitational energy density of the vacuum is very small .

3.2 COMPARISON OF ρ_{vac} TO ρ_g

The universe consists of visible as well as a comparatively larger amount of dark matter. Since the energy density would be affected most by the larger amount of matter, the present estimate for the amount of dark matter[16] will be used for this comparison. Using $\rho_g < 10^{-26}$ kg / m^3 = $2.9 \times 10^{-44}\ f^{-4}$ for the amount of dark matter in the universe and comparing with Eq. (1.22) , where $\rho_{vac} = 8.4 \times 10^{76}\ f^{-4}$, we obtain a discrepancy of 120 orders of magnitude. This is one of the largest discrepancies between two theories in all of science.

3.3 EINSTEIN'S FIELD EQUATION

The field equation of general relativity is given by[17]

$$G_{ab} = 8\pi\, \tau_{ab}\ , \tag{3.7}$$

where τ_{ab}, the energy-momentum tensor, corresponds to the distribution of matter and G_{ab}, the metric tensor, corresponds to the geometry of space. The field equations are difficult to solve because they are non-linear and do not follow the superposition principle. This non-linearity occurs because a gravitational field contains energy which, according to the equivalence of mass and energy, is also a source of gravitational fields. The essence of general relativity is that gravity couples universally with energy. This principle must apply to the enormous zero-point energy of the vacuum, yet the gravitation effect is not observed. Perhaps an analysis of the nature of this interaction may provide the essential information necessary for a workable theory of quantum gravity.

3.4 QUANTUM FIELD THEORY

Quantum field theory combines special relativity and quantum mechanics. In quantum mechanics the wave function ψ ($\mathbf{r}$,t) is a field, which is used to predict the results of observation. In quantum field theory, however, the wave function ψ ($\mathbf{r}$,t) is quantized and becomes an operator $\psi(\mathbf{r},t)$, which satisfies certain commutation relations.[18]
This method is called "second quantization" and requires the classical coordinates of quantum mechanics to become quantum operators. Since ψ ($\mathbf{r}$,t) is a field the quantum operators possess continuous rather than discrete indicies. Thus, quantum field theory has an infinite number of degrees of freedom and frequently leads to infinities in computation. As an example consider the zero-point energy of Eq. (1.22)

$$\rho_{vac} = (\omega_{max})^4 / 8\pi^2, \tag{3.8}$$

ρ_{vac} can only be determined for a particular finite value of ω_{max}, since $\rho_{vac} \rightarrow \infty$ as $\omega_{max} \rightarrow \infty$. Quantum field theory only requires differences in energies; so, the infinite solutions can be eliminated by arbitrarily defining the vacuum energy as a zero point of reference. It is seen then that in quantum field theory, infinite energies can be easily defined away. However, since gravity couples universally with matter and energy, the infinite energy of the electromagnetic vacuum must be considered in general relativity. Currently there is no acceptable explanation as to why such a large zero-point vacuum energy has such a small gravitational effect.

CHAPTER IV

THE COSMOLOGICAL CONSTANT PROBLEM

4.1 COSMOLOGICAL PRESSURE

The large electromagnetic energy density of the vacuum suggests that the universe possesses a pressure which is analogous to a cosmological constant. To demonstrate this let us treat the universe as a gas undergoing an adiabatic expansion with pressure P, energy density μ, internal energy E, and a zero external force.

From the first law of thermodynamics the work done is

$$dE = -PdV, \tag{4.1}$$

and

$$dE = \mu\, dV, \tag{4.2}$$

which combine to yield

$$P = -\mu. \tag{4.3}$$

According to Eq.(4.3) the energy density of the universe is equivalent to a negative pressure which can oppose the attractive force of gravity and cause the universe to expand. Also, Eq. (4.2) gives

$$dE / dV = \mu . \tag{4.4}$$

This result indicates that the work, which expands the universe, goes into maintaining a constant value for the cosmological constant. If the cosmological constant is as large as the value predicted by Eq. (1.22), the universe would be expanding at a rate of 10^{60} times its present value.[19] We are then faced with the question of whether the cosmological constant does exist and if so, what is its value? This is the cosmological constant problem.

4.2 THE AGE PROBLEM

The equation for the expansion of the universe R(t) is[20]

$$H^2 \equiv (\{dR/dT\}/ R)^2 = 8\pi G\rho_m / 3 \;+\; \Lambda / 3 \;- k / R^2, \tag{4.5}$$

where ρ_m is the mass density of the universe, $k = {}^-1, 0, {}^+1$ is the curvature for a universe which is respectively open, flat or closed; H is Hubbell's constant with a present value of

H_o, and G is the universal gravitational constant. According to Eq. (4.5) there are three terms which govern the expansion of the universe: a matter term, a cosmological constant term, and a curvature term.

Let

$$\Omega_m \equiv 8\pi G\rho_m / \{3\, H_o^{\,2}\}\,, \tag{4.6}$$

$$\Omega_\Lambda \equiv \Lambda / \{3\, H_o^{\,2}\}, \tag{4.7}$$

$$\Omega_k \equiv - k / \{R_o^2 H_o^{\,2}\}, \tag{4.8}$$

then

$$\Omega_m + \Omega_\Lambda + \Omega_k = 1. \tag{4.9}$$

The present cosmology model predicts an age for the universe, which is much younger than the oldest stars. This problem can be resolved by choosing a sufficiently large cosmological constant in Eq. (4.3) for a flat universe. Assume that there is no cosmological constant and that the universe is flat. In this case the expansion of the universe will be governed by the amount of matter it contains. The more massive the universe the younger it would be. The expansion of a massive universe will eventually be slowed down by the force of gravity. This implies that the universe expanded at a faster rate in the past than in the present and therefore took less time to reach its present size. Thus it will be younger. On the other hand assume that there is a cosmological constant and that the universe is flat. Since the cosmological constant is equivalent to a pressure, the force provided by the pressure will become greater as the universe expands. Thus the universe will expand at a faster rate in the present than it did in the past. This will allow for an older universe. Thus by choosing the proper combination of Ω_m and Ω_Λ in Eq. (4.9), with $\Omega_k = 0$, we can adjust our models to resolve the age problem; the larger the cosmological constant, the older the universe. This implies that the cosmological constant is physically significant and requires further study.

4.3 SEARCHING FOR THE COSMOLOGICAL CONSTANT

The expanding universe has a decreasing mass density and a constant cosmological constant. This relationship should provide observational effects that may help in determining the value of the cosmological constant. There are several experimental techniques presently available for the measurement of the cosmological constant; however, two of these seem most promising; the study of gravitational lensing and the study of high redshift supernovae.

The theory of general relativity predicts that light, from an object, will bend in a gravitational field and produce gravitational lenses. These lenses will then produce multiple images of the object. The difference in travel time from the source to the observer of two distinct rays is inversely proportional to Hubbell's constant.[21]

The method of gravitational lensing studies the images of quasars, which are the furthest visible objects in the universe, that are lensed by elliptical galaxies. This method is unique because it is independent of the dynamical state of the object that serves as the lens. It is expected that high values of the cosmological constant will produce larger numbers of gravitational lenses.[22] Thus far only a few gravitational lenses have been observed. This suggests that the cosmological constant is small.

Another experimental method is the study of type Ia supernovae[23] explosions. These explosions occur when a dying white dwarf star pulls too much gas off a neighboring red giant[24] and undergoes a thermonuclear explosion. The study of type Ia supernovae is unique because all type Ia explosions are the same; type Ia supernovae explosions which occurred when the universe was young are the same as those which occur today. By comparing the brightness of distant supernovae to those which are nearby, we are able to determine their distances. Using their distance and redshift, the Hubble constant can be determined. Recent data from both methods indicates that Ω_{Λ_0} is less than $0{\cdot}5$ within a 95% confidence interval, this implies that the universe will keep expanding. However, both of these methods are quite new and require further refining.

CHAPTER V

SUMMARY AND CONCLUSION

The purpose of this thesis was to demonstrate the ideas involved in the cosmological constant problem. This was accomplished by comparing the zero-point energy density of the vacuum to the observed energy density and showing that these differ by 120 orders of magnitude. It was also shown that the cosmological constant could act as a pressure to expand the universe and that the work done in an adiabatic expansion is precisely the amount of energy necessary to keep the cosmological constant a constant.

The question of course is, *does the cosmological constant exist*? This problem involves two of the most successful physical theories. Quantum field theory which is accurate to one part in 10^{11} and general relativity with an accuracy[25] of one in 10^{14} should not disagree by 120 orders of magnitude. Perhaps this is a clue that both theories are approximations of a more fundamental theory.

APPENDIX

APPENDIX A

DERIVATION OF EQ. (14)

Given:

$$\nabla \cdot \{ \mathbf{C} \, \mathrm{X} \, \mathbf{D} \} = \mathbf{D} \cdot \nabla \mathrm{X} \, \mathbf{C} - \mathbf{C} \cdot \nabla \mathrm{X} \, \mathbf{D}. \tag{A.1}$$

$$\nabla \cdot \mathbf{A} = 0. \tag{A.2}$$

$$\nabla^2 \mathbf{A} + \mathbf{k}^2 \mathbf{A} = \mathbf{0} \text{ (Hemholtz equation).} \tag{A.3}$$

$$\mathbf{B} = \nabla \mathrm{X} \; \mathbf{A} \tag{A.4}$$

$$\mathbf{A} = \mathbf{A}_\circ \; e^{-i(\omega t + \mathbf{k} \cdot \mathbf{x})} \tag{A.5}$$

Show:

$$\mathbf{B} \cdot \mathbf{B}^* = \mathbf{k}^2 \; \mathbf{A} \cdot \mathbf{A}^* \tag{A.6}$$

Proof:

$$\mathbf{B} \cdot \mathbf{B}^* = \{ \nabla \mathrm{X} \, \mathbf{A} \} \cdot \{ \nabla \mathrm{X} \, \mathbf{A}^* \} \tag{A.7}$$

$$= \mathbf{A}^* \cdot \nabla \times \nabla \times \mathbf{A} + \nabla \cdot \{\mathbf{A} \times \nabla \times \mathbf{A}^*\} \qquad \text{(A.8)}$$

$$= \mathbf{A}^* \cdot \{ \nabla \cdot \nabla \mathbf{A} - \nabla^2 \mathbf{A}\} + \nabla \cdot \{\mathbf{A} \times \nabla \times \mathbf{A}^*\}. \qquad \text{(A.9)}$$

Substituting (A.2) and (A.3) into (A.9) we obtain

$$\{ \nabla \times \mathbf{A} \} \cdot \{\nabla \times \mathbf{A}^*\} = \mathbf{k}^2 \mathbf{A} \cdot \mathbf{A}^* + \nabla \cdot \{\mathbf{A} \times \nabla \times \mathbf{A}^*\}. \qquad \text{(A.10)}$$

Integrating (A.10) gives

$$\int ((\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{A}^*))\, dV = \oint \nabla \cdot (\mathbf{A} \times \nabla \times \mathbf{A}^*)\, dV$$

$$+ \int \mathbf{k}^2 \mathbf{A} \cdot \mathbf{A}^*\, dV. \qquad \text{(A.11)}$$

Using the divergence theorem, we have

$$\oint \nabla \cdot (\mathbf{A} \times \nabla \times \mathbf{A}^*)\, dV = \oint d\mathbf{S} \cdot (\mathbf{A} \times (\nabla \times \mathbf{A}^*)). \qquad \text{(A.12)}$$

This surface integral vanishes due to the periodic boundary conditions on **A** and **A***.

Thus (A.11) becomes

$$\int ((\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{A}^*))\, dV = \int \mathbf{k}^2 \mathbf{A} \cdot \mathbf{A}^*\, dV \ , \tag{A.13}$$

or

$$\mathbf{B} \cdot \mathbf{B}^* = \{ \nabla \times \mathbf{A} \} \cdot \{ \nabla \times \mathbf{A}^* \} = \mathbf{k}^2 \mathbf{A} \cdot \mathbf{A}^* \tag{A.14}$$

APPENDIX B

In Appendix B Newton's Law of gravitation will be used to derive the force on a mass located anywhere within the hollowed-out region of a solid sphere.

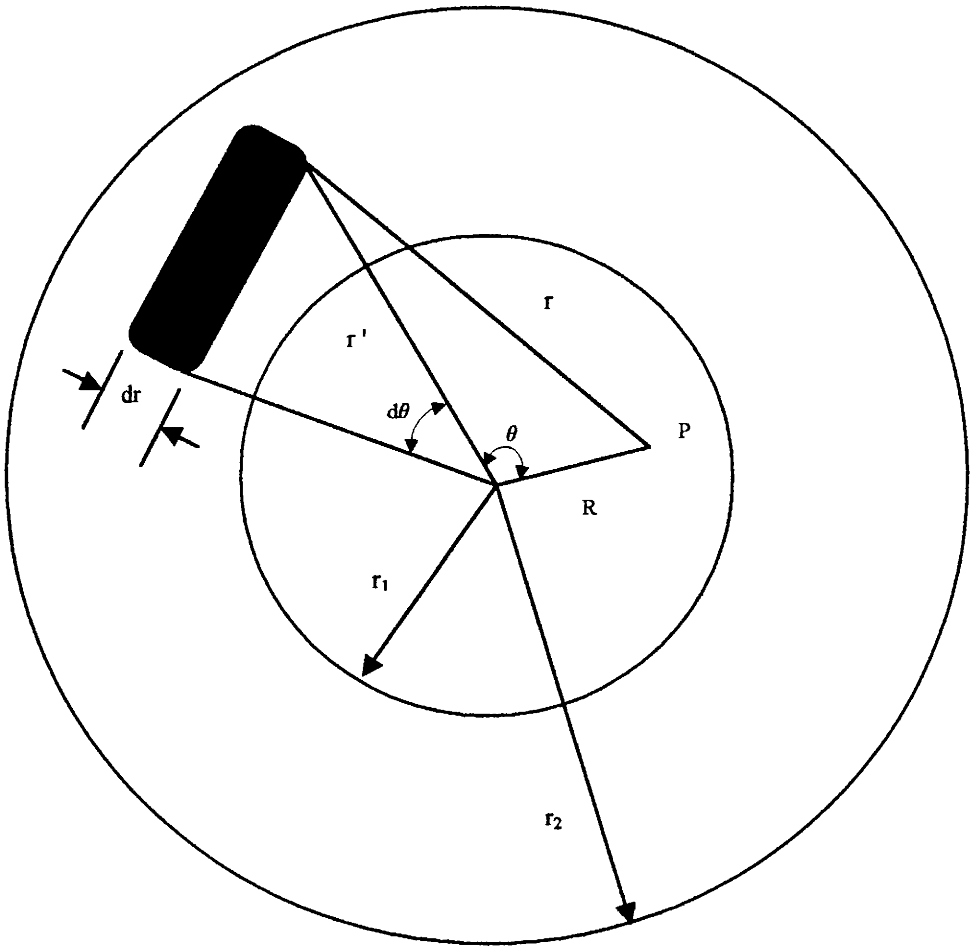

Figure B-1. A hollowed-out sphere of radius r_1 within a solid sphere of radius r_2, which contains a uniform distribution of matter.

Consider the hollowed-out sphere in figure B-1. The potential Φ at the field point P (where the prime denotes a dummy variable of integration) is given by[26]

$$\Phi = -G \int \{\rho\,(r') / r\} dv' . \tag{B.1}$$

For a uniform mass distribution

$$\rho\,(r') = \rho . \tag{B.2}$$

In spherical coordinates

$$\Phi = \quad -G\rho \iiint \{r'^{\,2} \sin\theta \; dr' \; d\theta \; d\phi\} / r , \tag{B.3}$$

upon integrating with respect to ϕ from 0 to 2π, we have

$$\Phi = \; -2\pi G\rho \int r'^{\,2} \, dr' \int \{ \sin\theta \; d\theta \} / r . \tag{B.4}$$

The law of cosines gives

$$r^2 = r'^2 + R^2 - 2r' \, R\cos\theta , \tag{B.5}$$

since R is constant for a given r', differentiation yields,

$$2rdr = 2r'R\sin\theta d\theta \tag{B.6}$$

substituting Eq. (B.6) into Eq. (B.4) and integrating first with limits on r from r' - R to r' + R and then with limits on r' from r_1 to r_2 yields,

$$\Phi = \{-2\pi G\rho / R\}\int r'\ dr' \int dr$$

$$= -4\ G\rho \int r'dr'$$

$$= -2\pi G\rho\ (\ r_2{}^2 - r_1{}^2\ )\ . \tag{B.7}$$

According to Eq. (B.7) the potential inside the hollowed-out region is constant, therefore the force on any mass placed within that region is zero.

APPENDIX C

In this appendix the Planck time, the Planck mass and the Planck energy will be given. Also, the unit conversions for Eq. (1.22) and Eq. (3.6) will be determined.

PLANCK UNITS

The Planck time t_p is obtained from a combination of fundamental physical constants. It contains Newton's gravitational constant G, Planck's constant $\hbar$ and the speed of light c. For $\hbar = 1.05 \times 10^{-34}$ J/s, $G = 6.672 \times 10^{-11}$ N m^2/kg^2,

and $c = 3.0 \times 10^8$m/s, we have

$$t_p = \{\hbar G / c^5\}^{1/2} = 5.39 \times 10^{-44} \text{ s.} \tag{C-1}$$

The Planck mass M_p is also determined by a combination of the fundamental constants and is given by,

$$M_p = \{\hbar c / G\}^{1/2} = 2.18 \times 10^{-5} \text{ g.} \tag{C-2}$$

The Planck energy E_p is then obtained from,

$$E_p = M_p c^2 = 1.221 \times 10^{19} \text{ Gev.} \tag{C-3}$$

THE UNIT CONVERSION OF EQ. (1.22)

Now let us consider the unit conversion of Eq. (1.22). From Eq. (1.21)

$$\rho_{vac} = (\omega_{max})^4 / 8\pi^2 . \tag{C-4}$$

Let us introduce the fermi (f) unit of length, where $1f = 1x\ 10^{-15}$m .

$$\text{Now } \hbar c = 3.162 \times 10^{-17} \text{ erg} \cdot \text{cm} = 197.3 \text{ Mev} \cdot \text{f}. \tag{C-5}$$

For $\omega_{max} = 10^{19}$ Gev, we have

$$\omega_{max} = \{10^{19} \text{ Gev}\}\{\ 1/(\ 197.3 \text{ Mev} \cdot \text{f})\ \}$$

$$= 5.08 \times 10^{9} / \text{f}. \qquad \text{(C-6)}$$

Substituting (C-6) into (C-4) yields,

$$\rho_{vac} = (\omega_{max})^4 / 8\pi^2$$

$$= (5.08 \times 10^9 / \text{f})^4 / (8\pi^2)$$

$$= 8.4 \times 10^{76}\ \text{f}^{-4}. \qquad \text{(C-7)}$$

THE UNIT CONVERSION OF EQ. (3.7)

From Eq. (3.6)

$$\rho_g < 2.3 \times 10^{-9} \text{ kg/m}^3. \qquad \text{(C-8)}$$

Using $E = mc^2$, we obtain

$$1 \text{ kg} = 9 \times 10^{16} \text{ J}. \qquad \text{(C-9)}$$

Also,

$$1 \text{ Mev} = 1.602 \times 10^{-13} \text{ J}, \qquad \text{(C-10)}$$

and

$$\hbar c = 197.3 \text{ Mev} \cdot \text{f}. \qquad \text{(C-11)}$$

Recalling that

$$1\text{f} = 1 \text{x } 10^{-15}\text{m}, \qquad \text{(C-12)}$$

and combining (C-8) thru (C-12), we obtain

$$(2.3 \times 10^{-9}\ \mathrm{kg/m^3})\ (9.0 \times 10^{16}\ \mathrm{J/kg})\ (1\mathrm{Mev}/\ 1.602 \times 10^{-13}\mathrm{J})(1 \times 10^{-15}\mathrm{m/\ f})$$

$$(\ 1/\ 197.3\ \mathrm{Mev{\cdot}f}) = \ 6.5 \times 10^{-27}\ \mathrm{f}^{-4}. \qquad \text{(C-13)}$$

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