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AN EXISTENCE RESULT FOR ELLIPTIC PROBLEMS WITH SINGULAR CRITICAL GROWTH

YASMINA NASRI

Abstract. We prove the existence of nontrivial solutions for the singular critical problem

$$-\Delta u - \mu \frac{u}{|x|^2} = \lambda f(x) u + u^{2^*-1}$$

with Dirichlet boundary conditions. Here the domain is a smooth bounded subset of $\mathbb{R}^N$, $N \geq 3$, and $2^* = \frac{2N}{N-2}$ which is the critical Sobolev exponent.

1. Introduction

This paper concerns the semilinear elliptic problem

$$\begin{gathered} -\Delta u - \mu \frac{u}{|x|^2} = \lambda f(x) u + u^{2^*-1} \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 3$ with $0 \in \Omega$; λ and μ are positive parameters with $0 \leq \mu < \overline{\mu} := (\frac{N-2}{2})^2$, $\overline{\mu}$ is the best constant in the Hardy inequality, $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent and f is a positive measurable function which will be specified later.

In recent years, many people have paid much attention to the existence of nontrivial solutions for singular problems we cite [4, 5, 7, 8] and the references cited therein.

For $f(x) = 1$, Jannelli [7] obtained the following results:

If $0 \leq \mu \leq \overline{\mu} - 1$, then (1.1) has at least one solution $u \in H_0^1(\Omega)$ for all $0 < \lambda < \lambda_1(\mu)$ where $\lambda_1(\mu)$ is the first eigenvalue of the operator $(-\triangle - \frac{\mu}{|x|^2})$ in $H_0^1(\Omega)$.

If $\overline{\mu} - 1 < \mu < \overline{\mu}$, then (1.1) has at least one solution $u \in H_0^1(\Omega)$ for all $\mu^* < \lambda < \lambda_1(\mu)$ where

$$\mu^* = \min_{\varphi \in H_0^1(\Omega)} \frac{\int_\Omega \frac{|\nabla \varphi(x)|^2}{|x|^{2\sigma}} dx}{\int_\Omega \frac{|\varphi(x)|^2}{|x|^{2\sigma}} dx}$$

and $\sigma = \sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu}$.

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If $\overline{\mu}-1<\mu<\overline{\mu}$ and $\Omega=B(0,R)$ then (1.1) has no solution for $\lambda\leq\mu^*$.

If $\lambda\leq 0$ and Ω is star shaped then (1.1) has no nontrivial solutions using Pohozaev-type identity.

For the quasi-linear form of (1.1) the problem has been studied by [5] for $\mu=0$ and $f(x)=\frac{1}{|x|^q}$ where $0\leq q<p$. The purpose of the present paper is to extend (partially) the results obtained by [7] to the case where f can be singular.

This paper is organized as follows. In section 2, we recall some preliminaries results. In section 3, we give the proof of our theorem using mountain pass Theorem.

2. Notation and Preliminaries

We make use the following notation:

$L^p(\Omega)$, $1\leq p\leq\infty$, denote Lebesgue spaces, the norm L^p is denoted by $\|\cdot\|_p$ for $1\leq p\leq\infty$;

$D^{1,2}(\mathbb{R}^N)$ denotes the closure space of $C_0^\infty(\mathbb{R}^N)$ with respect the norm $\|\cdot\|_{D^{1,2}(\mathbb{R}^N)}:=\big(\int_{\mathbb{R}^N}|\nabla u|^2dx\big)^{1/2}$;

$B_r(0)$ is the ball centred at 0 with radius r;

C, C_1, C_2 will denote various positive constants;

On $H_0^1(\Omega)$ we use the norm

$$\|u\|_\mu=\Big(\int_\Omega(|\nabla u|^2-\mu\frac{u^2}{|x|^2})dx\Big)^{1/2}.$$

By Hardy's inequality [6], this norm is equivalent to the usual norm of $H_0^1(\Omega)$. Let

$$\mathcal{F}=\big\{f:\Omega\to\mathbb{R}^+:\lim_{|x|\to 0}|x|^2f(x)=0\quad\text{with } f\in L^\infty_{\rm loc}(\Omega\backslash\{0\})\big\};$$

for $0\leq\beta<2$, we set

$$\mathcal{F}_{2,\beta}=\big\{f\in\mathcal{F}:0<\lim_{|x|\to 0}|x|^\beta f(x)<\infty\big\}.$$

Now, we recall the following results.

Lemma 2.1 ([4]). *Let $0\leq\mu<\overline{\mu}=(\frac{N-2}{2})^2$, $\lambda\in\mathbb{R}^+$, $f\in\mathcal{F}$. Then the eigenvalue problem*

$$\begin{gathered}-\Delta u-\mu\frac{u}{|x|^2}=\lambda f(x)u\quad\text{in }\Omega\\ u=0\quad\text{on }\partial\Omega\end{gathered}$$

admits a nontrivial weak solutions in $H_0^1(\Omega)$ corresponding to $\lambda\in(\lambda_\mu^k(f))_{k=1}^\infty$ where $0<\lambda_\mu^1(f)<\lambda_\mu^2(f)\leq\lambda_\mu^3(f)\leq\dots\to+\infty$.

Lemma 2.2 ([4]). *Let Ω be a bounded domain in $\mathbb{R}^N$ and $f\in\mathcal{F}$. Then the embedding $H_0^1(\Omega)\hookrightarrow L^2(\Omega,f\,dx)$ is compact.*

Lemma 2.3 ([4]). *Let $2_\beta^*=\frac{2(N-\beta)}{N-2}$, if $f\in\mathcal{F}_{2,\beta}$, $0\leq\beta<2$; then the embedding $H_0^1(\Omega)\hookrightarrow L^q(\Omega,fdx)$ is (i) continuous for all $2\leq q\leq 2_\beta^*$, (ii) compact for $2\leq q<2_\beta^*$.*

Now, we give some examples of function $f\in\mathcal{F}$ having lower order singularity than $|x|^{-2}$ at the origin:

(a) Any bounded function.
(b) In a small neighbourhood of 0, f is $|x|^{-\beta}$ for $0<\beta<2$.

(c) $f(x) = |x|^{-\beta}/|\log|x||$ in a small neighbourhood of 0.

Definition 2.4. Let $c \in \mathbb{R}$, E be a Banach space and $I \in C^1(E, \mathbb{R})$. We say that I satisfies the Palais-Smale condition at the level c, for short $(PS)_c$, if every sequence $(u_n)_n$ in E such that $I(u_n) \to c$ and $I'(u_n) \to 0$ as $n \to +\infty$ in E' (dual of E), has a convergent subsequence in E.

Definition 2.5. A function u in $H_0^1(\Omega)$ is said to be a weak solution of (1.1) if u satisfies

$$\int_\Omega \Big(\nabla u \nabla v - \mu \frac{uv}{|x|^2} - \lambda f(x) uv dx - u^{2^*-1} v\Big) dx = 0 \quad \text{for all } v \in H_0^1(\Omega).$$

It is well known that the nontrivial solutions of (1.1) are equivalent to the non zero critical points of the energy functional

$$J_{\lambda,\mu}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\mu}{2}\int_\Omega \frac{u^2}{|x|^2} dx - \frac{\lambda}{2}\int_\Omega f(x) u^2 dx - \frac{1}{2^*}\int_\Omega |u|^{2^*} dx.$$

Define the constant

$$S_\mu = \inf_{u \in D^{1,2}(\mathbb{R}^N)\backslash\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx - \mu \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx}{\big(\int_{\mathbb{R}^N} |u|^{2^*}\big)^{2/2^*}}.$$

It is known that S_μ is achieved by the family of functions

$$u_\varepsilon^* = \frac{C_\varepsilon}{(\varepsilon |x|^{\sigma'/\sqrt{\overline{\mu}}} + |x|^{\sigma/\sqrt{\overline{\mu}}})^{\sqrt{\overline{\mu}}}}$$

where $C_\varepsilon = (4\varepsilon N(\overline{\mu} - \mu)/(N-2))^{\frac{\sqrt{\overline{\mu}}}{2}}$, $\sigma = \sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu}$ and $\sigma' = \sqrt{\overline{\mu}} - \sqrt{\overline{\mu} - \mu}$, see [8] for the details.

Note that u_ε^* satisfies

$$-\Delta u - \mu \frac{u}{|x|^2} = |u|^{2^*-2} u \quad \text{for } u \in D^{1,2}(\mathbb{R}^N)\backslash\{0\}.$$

Hence, we have

$$\|u_\varepsilon^*\|_\mu^2 = \|u_\varepsilon^*\|_{2^*}^{2^*} = (S_\mu)^{N/2}.$$

Let $0 \le \phi(x) \le 1$ be a function in $C_0^\infty(\Omega)$ defined as

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \le R \\ 0 & \text{if } |x| \ge 2R, \end{cases}$$

where $B_{2R}(0) \subset \Omega$. Set

$$u_\varepsilon = \phi(x) u_\varepsilon^* \quad \text{and} \quad v_\varepsilon = \frac{u_\varepsilon}{\|u_\varepsilon\|_{2^*}}, \tag{2.1}$$

so that $\|v_\varepsilon\|_{2^*}^{2^*} = 1$.

In the present paper we prove the following result.

Theorem 2.6. *Let $f \in \mathcal{F}_{2,\beta}$ and $0 \le \beta < 2$. If $0 \le \mu \le \overline{\mu} - (\frac{2-\beta}{2})^2$ and $0 < \lambda < \lambda_\mu^1(f)$, then* (1.1) *has at least one positive solution.*

3. Proof of the main theorem

First, we establish some lemmas.

Lemma 3.1. *Assume that $f \in \mathcal{F}_{2,\beta}$ and $0 < \lambda < \lambda_\mu^1(f)$. Then $J_{\lambda,\mu}$ satisfies $(PS)_c$ for all $c < (S_\mu)^{N/2}/N$.*

Proof. Let $(u_n)_n$ be a sequence such that

$$J_{\lambda,\mu}(u_n) \to c \quad \text{and} \quad J'_{\lambda,\mu}(u_n) \to 0 \quad \text{in } [H_0^1(\Omega)]' \text{ as } n \to +\infty. \tag{3.1}$$

We remark that

$$2J_{\lambda,\mu}(u_n) - \langle J'_{\lambda,\mu}(u_n), u_n \rangle = (1 - \frac{2}{2^*})\|u_n\|_{2^*}^{2^*} \le 2c + o(1), \tag{3.2}$$

combining (3.1) and (3.2) we show that (u_n) is bounded in $H_0^1(\Omega)$.

From Lemmas 2.2 and 2.3, and the reflexivity of $H_0^1(\Omega)$ we extract a subsequence, still denoted u_n such that

$$\begin{gathered} u_n \to u \quad \text{weakly in } H_0^1(\Omega) \\ u_n \to u \quad \text{in } L^r(\Omega) \text{ if } 1 < r < 2^*, \\ u_n \to u \quad \text{almost everywhere,} \\ \frac{u_n}{x} \to \frac{u}{x} \quad \text{weakly in } L^2(\Omega), \\ u_n \to u \quad \text{strongly in } L^2(\Omega, f dx). \end{gathered} \tag{3.3}$$

From (3.3) we deduce that

$$\langle J'_{\lambda,\mu}(u), \varphi \rangle = 0 \quad \text{for all } \varphi \in H_0^1(\Omega), \tag{3.4}$$

hence u is a solution of (1.1).

Denote $v_n := u_n - u$, then the Brezis-Lieb lemma [2] implies

$$\begin{gathered} \|\nabla u_n\|_2^2 = \|\nabla u\|_2^2 + \|\nabla v_n\|_2^2 + o(1); \\ \|u_n\|_{2^*}^{2^*} = \|u\|_{2^*}^{2^*} + \|v_n\|_{2^*}^{2^*} + o(1); \\ \int_\Omega \frac{u_n^2}{|x|^2} dx = \int_\Omega \frac{u^2}{|x|^2} dx + \int_\Omega \frac{v_n^2}{|x|^2} dx + o(1). \end{gathered} \tag{3.5}$$

Using (3.1), (3.5) and lemma 2.2, we obtain

$$J_{\lambda,\mu}(u) + \frac{1}{2}\|v_n\|_\mu^2 - \frac{1}{2^*}\|v_n\|_{2^*}^{2^*} = c + o(1), \tag{3.6}$$

and

$$\|u\|_\mu^2 = \|u\|_{2^*}^{2^*} + \lambda \int_\Omega f(x) u^2 dx - \|v_n\|_\mu^2 + \|v_n\|_{2^*}^{2^*} + o(1).$$

From (3.4) it follows that

$$\|v_n\|_\mu^2 - \|v_n\|_{2^*}^{2^*} = o(1).$$

We may therefore assume that

$$\|v_n\|_\mu^2 \to a \quad \text{and} \quad \|v_n\|_{2^*}^{2^*} \to a,$$

by the definition of S_μ, we have

$$S_\mu \|v_n\|_{2^*}^2 \le \|v_n\|_\mu^2,$$

in the limit we have

$$S_\mu a^{2/2^*} \le a,$$

it follows that either $a = 0$ or $a \geq (S_\mu)^{N/2}$.

If $a \geq (S_\mu)^{N/2}$ passing in the limit in (3.6) we obtain

$$J_{\lambda,\mu}(u) + \frac{1}{N}a = c$$

using the assumption $c < \frac{1}{N}(S_\mu)^{N/2}$, we find

$$J_{\lambda,\mu}(u) < 0. \tag{3.7}$$

On the other hand, from (3.4) we obtain

$$J_{\lambda,\mu}(u) = \frac{1}{N}\|u\|_{2^*}^{2^*} \geq 0,$$

which is a contradiction with (3.7). Then $u_n \to u$ strongly in $H_0^1(\Omega)$. □

Lemma 3.2. *Assume that $f \in \mathcal{F}_{2,\beta}$ then 1/ There exist α, $\delta > 0$ such that $J_{\lambda,\mu}(u) \geq \alpha$ for all $u \in H_0^1(\Omega)$ such that $\|u\|_\mu = \delta$ for all $0 < \lambda < \lambda_\mu^1(f)$. 2/ $J_{\lambda,\mu}(v) < 0$ for all $v \in H_0^1(\Omega)$ such that $\|v\|_\mu > \delta$.*

Proof. Using the definition of S_μ and the fact that $0 < \lambda < \lambda_\mu^1(f)$, we obtain

$$J_{\lambda,\mu}(u) \geq \frac{1}{2}\big(1 - \frac{\lambda}{\lambda_\mu^1(f)}\big)\|u\|_\mu^2 - \frac{1}{2^*(S_\mu)^{2^*/2}}\|u\|_\mu^{2^*}.$$

So for $\delta > 0$ sufficiently small there exists $\alpha > 0$ such that

$$J_{\lambda,\mu}(u) \geq \alpha \quad \text{for } \|u\|_\mu = \delta.$$

For $t > 0$,

$$J_{\lambda,\mu}(tu) = \frac{t^2}{2}(\|u\|_\mu^2 - \int_\Omega f(x)u^2dx) - \frac{t^{2^*}}{2^*}\|u\|_{2^*}^{2^*}dx,$$

as $t \to +\infty$ we have $J_{\lambda,\mu}(tu) \to -\infty$. Then there exists $v \in H_0^1(\Omega)$ such that $J_{\lambda,\mu}(v) < 0$ for $\|v\|_\mu > \delta$. □

Lemma 3.3. *Assume that $0 < \lambda < \lambda_\mu^1(f)$ and $0 \leq \mu \leq \overline{\mu} - (\frac{2-\beta}{2})^2$. Then*

$$\sup_{0 \leq t < \infty} J_{\lambda,\mu}(tv_\varepsilon) < \frac{1}{N}(S_\mu)^{N/2}$$

provided $\varepsilon > 0$ is a small enough.

Proof. Consider the functions

$$g\,(t) := J_{\lambda,\mu}(tv_\varepsilon) = \frac{t^2}{2}(\|v_\varepsilon\|_\mu^2 - \lambda\int_\Omega f(x)v_\varepsilon^2dx) - \frac{t^{2^*}}{2^*},$$

where v_ε is the extremal function defined in (2.1). Note that $\lim_{t\to+\infty} g(t) = -\infty$ and $g(t) > 0$ when t is close to 0. So that $\sup_{t\geq 0} g(t)$ is attained for some $t_\varepsilon > 0$. From

$$0 = g'(t_\varepsilon) = t_\varepsilon\big(\|v_\varepsilon\|_\mu^2 - \lambda\int_\Omega f(x)v_\varepsilon^2dx\big) - t_\varepsilon^{2^*-1}\|v_\varepsilon\|_{2^*}^{2^*},$$

we have

$$t_\varepsilon = \Big[\|v_\varepsilon\|_\mu^2 - \lambda\int_\Omega f(x)v_\varepsilon^2dx\Big]^{\frac{1}{2^*-2}}.$$

Thus,

$$g\,(t_\varepsilon) = \frac{1}{N}\Big(\|v_\varepsilon\|_\mu^2 - \lambda\int_\Omega f(x)v_\varepsilon^2dx\Big)^{\frac{2^*}{2^*-2}}.$$

Then as in [7] (see also [3]), we have the following estimates:

$$\int_{\Omega}\Big(|\nabla v_{\varepsilon}|^2dx-\mu\frac{{v_{\varepsilon}}^2}{|x|^2}\Big)dx=S_{\mu}^{\frac{N}{2}}+C\varepsilon^{\frac{N-2}{2}};$$

since $f\in\mathcal{F}_{2,\beta}$, there exist $r>0$ and $C_1,C_2>0$ such that $K_1|x|^{-\beta}\le f(x)\le K_2|x|^{-\beta}$ on $B_R(0)$. Thus

$$C_1\varepsilon^{\frac{\sqrt{\overline{\mu}}}{2\sqrt{\overline{\mu}-\mu}}(2-\beta)}\le\int_{\Omega}f(x)v_{\varepsilon}^2dx\le C_2\varepsilon^{\frac{\sqrt{\overline{\mu}}}{2\sqrt{\overline{\mu}-\mu}}(2-\beta)}\quad\text{if } \mu<\overline{\mu}-(\frac{2-\beta}{2})^2;$$

$$C_1\varepsilon^{\frac{N-2}{2}}|\log\varepsilon|\le\int_{\Omega}f(x)v_{\varepsilon}^2dx\le C_2\varepsilon^{\frac{N-2}{2}}|\log\varepsilon|\quad\text{if } \mu=\overline{\mu}-(\frac{2-\beta}{2})^2.$$

Consequently,

$$g(t_{\varepsilon})\le\begin{cases}\frac{1}{N}S_{\mu}^{\frac{N}{2}}+C\varepsilon^{\frac{N-2}{2}}-C_1\varepsilon^{\frac{N-2}{2}}|\log\varepsilon| & \text{if } \mu=\overline{\mu}-(\frac{2-\beta}{2})^2,\\ \frac{1}{N}S_{\mu}^{\frac{N}{2}}+C\varepsilon^{\frac{N-2}{2}}-C_1\varepsilon^{\frac{\sqrt{\overline{\mu}}}{2\sqrt{\overline{\mu}-\mu}}(2-\beta)} & \text{if } \mu<\overline{\mu}-(\frac{2-\beta}{2})^2.\end{cases}$$

Therefore, for $\varepsilon>0$ sufficiently small and $\mu\le\overline{\mu}-(\frac{2-\beta}{2})^2$ we get

$$\sup_{t\ge0}J_{\lambda,\mu}(tv_{\varepsilon})<\frac{1}{N}S_{\mu}^{N/2}.$$

□

Proof of Theorem 2.6. From Lemmas 3.1, 3.2 and 3.3, $J_{\lambda,\mu}$ satisfies all assumptions of mountain pass Theorem [1], then c is a critical value i.e. there exists $u\in H_0^1(\Omega)$ such that $J'_{\lambda,\mu}(u)=0$ and $J_{\lambda,\mu}(u)=c>0$. Since $J_{\lambda,\mu}(u)=J_{\lambda,\mu}(|u|)=c$, thus problem (1.1) admits a positive solution. □

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Yasmina Nasri
Université de Tlemcen, département de mathématiques, BP 119 Tlemcen 13000, Algérie
E-mail address: y_nasri@mail.univ-tlemcen.dz